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„2-Dimensional Extended TQFTs and the Cobordism Hypothesis"

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Abstract

Motivated by topological symmetries of quantum mechanics, we study 2-dimensional extended topological quantum field theories (TQFTs). A 2-dimensional extended TQFT with orientation is a symmetric monoidal 2-functor $F : Bord^{or}_{2,1,0} \to \mathcal{B}$. A class of such theories is the state-sum TQFTs where the codomain $\mathcal{B} = Alg^2_{\Bbbk}$ is the symmetric monoidal 2-category of algebras, bimodules and bimodule maps. Fully dualisable objects in such a category are separable symmetric Frobenius algebras. The cobordism hypothesis states that the full 2-subgroupoid of fully dualisable objects in $Alg^2_{\Bbbk}$ classify 2-dimensional oriented TQFTs. We explicitly show this classification.

Abstrakt

Motiviert durch topologische Symmetrien der Quantenmechanik untersuchen wir zweidimensionale erweiterte topologische Quantenfeldtheorien (TQFTs). Eine zweidimensionale erweiterte TQFT mit Orientierung ist ein symmetrischer monoidaler 2-Funktor $F : Bord^{or}_{2,1,0} \to \mathcal{B}$. Eine Klasse solcher Theorien sind die Zustands-Summen-TQFTs, wobei die Zielmenge $\mathcal{B} = Alg^2_{\Bbbk}$ die symmetrische monoidale 2-Kategorie der Algebren, Bimodule und Bimodulabbildungen ist. Vollständig dualisierbare Objekte in einer solchen Kategorie sind separable symmetrische Frobenius-Algebren. Die Cobordismus-Hypothese besagt, dass der volle 2-Untergruppoid der vollständig dualisierbaren Objekte in $Alg^2_{\Bbbk}$ zweidimensionale orientierte TQFTs klassifiziert. Wir zeigen diese Klassifikation explizit.

Acknowledgements

A special thanks to my advisor Nils, not only for his scientific insight and for teaching me the beauty of abstract mathematics, but also for being a good friend and supporting me in my most frustrated moments.

Another special thanks goes to my 'mama' who took care of me under very tough circumstances.

Contents

1 Introduction

1.1 Motivation

The main subject of this thesis is 2-dimensional extended topological quantum field theories (TQFTs). They are *symmetric monoidal 2-functors* from symmetric monoidal 2-category of extended bordisms to a symmetric monoidal 2-category. n-dimensional bordisms between $(n-1)$-dimensional smooth manifolds form an equivalence relation. Such equivalence classes of bordisms form a bordism group with the disjoint union regarded as the group multiplication. The symmetric monoidal 2-category of extended bordisms 'categorifies' bordism groups which are important notions in algebraic topology. One can think of extended TQFTs as 'representations of bordisms' on algebraic structures, generalizing representations of groups on vector spaces.

Among many applications listed at the end of this subsection, extended TQFTs encode topological symmetries of quantum field theories (QFTs). Here, we motivate extended TQFTs by giving an example of a 1-dimensional QFT, namely quantum mechanics, in a language that can be 'generalized' to n-dimensional QFTs.

The reason for starting with quantum mechanics is as follows: higher categorical computation involves systematically studying the intricate web of dots, arrows, and arrows between arrows, extending indefinitely or stopping at a certain point. While the elegance and enthusiasm of studying this may captivate some, it can also serve as a source of frustration for others. However, when viewed through the lens of physics, it swiftly transcends into a realm of profound philosophical depth. Weatherall provides a compelling example, comparing Galilean space-time with Newtonian space-time, Newtonian gravitation with geometrized Newtonian gravitation, Maxwell's electromagnetism with Yang-Mills theory, and geometric general relativity with Einstein algebras through a forgetful functor [29]. Imagine the implications of a 2-functor!

In our current understanding of theoretical physics, any physical theory can be formulated by some field theory. This is because of the fact that all fundamental forces we know of are formulated in terms of field theories.

To motivate field theories from a physical point of view, we will relax mathematical rigor in this subsection and start with an example of a quantum mechanical system. A 'quantum particle' on a Riemannian manifold (N, g) 'takes all paths' between two points with a statistical distribution, each of which has equal possibility in magnitude but with a phase factor of $e^{i/\hbar S[x]}$ where $S[x]$ is the action functional. The probability of a particle that was measured to be at $p_1 \in N$ at time t_1 and will be measured at p_2 at time t_2 can be computed by the partition function $Z(t_1, p_1; t_2, p_2)$. If $N = \mathbb{R}^n$, we consider the graph of paths divided into infinitesimal regions $dx_1 \dots dx_n \dots$ and compute the partition function

$$Z(t_1, p_1; t_2, p_2) = \int_{x(t_1)=p_1}^{x(t_2)=p_2} e^{i/\hbar S[x]} Dx \tag{1.1}$$

where the measure $Dx = dx_1 dx_2 \dots dx_n dx_{n+1} \dots$ is possibly ill-defined. If we take space to be discrete so that the integral becomes a sum, the phase factor in the integral leads to the assumption that the action can be written as $S[x] = \sum_i S(x_i, x_{i+1})$ and that $S(x_i, x_{i+1}) = \delta S|_{t_i}^{t_{i+1}}$ which means that the particle takes the classical path in a small interval extremizing the action.

This is the path integral approach to *canonical quantization* using the Lagrangian of the system. This approach can be made precise using random walk techniques with the stochastic measure [13, Section 2]. We can switch to Hamilton's formalism by computing the path integral for $Z(t_1, p_1, t, x(t))$ and see that this integral is proportional to the wave function [10, Section 5].

Equivalently, one can use Hamilton's point of view for quantization in the first place in which one replaces Poisson brackets with commutators. The fact that a quantum particle follows the classical path in the infinitesimal regions allows us to consider the Hamiltonian which is constant of motion under time translation as a time translation generator and leads to the Schrödinger equation in the non-relativistic case

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi \tag{1.2}$$

where $\psi \in \mathcal{H}$ denotes the wave function belonging to the Hilbert space $\mathcal{H} = L^2(N, \mu_g)$. The Hilbert space is the space of possibly infinite dimensional square integrable complex functions on (N, g) with the measure μ_g. The 1-parameter group of unitary automorphisms of the Hilbert space $U(t) = e^{-it/\hbar H} \in Aut(\mathcal{H})$ generated by the Hamiltonian together with an initial state $\psi(x, 0)$ gives a solution to the Schrödinger equation $\psi(x, t) = e^{-it/\hbar H}\psi(x, 0)$.

To summarize physical ideas developed above, the data which precisely defines a quantum mechanical system is a Hilbert space $\mathcal{H}$ and a self-adjoint operator H. We observe that the background field (N, g) is used in the path integral 'to sum over paths in N' to form the Hilbert space. The time translation operator $U(t)$ depends on the Hamiltonian and the magnitude of t. It has the property that $U(t_2 + t_1) = U(t_2) \circ U(t_1)$. This is the *causality/locality principal* which is a fundamental principal of physics and all physically relevant field theories. Furthermore, it is associative and $lim_{\epsilon \to 0} U(\epsilon) = id_{\mathcal{H}}$. In addition, if we have N particles moving on (N, g) without interaction, their state-space can be described by the tensor product of N copies of the Hilbert spaces. This is the superposition principal of fields and waves.

This motivates us to model quantum mechanics as a monoidal functor from some monoidal category of 1-dimensional geometries to some monoidal category of algebras. A category consists of objects together with a set of morphisms between pairs of objects such that morphisms are composable, for every object there is an identity morphism and the composition is associative. A functor between two categories assigns objects to objects, morphisms to morphisms, identities to identities, such that this assignment is compatible with compositions of both categories. This means that a functor $F : \mathcal{A} \to \mathcal{B}$ satisfies $F(g \circ_{\mathcal{A}} f) = F(g) \circ_{\mathcal{B}} F(f)$. For detailed introduction to categories, readers may refer to [19], [20]. Monoidal categories and functors will be discussed in Section 2.1.1. For now, we regard a monoidal structure on a category as disjoint union of geometrical/topological spaces or tensor product of algebraic spaces in which the empty set and the ground field are monoidal units, respectively. A monoidal functor is a functor that respects the monoidal structure.

Therefore, the monoidal functor definition of quantum mechanics essentially captures causality and superposition principals. This is important because causality and superposition principles are fundamental concepts in physics. Physicists often construct possibly ill-defined complicated theories from better understood simpler theories by relaxing some axioms and keeping more fundamental ones. This suggests that an n-dimensional (quantum) field theory should be a monoidal functor from a monoidal category of n-dimensional geometries to a monoidal category of algebras.

Next, we discuss the domain of this functor. Roughly, it is a geometric category $Bord_{1,0}^{Riem,or}$ whose objects are oriented points and morphisms are 1-dimensional oriented manifolds of length t, composition is gluing of 1-manifolds along their common boundary (points) and disjoint union is the monoidal product, empty set is the monoidal unit. The issue here is gluing. First of all, we cannot smoothly glue 1-manifolds along a point; we need to define objects as points together with collars that are germs embedded in 1-dimensional Rimeannian manifolds. Such germs, in fact, capture infinitesimal time translation, i.e. the Hamiltonian. This lets us to define a smooth composition but it is still not associative since there is no unique smooth structure on the glued manifold. However, as it will be discussed in Section 3.1, this is sufficient to define *a topological field theory* by taking diffeomorphism classes of 1-bordisms. Here, in the geometrical case, one needs a certain equivalence class of morphisms to have a well-defined associative composition. Furthermore, to define a sensible quantum mechanical system, one needs to include a background field (N, g) which represents space-time. This is achieved by endowing 1-dimensional bordism category with sheaves and stacks smoothly attached to every small neighborhood of 1-manifolds. This can be thought as smooth fibering over a dynamical 1-dimensional manifold in which fields live in, and gluing can be defined by fiber products. An interested reader is referred to a recent paper [24].

The codomain is the category of Hilbert spaces where objects are Hilbert spaces, morphisms are linear maps respecting the structure of the Hilbert space [16]. To sum up, a quantum mechanical system is a monoidal functor $F : Bord^{Riem,or}(N, g) \to Hilb$:

- A point with a + orientation is mapped to a Hilbert space $\mathcal{H}$ and negatively oriented point is mapped to the dual space $\mathcal{H}^*$.
- A bordism with length τ between two points is mapped to a Wick-rotated linear map $U(\tau) = e^{-i\tau H/\hbar}$.
- Gluing of 1-manifolds is mapped to composition of linear maps
- The monoidal unit $\emptyset$ is mapped to the ground field $\mathbb{C}$.

- The disjoint union is mapped to the tensor product.

We now take our attention to symmetries. Traditionally in physics literature, symmetries are understood as a set of invertible tranformations of the model that leaves the action functional invariant. There are two types of symmetries. First, symmetries of space-time e.g. group of isometries of a Riemannian manifold. Second, internal symmetries that are internal to the model e.g. $U(1)$-symmetry of electromagnetism.

Classically, the Noether procedure associates conserved quantities (Noether charges) to symmetries. In quantum systems, on the other hand, a Noether charge is treated as an operator on the Hilbert space as a generator of the symmetry that commutes with the Hamiltonian.. More precisely, an invertible symmetry in a quantum mechanical system is expressed by a Lie group G together with a representation $\rho : G \to GL(\mathcal{H})$ such that elements of representation of its Lie algebra commute with the Hamiltonian. Let $\mathfrak{g}$ be the Lie algebra of G and $\rho' : \mathfrak{g} \to GL(\mathcal{H})$ be its representation on the Hilbert space. The Hamiltonian commutes with elements of representations of the Lie algebra: $[H, \rho'(L)] = 0$ for all $L \in \mathfrak{g}$. Here, $\rho'(L)$ corresponds to the Noether charge.

The case we are interested in is internal topological symmetries, including non-invertible symmetries. Since we described quantum mechanical system 'globally', such symmetries appear more naturally. A non-invertible symmetry amounts to 'a group without inverses', thus, an algebra (Definition 2.1.2), and a representation amounts to modules over algebras (Definition 2.1.7). To make this connection clearer, we consider the group algebra of a finite group in Example 2.1.9 and see that modules over group algebras are precisely the representations of groups. We expect a non-invertible topological symmetry of a quantum mechanical system to be an algebra A together with a left A-module $(\mathcal{H}, l)$, where $l : A \otimes \mathcal{H} \to \mathcal{H}$ is the left A-action, such that it commutes with the propagator in the sense that $U(t) \circ l = l \circ (1_A \otimes U(t))$ as maps $A \otimes \mathcal{H} \to \mathcal{H}$.

In functorial words, let F be a 1-dimensional quantum field theory with $F(+) = \mathcal{H}$, let A be a $\mathbb{C}$-algebra and let A be a right A-module over itself. The A-symmetry on $\mathcal{H}$ can be realized as the 'sandwich'

$$A \otimes_A \mathcal{H} \cong \mathcal{H} \tag{1.3}$$

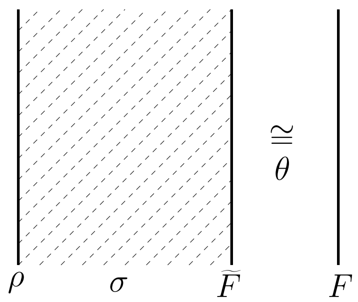
where $\otimes_A$ is the relative tensor product defined in Definition 2.2.3. Furthermore, if B is another algebra of a symmetry of the Hilbert space, both symmetries can be expressed as $(A \otimes B) \otimes_{A \otimes B} \mathcal{H} \cong \mathcal{H}$.

Turns out that such topological symmetries of n-dimensional field theories are 'controlled' by $(n + 1)$-dimensional extended topological field theories. We already encountered an example of a topological quantum field theory by taking diffeomorphism classes of bordisms in dimension 1. Topological field theories are mathematically well-understood, easier to work with as there is no geometrical structure like length or Riemannian metric on the bordism category. However, they can be endowed with extra topological structures like orientation, topological groups, or more general tangential structures. An n-dimensional topological quantum field theory is a symmetric monoidal functor from n-dimensional bordism category to a symmetric monoidal target category which we can take to be the category of vector spaces (Definition 3.3.1).

To accommodate extended TQFTs, higher categories are necessary. A 2-category, in addition to objects and (1)-morphisms of a category, introduces an additional layer of composable 2-morphisms, ensuring compatibility with the axioms of 1-morphisms. Just as with 1-categories, 2-categories can be endowed with a monoidal structure, subject to appropriate compatibility conditions. Section 2.2 delves into the discussion of symmetric monoidal 2-categories. One can iterate this by endowing $(n-1)$-category with n-morphisms to get an n-category.

Extending the previously discussed n-dimensional symmetric monoidal bordism category to a symmetric monoidal 2-category involves allowing the boundaries of n-manifolds to possess codimension 2 corners. Roughly, $(Bord_{n,n-1,n-2}, \sqcup, \emptyset)$ is a symmetric monoidal 2-category, where objects are $(n-2)$-dimensional manifolds, 1-morphisms are manifolds with boundaries, and 2-morphisms are n-manifolds with boundary and corners. Extended TQFTs, are symmetric monoidal 2-functors from this bordism category. One can continue this extension further all the way to codimension n corners, to get a fully extended bordism category $Bord_{n+1,n,\ldots 1}$.

Motivated by this, consider following schematic



(1.4)

Following the notation of [5], $\sigma : Bord^{\mathcal{F}}_{n+1,n,\ldots,1\to\mathcal{C}}$ is an $(n+1)$-dimensional *fully extended TQFT* with tangential structure $\mathcal{F}$, ρ is the *topological right regular boundary theory*, $\tilde{F}$ is an n-dimensional possibly geometrical *left boundary theory*, F is the n-dimensional quantum field theory. The tuple (σ, ρ) is the abstract symmetry data, $(\tilde{F}, \theta)$ is the concrete realization of the symmetry (σ, ρ). Topological (also called categorical, generalized) symmetries of a quantum field theory is encoded in

$$\rho \otimes_\sigma \tilde{F} \cong F \tag{1.5}$$

In this thesis, we understand this general statement in the case $\sigma : Bord^{or}_{2,1,0} \to Alg^2_{\mathbb{C}}$ is an oriented extended TQFT. $Alg^2_{\mathbb{C}}$ is the symmetric monoidal 2-category of algebras, bimodules and bimodule maps. Such extended TQFTs with target $Alg^2_{\mathbb{C}}$ are classified by separable symmetric Frobenius algebras by the cobordism hypothesis and are called state sum (or semi simple) models. We will discuss state sum construction in Example 3.3.4.

Let now $\sigma(+) = A$ be a separable symmetric Frobenius algebra, $\rho(+) = A_A : A \to \mathbb{C}$ and $\tilde{F}(+) = {}_A\mathcal{H} : \mathbb{C} \to A$ are 1-morphisms, $F(+) = \mathcal{H}$ is the quantum mechanical system we partially defined. The A symmetry of the quantum mechanical system is encoded in (1.3).

Equation (1.5) lets us study symmetries of possibly ill-defined n-dimensional field theories in an $(n+1)$-dimensional topological field theory setting which is well-defined and studied extensively in the mathematics literature. Moreover, topological defects can be encoded in this formalism. An introduction to defects can be found here [4].

In addition to this, studying 2-dimensional extended TQFTs provides a rich framework for understanding mathematical structures such as manifolds, knots, and invariants. They offer a rigorous mathematical formalism to study topological properties of spaces and their invariants, which has applications in various branches of mathematics, including topology, geometry, and algebraic geometry. TQFTs also serve as toy models for understanding QFTs. 3-dimensional TQFTs through Chern Simons - Reshetikhin Turaev theory relates Jones polynomials of knot invariants to field theories [30].

Moreover, 2-dimensional TQFTs, in particular, can describe certain phases of matter, such as topological insulators and topological superconductors, which exhibit interesting quantum properties and potential applications in quantum computing and electronics. Their application include fractional quantum Hall effect. [13] applies TQFTs to classify invertible gapped phases of matter.

Finally, studying TQFTs can reveal deep connections between seemingly distinct physical theories through dualities and symmetries. These connections lead to new insights into the nature of quantum field theories and their behavior under different conditions.

1.2 Outline

Section 2.1 includes a review of symmetric monoidal categories and functors with graphical calculus. Two examples $\mathbb{k}$-vector spaces and $\mathbb{k}$-algebras are presented and used to generalize to the symmetric monoidal category of algebras over a monoidal category $\mathcal{C}$. Later, Frobenius algebras, semi-simple algebras and their center and modules are defined. Examples of matrix algebras and group algebras are discussed and finally duality in general symmetric monoidal 2-categories is defined.

In Section 2.2, we move one dimension higher and first define 2-categories, followed by symmetric monoidal 2-categories. Even though the data of symmetric monoidal categories are very complicated, they can be precisely presented by 3-dimensional graphics. An example of a symmetric monoidal 2-category is algebras, bimodules and bimodule maps, also known as the Morita category denoted by $Alg^2_{\mathbb{k}}$, which is constructed in detail. Furthermore, fully dualisable objects of a symmetric monoidal 2-category is defined and shown that in the case of Morita category, they correspond to separable symmetric Frobenius algebras.

In chapter 3, we turn to topology. In Section 3.1, firstly n-dimensional manifolds with boundary and the bordism category $Bord_{n,n-1}$ are defined. Secondly, we define orientation and restrict our attention to 2-dimensional case in which we have comprehensive classification results. Thirdly, we extend our discussion to n-dimensional manifolds with boundary and corners of codimension 2. We explore horizontal and vertical gluing to define the once-extended symmetric monoidal 2-category of $Bord_{n,n-1,n_2}$. Fourthly, we return to dimension 2 and provide generators-relations description of $Bord^{or}_{2,1,0}$.

In Section 3.2, we generalize orientation to an arbitrary group structure on the bordism category. To do so, we define principal fiber bundles and demonstrate that the category of principal G-bundles over a manifold M and bundle morphisms form a groupoid $Bun_G M$. Isomorphism classes of principal G-bundles are shown to be in bijection with the homotopy class of continuous maps $[M, BG]$, where BG is the classifying space of the group G. Consequently, we endow the bordism category with a G-structure.

In Section 3.3, we define TQFTs and give classification results. An n-dimensional closed TQFT with a G-structure is a symmetric monoidal functor $F : Bord^G_{n,n-1} \to \mathcal{C}$. 1-dimensional oriented TQFTs are classified by full subcategory of dualisable objects in $\mathcal{C}$. This is 1-dimensional cobordism hypothesis. In dimension 2, closed TQFTs are classified by commutative Frobenius algebras. The heuristic description of state sum models are given. They are formed by taking center of a semisimple algebra and they may be used as a playground to understand algebraic, geometric and gauge theoretic aspects as they include Dijkgraaf-Witten theories when the semisimple algebra is the group algebra of a finite group G.

Lastly, n-dimensional once extended TQFTs are defined and a sketch proof of classification of the oriented theory in dimension 2 with target $Alg^2_{\mathbb{k}}$ is presented. Such TQFTs are classified by full subcategory of fully dualizable objects in $Alg^2_{\mathbb{k}}$, that is, separable symmetric Frobenius algebras. The proof is the direct result of generators and relations description of the 2-dimensional extended bordism category. This is the 2-dimensional cobordism hypothesis and the main topic of this thesis. 1 and 2-dimensional classification results are special situation of more general hypothesis [25] in the (∞, n)-setting. It is the cobordism hypothesis which states that a fully extended TQFT, that is, an (∞, n) functor $F : Bord^G_{\infty,n} \to \mathcal{C}$ is fully determined by its value on a point.

2 Categorical and Algebraic Preliminaries

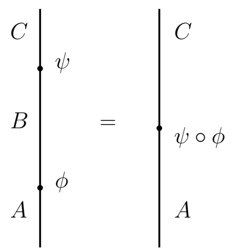
2.1 Symmetric Monoidal Categories

In this section, symmetric monoidal categories are defined with duals and graphical calculus.

2.1.1 Definitions and Examples

We start with a review of symmetric monoidal categories and functors which describe closed topological quantum field theories. Additionally, we introduce graphical calculus as a precise representation of categories. For readers seeking a foundational understanding of category theory, a comprehensive introduction is available in [22]. Those interested in a detailed exploration of symmetric monoidal categories and functors can refer to the works of [20] and [17].

Let $\mathcal{C}$ be a category. Objects and morphisms of $\mathcal{C}$ can diagramatically be represented by points and lines with labels. Composition of morphisms are read from bottom to top which we may interpret as *the arrow of time*. For $A, B, C \in \mathcal{C}$, and $\phi \in Hom_{\mathcal{C}}(A, B)$ and $\psi \in Hom_{\mathcal{C}}(B, C)$, composition of morphisms is depicted as



(2.1)

whereas identity morphism $1_A \in Hom_{\mathcal{C}}(A, A)$ is identified with the object $1_A \hat{=} A$.

A monoidal category is roughly a category endowed with a (weakly) associative unital product. More precisely, a **monoidal category** is a tuple $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$:

i. $\mathcal{C}$ is a category.

ii. $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a functor called monoidal product.

iii. $\mathbb{1} \in \mathcal{C}$ is an object called monoidal unit.

iv. There is a family of natural isomorphisms; associators $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ for all $A, B, C \in \mathcal{C}$, left and right unitors $l_A : \mathbb{1} \otimes A \to A$ and $r_A : A \otimes \mathbb{1} \to A$ for all $A \in \mathcal{C}$, such that the pentagon and the triangle diagrams commute [17, Def. 1.2.1]. These diagrams ensure that associatiors and unitors are compatible.

We will omit associators and unitors in the notation and refer to a monoidal category as $(\mathcal{C}, \otimes, \mathbb{1})$ or just $\mathcal{C}$.

A **monoidal functor** between two monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$ consists of

i. a functor $F : \mathcal{C} \to \mathcal{C}'$,

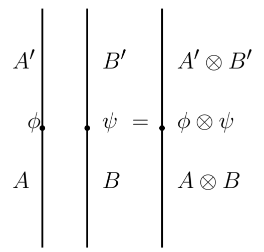
ii. natural isomorphisms $\phi_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$ for all $A, B \in \mathcal{C}$,

iii. a natural isomorphism $\phi : \mathbb{1}' \to F(\mathbb{1})$

such that one hexagon and two rectangular diagrams commute [17, Def. 1.2.14]. These commutative diagrams ensure that natural isomorphisms are compatible with monoidal structures of both categories.

Monoidal categories $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes, \mathbb{1})$ are **monoidally equivalent** if there are monoidal functors $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}$ such that $G \circ F \cong Id_{\mathcal{C}}$ and $F \circ G \cong Id_{\mathcal{C}'}$ where Id stands for identity monoidal functor from monoidal category to itself. By [8, Theorem 2.8.5], any monoidal category is monoidally equivalent to a strict monoidal category where all associators and unitors are replaced by equality signs and coherence axioms are redundant. Thus, disregarding associators and unitors is justified without loss of generality.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category. For objects $A, A', B, B' \in \mathcal{C}$ and morphisms $\phi : A \to A'$, $\psi : B \to B'$, the monoidal product of objects and morphisms is depicted by

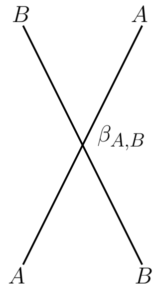


(2.2)

A monoidal category is called **braided** if there exists a braiding, that is, natural isomorphisms $\beta_{A,B} : A \otimes B \to B \otimes A$ for all $A, B \in \mathcal{C}$ such that one triangle, two hexagon diagrams commute ensuring that braiding is coherent with associators and unitors of the monoidal category [17, Eq. 1.2.36-37-38]. A braided monoidal category is called **symmetric** if $\beta_{A,B} = \beta^{-1}_{B,A}$.

The braiding is graphically depicted by over and under crossings of diagonal lines representing objects. If the braided monoidal category is symmetric (which is what we are interested in for TQFTs), then over and under crossings are not distinguished. For

$A, B \in \mathcal{C}$, $\beta_{A,B}$ is depicted as



(2.3)

A **braided monoidal functor** $F : (\mathcal{C}, \otimes, \mathbb{1}, \beta) \to (\mathcal{C}', \otimes', \mathbb{1}', \beta')$: between braided monoidal categories is a monoidal functor such that one triangle and one hexagon diagram commutes [17, Eq. 1.2.27-28]. These diagram ensure that the braiding is compatible with natural isomorphisms of associators and unitors of the monoidal category. If braided monoidal categories $\mathcal{C}$ and $\mathcal{C}'$ are symmetric, then we call F a **symmetric monoidal functor**.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$ be monoidal categories. A natural transformation between two monoidal functors $F, F' : \mathcal{C} \to \mathcal{C}'$ is called a **monoidal natural transformation** if it respects monoidal structures. This means that the rectangular diagram for monoidal products and the triangle diagram for monoidal units commute [17, Eq. 1.2.20].

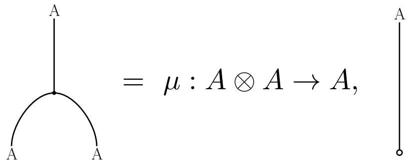
Example 2.1.1.

i. Symmetric monoidal category of vector spaces $(Vect_{\mathbb{k}}, \otimes, \mathbb{k})$*: Objects are* $\mathbb{k}$*-vector spaces and morphisms are linear maps. The monoidal product is the tensor product of vector spaces and the ground field* $\mathbb{k}$ *is the tensor unit. Associators are the canonical isomorphisms* $(V_1 \otimes V_2) \otimes V_3 \cong V_1 \otimes (V_2 \otimes V_3)$ *for all* $V_1, V_2, V_3 \in Vect_{\mathbb{k}}$*, left-right unitors are* $V \otimes \mathbb{k} \cong V \cong \mathbb{k} \otimes V$ *for all* $V \in Vect_{\mathbb{k}}$*. It is also symmetric monoidal category since we have canonical isomorphisms* $V_1 \otimes V_2 \cong V_2 \otimes V_1$ *for all* $V_1, V_2 \in Vect_{\mathbb{k}}$*.*

ii. Symmetric monoidal category of $\mathbb{k}$*-algebras* $(Alg_{\mathbb{k}}, \otimes, \mathbb{k})$*: Objects are associative unital* $\mathbb{k}$*-algebras, that is a tuple* (A, μ, η) *where* $A \in Vect_{\mathbb{k}}$ *is a vector space together with a linear map* $(\mu : A \otimes A \to A)$*, denoted by* $(a_1 \otimes a_2 \mapsto a_1.a_2)$ *with a unit* $(\mathbb{k} \to A, 1 \mapsto e)$ *such that* $a.e = a = e.a$ *for all* $a \in A$ *and* $(a_1.a_2).a_3 = a_1.(a_2.a_3)$ *for all* $a_1, a_2, a_3 \in A$*, and morphisms are linear maps that preserve algebra structure,* $\xi : V \to V'$ *in* $Vect_{\mathbb{k}}$ *such that* $\xi(v_1.v_2) = \xi(v_1).\xi(v_2)$ *and* $\xi(e) = e'$*. The monoidal product is the tensor product of underlying vector spaces where the multiplication in the tensor product space defined by* $((a_1 \otimes a_2) \otimes (a'_1 \otimes a'_2)) \mapsto (a_1.a'_1) \otimes (a_2.a'_2)$*. The tensor unit is the ground field* $\mathbb{k}$ *viewed as a* $\mathbb{k}$*-algebra over itself. The natural isomorphisms are induced from that of* $Vect_{\mathbb{k}}$*.*

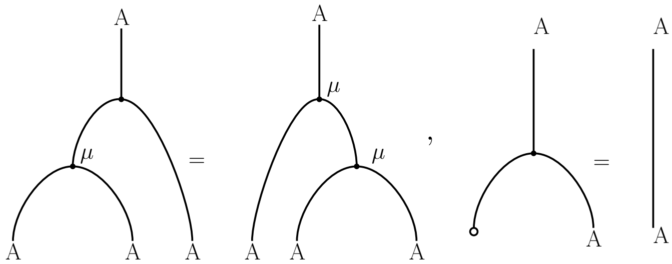
Note that an object of $Alg_{\mathbb{k}}$ is defined by a vector space V, together with an associative unital multiplication $V \otimes V \to V$. Next, we generalize $\mathbb{k}$-algebras to algebras over any (symmetric) monoidal category.

Definition 2.1.2. *Let* $(\mathcal{C}, \otimes, \mathbb{1})$ *be a symmetric monoidal category.* $(Alg(\mathcal{C}), \otimes, \mathbb{1})$ *is a monoidal category whose objects are tuples* (A, μ, η) *where* $A \in \mathcal{C}$*, multiplication* μ *and unit* η *are morphisms in* $\mathcal{C}$



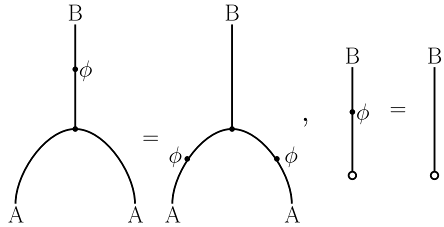
$$= \mu : A \otimes A \to A, \qquad = \eta : \mathbb{1} \to A \tag{2.4}$$

such that multiplication is associative and unital



(2.5)

and morphisms are morphisms of $\mathcal{C}$ *with the following compatibility condition: for* $A, B \in \mathcal{C}$*,* $\phi \in Hom_{Alg(\mathcal{C})}(A, B)$ *is a morphism in* $\mathcal{C}$ *such that*

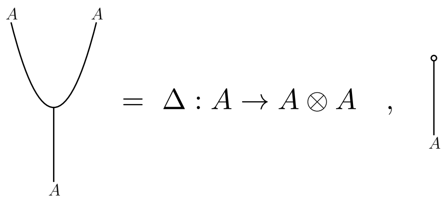


(2.6)

Multiplication and unit are defined on monoidal product pointwise, i.e. for $A, B \in Alg(\mathcal{C})$*,* $\mu_{A\otimes B} = \mu_A \otimes \mu_B$ *and* $\eta_{A\otimes B} = \eta_A \otimes \eta_B$*.*

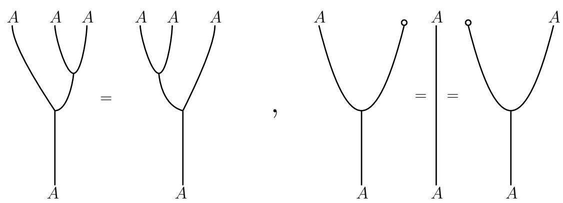
In this notation, the category of ordinary $\mathbb{k}$-algebras is $Alg_{\mathbb{k}} = Alg(Vect_{\mathbb{k}})$.

A **coalgebra** (A, Δ, ϵ) in $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with morphisms in $\mathcal{C}$



$$= \Delta : A \to A \otimes A \quad , \qquad = \epsilon : A \to \mathbb{1} \tag{2.7}$$

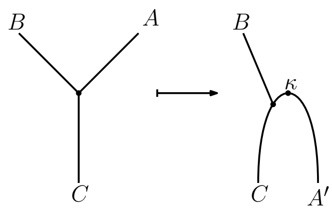
such that comultiplication is coassociative and counital



(2.8)

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a monoidal category and $\kappa \in Hom_{\mathcal{C}}(A \otimes A', \mathbb{1})$ be a morphism. κ is called a **non-degenerate pairing** if for all $B, C \in \mathcal{C}$, the map between morphisms

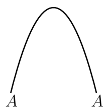
$$Hom_{\mathcal{C}}(C, B \otimes A) \to Hom_{\mathcal{C}}(C \otimes A', B)$$



(2.9)

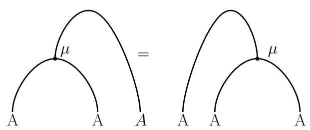
is an isomorphism.

Definition 2.1.3. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \beta)$ be a symmetric monoidal category and $(A, \mu, \eta) \in Alg(\mathcal{C})$ be an algebra over $\mathcal{C}$. A* ***Frobenius algebra*** *is a tuple (A, μ, η, κ) where*



$$= \kappa : A \otimes A \to \mathbb{1} \tag{2.10}$$

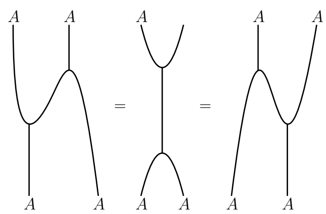
is a non-degenerate pairing compatible with the multiplication in the sense that



(2.11)

A Frobenius algebra is called ***symmetric*** *if $\kappa = \kappa \circ \beta$.*

Lemma 2.1.4. *Let $(A, \mu, \eta) \in Alg(\mathcal{C})$. Then the following are equivalent:*
1 - (A, μ, η, κ) is a Frobenius algebra.
2 - $(A, \mu, \eta, \Delta, \epsilon)$ is both an algebra and coalgebra such that the Frobenius relations hold



(2.12)

Proof. follows from [20, Lemma 2.2.4 and Proposition 2.3.24] □

Definition 2.1.5. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \beta)$ be a symmetric monoidal category.*

- *i. $(FrobAlg(\mathcal{C}), \otimes, \mathbb{1})$ is a symmetric monoidal category. Objects are Frobenius algebras, a mophism from (A, μ, η, κ) to $(A', \mu', \eta', \kappa')$ is an algebra morphism compatible with the Frobenius pairing; $\kappa' \circ (f \otimes f) = \kappa$.*

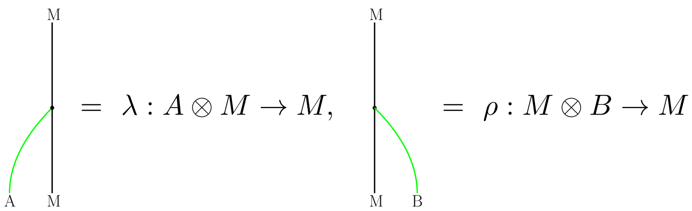
ii. $(cFrobAlg(\mathcal{C}), \otimes, \mathbb{1})$ *is a symmetric monoidal category where objects are commutative Frobenius algebras, that is,* $(A, \mu, \eta, \kappa) \in FrobAlg(\mathcal{C})$ *such that* $\mu = \mu \circ \beta$ *and morphisms are induced from* $FrobAlg(\mathcal{C})$*.*

Definition 2.1.6. *Let* $(\mathcal{C}, \otimes, \mathbb{1})$ *be a symmetric monoidal category.*

i. *A Frobenius algebra* $(A, \mu, \eta, \Delta, \epsilon)$ *is called* Δ*-separable if* $\mu \circ \Delta = 1_A$*.*

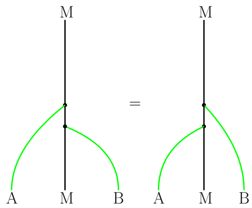
ii. $(ssFrob(\mathcal{C}), \otimes, \mathbb{1})$ *is a symmetric monoidal category whose objects are* Δ*-separable symmetric Frobenius algebras and morphisms are algebra morphisms compatible with the* Δ*-separable symmetric Frobenius structure and monoidal product is induced from* $\mathcal{C}$*.*

Definition 2.1.7. *Let* $(\mathcal{C}, \otimes, \mathbb{1})$ *be a monoidal category and let* $A, B \in Alg(\mathcal{C})$*. An* ***A-B bimodule*** *is a tuple* (M, λ, ρ)*, where* $M \in \mathcal{C}$ *together with left and right actions, in other words, morphisms in* $\mathcal{C}$



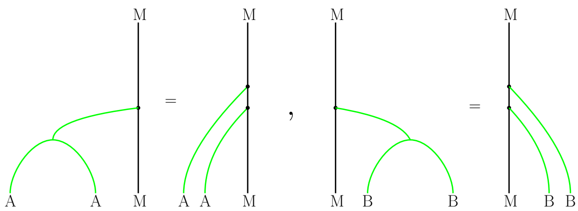
$$= \lambda : A \otimes M \to M, \qquad = \rho : M \otimes B \to M \tag{2.13}$$

such that left and right actions commute



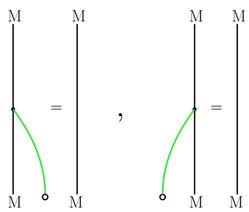
(2.14)

and compatible with the multiplication



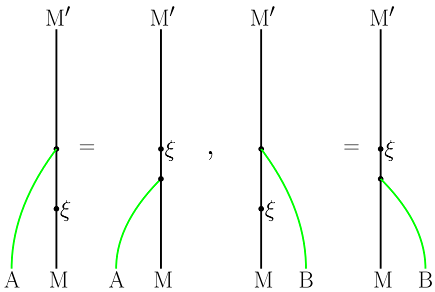
(2.15)

and the unit



(2.16)

A-B bimodules form a category denoted by ${}_AMod_B$ whose objects are A-B bimodules, and morphisms between two objects $M, M' \in {}_AMod_B$ are bimodule maps, i.e. morphisms $\xi : M \to M'$ in $\mathcal{C}$ such that



(2.17)

A **left A-module** is an A-$\mathbb{1}$ bimodule denoted by $Mod(A)$ or ${}_AMod$, and a **right B-module** is an $\mathbb{1}$-B bimodule denoted by Mod_B.

An algebra (A, μ, η) can be viewed as a left (or right) module over itself with the multiplication μ viewed as a left (or right) action on $A \in {}_AMod$ (or Mod_A).

Let A be a $\Bbbk$-algebra and $M \in Mod(A)$ be a left A-module. M is called **simple** if for each $m \in M \setminus \{0\}$, the action of A on the point m spans M i.e. $A.m = M$. In other words, there is only one *orbit* so that the action is *transitive*. An algebra A is called simple if it is simple as a left A-module.

An A-module M is called **semisimple** if it can be written as a direct sum of simple A-modules. Similarly, an algebra is called semisimple if it is semisimple as a left A-module.

The **center** of an algebra A is $Z(A) = Hom_{A,A}(A) = \{a \in A | a.b = b.a\} \forall b \in A$. If A is a Frobenius algebra, $Z(A)$ is also a Frobenius algebra since for all $a, b \in Z(A)$, $\mu(a, b) \in Z(A)$, $\Delta(a) \in Z(A) \otimes Z(A)$ and the unit η is by definition commutative and therefore in $Z(A)$. Moreover, Z maps algebras to (commutative) algebras, Frobenius algebras to (commutative) Frobenius algebras, separable algebras (which will be defined in (2.2.8)) to (commutative) separable algebras. However, it is not a functor since a morphism $f : A \to B$ restricted to $Z(A)$ is not necessarily mapped to $Z(B)$. Nevertheless, it is shown in [6] that it can be made into a *lax functor* between appropriate bicategories. The center of a semisimple algebra is what a state sum model TQFT assigns to an object as we will discuss in Section 3.

Example 2.1.8. *Let $(M_n(\Bbbk), \mu, \eta)$ be the algebra of $(n \times n)$ matrices where μ is the matrix multiplication and the unit $\eta = 1_n$ is the matrix with all entries zero, but diagonal entries are 1. It is simple since any left A-module is of the form $\Bbbk^n$. If $\Bbbk = \mathbb{C}$ (or any algebraically closed field), then by the Artin-Wedderburn theorem any semisimple $\mathbb{C}$-algebra A is isomorphic to direct sum of Matrix algebras:*

$$A \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}) \tag{2.18}$$

Take $A = \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})$ and let E_{ijk} be an $(n_k \times n_k)$ matrix for each value of $k \in \{1, \ldots, r\}$, whose only non-vanishing entry is i^{th} row and j^{th} column with value 1.

Clearly, they form a basis of $Mat_{n_k}(\mathbb{C})$. It is sufficient to determine values of the maps on the basis.

Any element $a \in A$ is of the form

$$a = \sum_{i,j=1}^{n_u} \sum_{u=1}^{r} \alpha_{iju} E_{iju} \tag{2.19}$$

for some $\alpha_{ijk} \in \mathbb{C}$.

Let $A = \bigoplus_{i=1}^{r} A_i$ with $A_i = M_{n_i}(\mathbb{C})$ simple algebras as above and let $V_i = \mathbb{C}^{n_i}$ be n_i-dimensional vector spaces. It is clear that $V_1, \ldots, V_r$ are simple left A-modules where A_j acts on V_i trivially for $i \neq j$ and V_i is also simple left A_i-module for all $i \in \{1, \ldots, r\}$. Therefore, the simple algebra A_i can be written as, $A_i = n_i V_i$ direct sum of n_i copies of V_i's for $n_i \in \mathbb{Z}_+$ and the semisimple algebra becomes $A = \bigoplus_{i=1}^{r} n_i V_i$.

Furthermore, by [9, Theorem 3.3.1], we see that any finite dimensional left A-module M is isomorphic to direct sum of arbitrary multiplicities of simple left A-modules:

$$M \cong \bigoplus_{i=1}^{r} \alpha_i V_i \tag{2.20}$$

for some $\alpha_i \in \mathbb{Z}_+$.

If $B \cong \bigoplus_{j=1}^{r} M_{d_i}(\mathbb{k})$, any finite dimensional A-B bimodule $M \in {}_A Mod_B$ of semisimple algebras $A \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{k})$ and $B \cong \bigoplus_{j=1}^{r} M_{d_i}(\mathbb{k})$ is of the form

$$M \cong \bigoplus_{i,j=1}^{r} \alpha_i \alpha_j \mathbb{k}^{n_i} \otimes \mathbb{k}^{d_i} \tag{2.21}$$

for some $\alpha_i, \alpha_j \in \mathbb{Z}_+$.

Multiplication and unit in terms of this basis are

$$\begin{aligned} \mu : A \otimes A &\to A \\ (E_{iju} \otimes E_{klv}) &\mapsto \delta_{u,v} \delta_{j,k} E_{ilv} \end{aligned} \tag{2.22}$$

$$\begin{aligned} \eta : \mathbb{k} &\to A \\ 1 &\mapsto \sum_{u=1}^{r} 1_{n_u} \end{aligned} \tag{2.23}$$

Matrix algebras can be endowed with the Frobenius algebra structure. Define a comultiplication and a counit map by

$$\begin{aligned} \Delta : A &\to A \otimes A \\ E_{iju} &\mapsto \sum_{k=1}^{n_u} \frac{1}{n_u} E_{iku} \otimes E_{kju} \end{aligned} \tag{2.24}$$

$$\begin{aligned} \epsilon : A &\to \Bbbk \\ \sum_{i,j=1}^{n_u}\sum_{u=1}^{r} \alpha^{iju} E_{iju} &\mapsto \sum_{u=1}^{r}\sum_{i=1}^{n_k} \alpha_{iiu} n_u \end{aligned} \tag{2.25}$$

Note that ϵ is the usual trace of matrices and non-degenerate pairing is $\kappa = \epsilon \circ \mu$.

The Frobenius relations hold:

$$A \otimes A \xrightarrow{\Delta \otimes 1_A} A \otimes A \otimes A \xrightarrow{1_A \otimes \mu} A \otimes A$$

$$(E_{iju} \otimes E_{\alpha\beta\gamma}) \mapsto (\sum_{k=1}^{n_u} \frac{1}{n_u} E_{iku} \otimes E_{kju}) \otimes E_{\alpha\beta\gamma} \mapsto \delta_{j\alpha}\delta_{u\gamma} \sum_{k=1}^{n_u} \frac{1}{n_u} E_{iku} E_{k\beta u} \tag{2.26}$$

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\Delta} A \otimes A$$

$$(E_{iju} \otimes E_{\alpha\beta\gamma}) \mapsto \delta_{u\gamma}\delta_{j\alpha} E_{i\beta u} \mapsto \delta_{u\gamma}\delta_{j\alpha} \sum_{i=1}^{n_u} \frac{1}{n_u} E_{iku} \otimes E_{k\beta u} \tag{2.27}$$

$$A \otimes A \xrightarrow{1_A \otimes \Delta} A \otimes A \otimes A \xrightarrow{\mu \otimes 1_A} A \otimes A$$

$$(E_{iju} \otimes E_{\alpha\beta\gamma}) \mapsto E_{iju} \otimes (\sum_{k=1}^{n_u} \frac{1}{n_\gamma} E_{\alpha k\gamma} \otimes E_{k\beta\gamma}) \mapsto \delta_{u\gamma}\delta_{j\alpha} \sum_{k=1}^{n_\gamma} \frac{1}{n_\gamma} E_{ik\gamma} \otimes E_{k\beta\gamma} \tag{2.28}$$

are equal. In fact, any map $E_{iju} \mapsto \alpha E_{iku} \otimes E_{kju}$ for some $\alpha \in \mathbb{C}$ and $1 \leq k \leq n_u$ satisfies Frobenius relations, thus, can be taken as Δ. However, the Δ we chose satisfies Δ-separability condition:

$$\mu \circ \Delta(E_{iju}) = \sum_{k=1}^{n_u} \frac{1}{n_u} \delta_{uu}\delta_{kk} E_{iju} = E_{iju} \tag{2.29}$$

Multiples of units are the center of matrix algebras. In other words, any $a \in Z(A)$ is of the form $a = \sum_{i=1}^{r} \alpha_i.1_n$ for some $\alpha_i \in \mathbb{C}$. The restriction of $\mu, \eta, \Delta, \epsilon$ to the center endows $Z(A)$ with commutative Frobenius algebra structure.

Example 2.1.9. *Let G be a finite group with neutral element e and $A = \mathbb{C}[G] = \{\sum_{g\in G} \kappa_g.g | \kappa_g \in \mathbb{C}\}$ group algebra with multiplication and unit*

$$\begin{aligned} \mu_G : A \otimes A &\to A \\ (\sum_{g\in G} \kappa_g g, \sum_{h\in G} \kappa'_h h) &\mapsto \sum_{g,h\in G} \kappa_g \kappa_h g.h \end{aligned} \tag{2.30}$$

$$\begin{aligned} \eta : \mathbb{C} &\to A \\ 1 &\mapsto e \end{aligned} \tag{2.31}$$

Group algebras can be endowed with a Frobenius structure:

$$\begin{aligned} \Delta : A &\to A \otimes A \\ h &\mapsto \frac{1}{|G|} \sum_{g\in G} h.g \otimes g^{-1} \end{aligned} \tag{2.32}$$

$$\begin{aligned} \epsilon : A &\to \mathbb{C} \\ \sum_{g\in G} k_g g &\mapsto \sum_{g\in G} k_g \end{aligned} \tag{2.33}$$

Frobenius relations hold since

$$(a \otimes b) \xmapsto{1_A \otimes \Delta} \frac{1}{|G|} \sum_{g\in G} a \otimes b.g \otimes g^{-1} \xmapsto{\mu \otimes 1_A} \frac{1}{|G|} \sum_{g\in G} a.b.g \otimes g^{-1} \tag{2.34}$$

$$(a \otimes b) \xmapsto{\mu} a.b \xmapsto{\Delta} \frac{1}{|G|} \sum_{g\in G} a.b.g \otimes g^{-1} \tag{2.35}$$

$$(a \otimes b) \xmapsto{\Delta \otimes 1_A} \frac{1}{|G|} \sum_{g\in G} a.g \otimes g^{-1} \otimes b \xmapsto{1_A \otimes \mu} \frac{1}{|G|} \sum_{g\in G} a.g \otimes g^{-1}.b \tag{2.36}$$

are equal since $\sum_{g\in G} g \otimes g^{-1}$ *commutes with all other elements of the group algebra.* $\mathbb{C}[G]$ *is* Δ*-separable since*

$$\mu \circ \Delta(h) = \frac{1}{|G|} \sum_{g\in G} h.g.g^{-1} = h \tag{2.37}$$

The center of the group algebra $Z(\mathbb{C}[G])$ *is therefore* Δ*-separable commutative Frobenius algebra. Since characteristic of* G *does not divide* $|G|$ *in the complex setting, then* $\mathbb{C}[G]$ *is semisimple by Maschke's theorem and by the Artin Wedderburn theorem:*

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} Mat_{n_i}(\mathbb{C}) \tag{2.38}$$

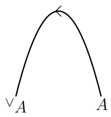
for some $r \in \mathbb{N}$*. Group algebra is* $|G|$*-dimensional vector space and therefore* $|G| = \sum_{i=1}^{r} n_i^2$*. This relation determines* r*. Therefore, the center of the group algebra is* r*-dimensional.*

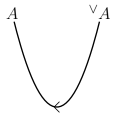
Modules are representations of algebras. If $A = \Bbbk[G]$ is the group algebra of a finite group G, then left A-modules can be identified with group representations of G (see [7, Proposition 2.41]).

2.1.2 Duality

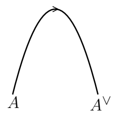
We now define duality in a (symmetric) monoidal category. Even though the braiding and the symmetry property are not needed for this definition, we require symmetry explicitly since TQFTs which are the main interest of this thesis are symmetric monoidal.

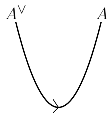
Definition 2.1.10. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a (symmetric) monoidal category. Left and right duality data for an object $A \in \mathcal{C}$ is $(A, {}^\vee A, ev_A, coev_A)$ and $(A, A^\vee, \widetilde{ev}_A, \widetilde{coev}_A)$, respectively where $A^\vee, {}^\vee A \in \mathcal{C}$. Evaluation and coevaluation maps are morphisms in $\mathcal{C}$*



$= ev_A : {}^\vee A \otimes A \to \mathbb{1}$


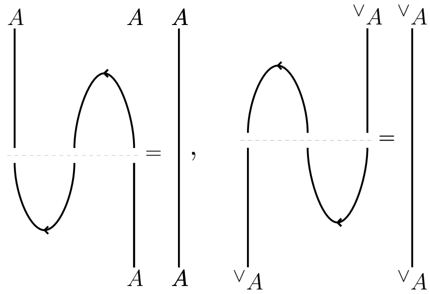
$= coev_A : \mathbb{1} \to A \otimes {}^\vee A$ (2.39)



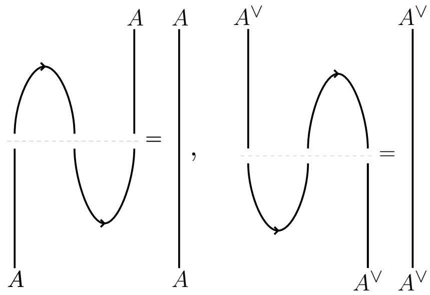
$= \widetilde{ev}_A : A \otimes A^\vee \to \mathbb{1}$


$= \widetilde{coev}_A : \mathbb{1} \to A^\vee \otimes A$ (2.40)

such that the composition satisfies the so-called Zorro identities:

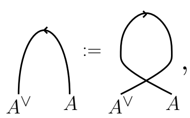


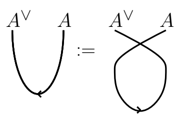
(2.41)



(2.42)

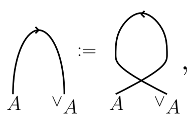
Let $(\mathcal{C}, \otimes, \mathbb{1}, \beta)$ be a symmetric monoidal category and $(A, A^\vee, \widetilde{ev}_A, \widetilde{coev}_A)$ be a right duality data. Then, $(A, A^\vee, \widetilde{ev}_A \circ \beta_{A^\vee, A}, \beta_{A^\vee, A} \circ \widetilde{coev}_A)$ is left duality data for $A \in \mathcal{C}$. Diagramatically,

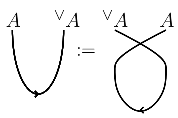
$ev_A =$


, $coev_A =$


(2.43)

Zorro identities follow from naturality of the braiding and right Zorro identities. Similarly, braiding and the right duality data realizes $A^\vee$ as a left dual:

$\widetilde{ev}_A =$


, $\widetilde{coev}_A =$


(2.44)

Example 2.1.11.

i. *In* $(Vect_{\Bbbk}, \otimes, \Bbbk)$, *finite dimensional objects are dualizable. For an n-dimensional vector space* $V \in Vect_{\Bbbk}$, $V^* := Hom_{\Bbbk}(V, \Bbbk)$ *is both left and right duals of* V. $\widetilde{ev}(v \otimes f) = f(v)$ *is the usual evaluation and coevaluation in terms of dual basis* $(e_1, \ldots, e_n)$ *and* $(f_1, \ldots, f_n)$ *of* V *and* V^*, $\widetilde{coev}(1) = \sum_{i=1}^{n} f_i \otimes e_i$.

$$v \otimes 1 \xmapsto{1_V \otimes \widetilde{coev}} v \otimes \sum_{i=1}^{n} f_i \otimes e_i \xmapsto{\widetilde{ev} \otimes 1_V} \sum_{i=1}^{n} f_i(v) e_i = v \tag{2.45}$$

other Zorro identities follow similarly.

ii. *Let* $(A, \mu, \eta, \Delta, \epsilon)$ *be a Frobenius algebra. Then, A has a dual in* $\mathcal{C}$ *with* ${}^{\vee}A = A = A^{\vee}$,

$$\widetilde{ev}_A = ev_A = \epsilon \circ \mu \qquad \widetilde{coev}_A = coev_A = \Delta \circ \eta \tag{2.46}$$

Frobenius relations together with unit and counit axioms correspond to Zorro identities.

iii. *Given a Frobenius algebra* $(A, \mu, \eta, \Delta, \epsilon)$ *and an isomorphism* $\phi : A \to A^{\vee} = {}^{\vee}A$ *in* $\mathcal{C}$, $\widetilde{ev}_A = \epsilon \circ \mu \circ (1_A \otimes \phi^{-1})$, $\widetilde{coev}_A = (\phi \otimes 1_A) \circ \Delta \circ \eta$ *defines right duality data in* $\mathcal{C}$, *whereas* $ev_A : \epsilon \circ \mu \circ (\phi \otimes 1_A)$, $coev_A : (1_A \otimes \phi^{-1}) \circ \Delta \circ \mu$ *defines a left duality data. Zorro moves follow from Frobenius identities and unit-counit axioms.*

2.2 Symmetric Monoidal 2-Categories

In this section, we will give a brief introduction to symmetric monoidal 2-categories. As symmetric monoidal categories and symmetric monoidal functors are fundamental to study closed TQFTs, symmetric monoidal 2-categories are the framework for once extended TQFTs. We will give the explicit definition of 2-categories but only explain the required data of symmetric monoidal 2-categories. For details and precise definitions, the reader can look up [28].

We note that since the monoidal product is associative and unital, just like morphisms of a category, we can think of them as another layer of morphisms of a higher category: composition of 1-morphisms of a 2-category with one object. Therefore, the data of a 2-category $\mathcal{B}$ with one object is given by $* \in \mathcal{B}$ as the object and the objects of a monoidal category as 1-morphisms together with monoidal product considered as (horizontal) composition of 1-morphisms and morphisms of the monoidal category as 2-morphisms. This is the idea of 2-categories, it is endowed with another layer of composable morphisms. To illustrate this idea, let us define $BVect_{\Bbbk}$ from the data of the monoidal category $(Vect_{\Bbbk}, \otimes, \Bbbk)$.

The data of $BVect_{\Bbbk}$ is the following:

1. an object $* \in BVect_{\Bbbk}$,

2. a Hom category, that is $Hom_{BVect_{\Bbbk}}(*, *) = Vect_{\Bbbk}$, that is,
   - 1-morphisms are $\Bbbk$-vector spaces with *horizontal composition* as the tensor product,
   - 2-morphisms are $\Bbbk$-linear maps with *horizontal composition* is the tensor product of linear maps and *vertical composition* is composition of linear maps,
   - associators and unitors are that of the monoidal category $Vect_{\Bbbk}$.

This is an example of a 2-category with one object. In other words, the trivial category with a single object $*$ is *enriched over* $Vect_{\Bbbk}$; the set $End(*)$ is replaced by a monoidal category $Vect_{\Bbbk}$. To motivate 2-categories further, consider $[Alg_{\Bbbk}]$ where objects are algebras and morphisms are isomorphism classes of bimodules where composition is relative tensor product over the intermediate algebra and on the other side, consider module category where objects are bimodules and morphisms are bimodule maps. These data can be put together to form a symmetric monoidal 2-category $Alg^2_{\Bbbk}$ as we will see in Example 2.2.4.

Definition 2.2.1. *A 2-category is a tuple $(\mathcal{B}, 1, \circ, \alpha, \lambda, \rho)$ with the following data:*

1. *There is a class of objects denoted by $ob(\mathcal{B})$ or $\mathcal{B}$.*
2. *For all objects $A, B \in \mathcal{B}$, there are Hom categories denoted by $Hom_{\mathcal{B}}(A, B)$. The data of $Hom_{\mathcal{B}}(A, B)$ is the following:*
   - *objects of $Hom_{\mathcal{B}}(A, B)$ are called 1-morphisms,*
   - *for all objects $X, Y \in Hom_{\mathcal{B}}(A, B)$, morphisms in $Hom_{\mathcal{B}}(A, B)$ called 2-morphisms denoted by $Hom_{Hom_{\mathcal{B}}(A,B)}(X, Y)$ or $Hom_{A,B}(X, Y)$,*
   - *for all 1-morphisms $X, Y, Z \in Hom(A, B)$ and for all 2-morphisms $\phi \in Hom_{A,B}(X, Y)$, $\psi \in Hom_{A,B}(Y, Z)$ composition of morphisms in $Hom_{\mathcal{B}}(A, B)$ denoted by $\psi.\phi \in Hom_{A,B}(X, Z)$ and called vertical composition of 2-morphisms and the identity 2-morphism denoted by $1_X \in Hom_{A,B}(X, X)$.*
3. *For each triple of objects $A, B, C \in \mathcal{B}$, there are functors $c_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \to \mathcal{B}(A, C)$ denoted by $\circ$ and called horizontal composition. The horizontal composition of 1-morphisms and 2-morphisms is the following:*
   - *for all objects $A, B, C \in \mathcal{B}$ and for all 1-morphisms $X \in \mathcal{B}(A, B)$ and $Y \in \mathcal{B}(B, C)$, $Y \circ X \in \mathcal{B}(A, C)$,*
   - *for all 1-morphisms $X_1, X_2 \in \mathcal{B}(A, B)$ and $Y_1, Y_2 \in \mathcal{B}(B, C)$ and for all 2-morphisms $f \in Hom_{A,B}(X_1, X_2)$, and for all $g \in Hom_{B,C}(Y_1, Y_2)$, the horizontal composition of 2-morphisms $g \circ f \in Hom_{A,C}(Y_1 \circ X_1, Y_2 \circ X_2)$,*
4. *For all objects $A \in \mathcal{B}$, there is a 1-morphism $1_A \in Hom_{\mathcal{B}}(A, A)$.*
5. *There are natural isomorphisms (called associators) between functors: for all $A, B, C, D \in \mathcal{B}$*

$$\alpha_{ABCD} : c_{ABD} \circ (c_{BCD} \times 1_{A,B}) \to c_{ACD} \circ (1_{C,D} \times c_{ABC}) \tag{2.47}$$

where $1_{A,B}$ is the trivial functor that takes each object and morphism to the same object and morphism, then the functors read

$$\begin{aligned} Hom_{\mathcal{B}}(C,D) \times Hom_{\mathcal{B}}(B,C) \times Hom_{\mathcal{B}}(A,B) &\to Hom_{\mathcal{B}}(A,D) \\ (Z,Y,X) &\mapsto (Z \circ Y) \circ X \\ (Z,Y,X) &\mapsto Z \circ (Y \circ X) \end{aligned} \tag{2.48}$$

6. *There are natural isomorphisms (called left and right unitors) between functors; for all $A, B \in \mathcal{B}$,*

$$\begin{aligned} \lambda_{AB} &: c_{ABB}(1_B \times 1_{A,B}) \to 1_{A,B} \\ \rho_{AB} &: c_{AAB}(1_{A,B} \times 1_A) \to 1_{A,B} \end{aligned} \tag{2.49}$$

   such that the triangle and the pentagon diagrams commute.[17, Eq. 2.1.4-5]

The commutative diagrams ensure that units and associators are in coherence.

Definition 2.2.2. *A 2-functor $(F, \phi, 1) : (\mathcal{B}, 1^{\mathcal{B}}, c^{\mathcal{B}}) \to (\mathcal{C}, 1^{\mathcal{C}}, c^{\mathcal{C}})$ between two 2-categories consists of the following:*

1. *There is an assignment $F : \mathcal{B} \to \mathcal{C}$ between objects.*

2. *There is a family of functors between Hom categories, for all $A, B \in \mathcal{B}$, $F_{A,B} : Hom_{\mathcal{B}}(A,B) \to Hom_{\mathcal{C}}(F(A), F(B))$.*

3. *There are natural isomorphisms for every $A, B, C \in \mathcal{B}$;*

$$\phi_{ABC} : c^{\mathcal{D}}_{F(A)F(B)F(C)} \circ (F_{BC} \times F_{AB}) \to F_{AC} \circ c^{\mathcal{C}}_{ABC}, \tag{2.50}$$

4. *There are unitors for all $A \in \mathcal{B}$*

$$1_A : 1^{\mathcal{C}}_{F(A)} \to F_{AA} \circ 1^{\mathcal{B}}_A \tag{2.51}$$

   such that the hexagon and two rectangle diagrams commute [17, Eq.4.1.3-4].

The commutative diagrams ensure that 2-functors are in coherence with associativitiy and unity natural transformations of 2-categories.

Let $(F, \phi, 1_F), (G, \psi, 1_G) : (\mathcal{B}, 1^{\mathcal{B}}, c^{\mathcal{B}}) \to (\mathcal{C}, 1^{\mathcal{C}}, c^{\mathcal{C}})$ be two 2-functors. A **pseudo natural transformation** $\sigma : F \to G$ is the following data:

i. For each $A \in \mathcal{B}$ there are 1-morphisms $\sigma_A : F(A) \to G(A)$.

ii. For every $A, B \in \mathcal{B}$ and $X \in Hom_{\mathcal{B}}(A,B)$, there are invertible 2-morphisms $\sigma_X : G(X) \circ \sigma_A \to \sigma_B \circ F(X)$

   such that the diagrams [28, Figure A.1] commute.

As we see moving from categories to 2-categories, the required data grows exponentially. We will endow the 2-category with a monoidal product and we have to make sure that the product is compatible with the structure of the 2-category. To do this, we need modifications that are transformations between pseudo natural transformations to make the data coherent. Furthermore, a monoidal 2-category can be used to construct a 3-category with one object just like a 2-category with one object can be constructed from the data of a monoidal category. This works even in higher categories. We now give a sketch of the definition of a symmetric monoidal 2-category.

Let $F, G : \mathcal{B} \to \mathcal{C}$ be 2-functors, $\sigma, \theta : F \to G$ be pseudo natural transformations. A **modification** between two pseudo natural transformations $\Gamma : \sigma \to \theta$ is a family of 2-morphisms $\Gamma_A : \sigma_A \to \theta_A$ for each $A \in \mathcal{B}$ such that [28, A.8] commutes.

A **symmetric monoidal 2-category** consists of

1. a 2-category $\mathcal{B}$,
2. a 2-functor $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$,
3. an object $\mathbb{1} \in \mathcal{B}$, monoidal unit
4. together with associators and unitors which are pseudo natural isomorphisms, (α, λ, ρ)
$$\begin{aligned} \alpha_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\ \lambda : \mathbb{1} \otimes A \mapsto \mathbb{1} \\ \rho : A \otimes \mathbb{1} \mapsto A, \end{aligned} \tag{2.52}$$
5. a symmetric braiding β is an invertible pseudo natural transformation $\otimes \to \otimes\tau$ where $\tau : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ is a 2-functor that flips objects, 1-morphisms and 2-morphisms. For all $A, B \in \mathcal{B}$
$$\beta_{A,B} : A \otimes B \to B \otimes A \tag{2.53}$$
is a 1-morphism equivalence and for all $A, B, C, D \in \mathcal{B}$ and for all $X_1, X_2 : A \to B$ and for all $Y_1, Y_2 : C \to D$
$$\beta_{\phi,\psi} : \phi \otimes \psi \to \psi \otimes \phi \tag{2.54}$$
is a 2-morphism equivalence,
6. together with 7 invertible modifications.

   These modifications ensure compatibility of associators, unitors and braiding. In addition, this data has to satisfy commutative diagrams as in [28, Def. 2.3].

Before giving examples, we will introduce relative tensor product which will be the horizontal composition of a 2-category $Alg^2_{\mathbb{k}}$.

Definition 2.2.3. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category, let $A \in Alg(\mathcal{C})$ and let $M \in Mod_A$ and $N \in {}_AMod$. The relative tensor product of M and N over A is an object $M \underset{A}{\otimes} N \in \mathcal{C}$ defined to be the coequalizer of the maps*

$$M \otimes (A \otimes N) \underset{(\lambda \otimes 1_N) \circ \alpha_{M,A,N}}{\overset{1_M \otimes \rho}{\rightrightarrows}} M \otimes N$$

in $\mathcal{C}$ where α is the associator. In other words, for any morphism $\phi : M \otimes N \to Z$ for an object $Z \in \mathcal{C}$ such that

$$\phi \circ (\rho_M \otimes 1_N)((M \otimes A) \otimes N) = \phi \circ (1_M \otimes \lambda_N)(M \otimes (A \otimes N)) \tag{2.55}$$

there exists a unique morphism $\widetilde{\phi} : M \otimes_A N \to Z$ in $\mathcal{C}$ up to isomorphism.

$$\begin{array}{ccccc} (M \otimes A) \otimes N & \underset{(1_M \otimes \lambda_N) \circ \alpha_{(M,A,N)}}{\overset{\rho \otimes 1_N}{\rightrightarrows}} & M \otimes N & \longrightarrow & M \otimes_A N \\ & & & \overset{\forall \phi}{\searrow} & \downarrow \exists ! \tilde{\phi} \\ & & & & Z \end{array} \tag{2.56}$$

Take $\mathcal{C} = Vect_{\mathbb{k}}$ and define the relative tensor product of two bimodules as above to be the quotient $M \otimes_A N := \{\sum_{i=1}^{k} m_i \otimes n_i \in M \otimes N | \sum_{i=1}^{k} (m.a) \otimes n = m \otimes (a.n) \forall a \in A\}$. If $i : M \otimes_A N \to M \otimes N$ is the inclusion, then for any $Z \in Vect_{\mathbb{k}}$ and for any linear map $\phi : M \otimes N \to Z$ such that $\phi((m.a) \otimes n) = \phi(m \otimes (a.n))$, we have a unique map $\tilde{\phi} := \phi \circ i$. Furthermore, if M is endowed with a left B-action and N is endowed with a right C-action for $B, C \in Alg_{\mathbb{k}}$ and if $\phi : M \otimes N \to Z$ is a morphism of B-C bimodules satisfying $\phi((m.a) \otimes n) = \phi(m \otimes (a.n))$, then $M \otimes N$ is clearly endowed with a B-C-bimodule structure.

It follows that there are canonical isomorphisms $M \otimes_A A \cong M$ given by $m \otimes_A a \mapsto m.a$, $m \mapsto m \otimes_A \eta$ where η is the unit in A and similarly $B \underset{B}{\otimes} M \cong M$.

The relative tensor product can be considered as composition of modules in the sense that given associative unital algebras, $A, B \in Alg(\mathcal{C})$ we can form a category $[Alg(\mathcal{C})]$ whose objects are associative unital algebras, and morphisms from A to B are isomorphism classes of B-A bimodules together with a relative tensor product as composition. For $M \in {}_BMod_A$ a B-A bimodule, $M \otimes_A A \cong M$ where A is viewed as an A-A bimodule, A can be considered as the identity morphism. Similarly, $B \otimes_B M \cong M$. This is an example of a category whose morphisms are not functions.

We can construct a monoidal category with relative tensor product. Let $(C, \otimes, \mathbb{1})$ be a monoidal category and $A \in Alg(\mathcal{C})$. The A-A bimodules and bimodules maps $({}_AMod_A, \otimes_A, A)$ form a monoidal category.

Example 2.2.4. $Alg_{\mathbb{k}}^2$

1. *Objects are associative, unital $\mathbb{k}$-algebras as discussed in Definition 2.1.2.*

2. *For all objects $A, B \in Alg_{\Bbbk}$, the Hom category is the category of B-A bimodules, $Hom_{Alg^2_{\Bbbk}}(A, B) := {}_BMod_A$ as defined in Equation (2.1.7). This has the following data:*

- *for all objects $A, B \in Alg^2_{\Bbbk}$, 1-morphisms from A to B are B-A-bimodules,*
- *for all 1-morphisms $M, N \in {}_BMod_A$, B-A bimodule maps as 2-morphisms,*
- *vertical composition of 2-morphisms are composition of bimodule maps with unit as the identity map.*

3. *Horizontal composition is the relative tensor product of bimodules over the intermediate algebra, that is, for all $A, B, C \in Alg^2_{\Bbbk}$ and $M \in {}_AMod_B$, $N \in {}_BMod_C$, the composition is $M \otimes_B N \in {}_AMod_C$ as in Definition (2.2.3):*

- *for $M_1, M_2 \in {}_AMod_B$, $N_1, N_2 \in {}_BMod_C$ and $f : M_1 \to M_2$, $g : N_1 \to N_2$, $f \otimes_B g : M_1 \otimes_B N_1 \to M_2 \otimes_B N_2$ is an A-C bimodule map.*

4. *For every $A \in Alg^2_{\Bbbk}$, unit 1-morphism of the horizontal composition is $A \in {}_AMod_A$ viewed as an A-A bimodule, relative tensor product is associative and unital with natural isomorphisms*
5. *Monoidal structure is the tensor product over the ground field $\Bbbk$ with the unit $\Bbbk$,*
6. *The 1-morphism components of the braiding are $\beta_{A,B} = A \otimes B \in {}_{B\otimes A}Mod_{A\otimes B}$, for all $A, B \in \mathcal{B}$. Since the composition $\beta_{B,A} \circ \beta_{A,B} = (B \otimes A) \otimes_{B\otimes A} (A \otimes B) \cong (A \otimes B)$ is the unit 1-morphism in ${}_{A\otimes B}Mod_{A\otimes B}$, the braiding is symmetric.*

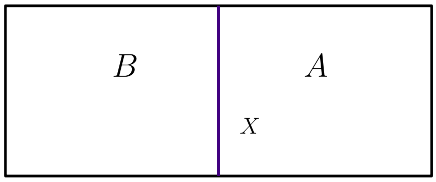
*A **symmetric monoidal 2-functor** $F : \mathcal{C} \to \mathcal{D}$ between two symmetric monoidal 2-categories $\mathcal{C}, \mathcal{D}$ consists of*

- *a 2-functor $F : \mathcal{C} \to \mathcal{D}$*
- *transformations $\xi : F(A) \otimes F(B) \to F(A \otimes B)$ and $\iota : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ which ensures that the monoidal product and the monoidal unit is compatible with the 2-functor*
- *together with four invertible modifications such that the diagrams [28, Figure 2.5, 2.6] commute.*

2.2.1 3d Graphical Calculus

In this subsection, we will introduce graphical calculus for symmetric monoidal 2-categories. For a detailed discussion for the graphical calculus, we refer to [2]. For a brief summary of graphical calculus for duals in 2-categories [3]. Realizing graphical representations of symmetric monoidal 2-categories pave the way to extended 2-dimensional bordism category in Section 3.1.

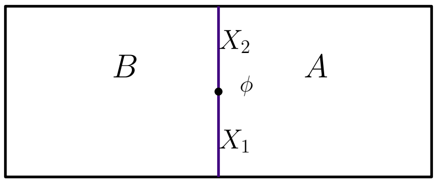
Objects of a symmetric monoidal 2-category $\mathcal{B}$ are depicted by 2-dimensional surfaces, 1-morphisms $X \in Hom_{\mathcal{B}}(A, B)$ are identified with $1_X \in Hom_{A,B}(X, X)$ and depicted as vertical lines.



$$\hat{=} \quad (X : A \to B) \tag{2.57}$$

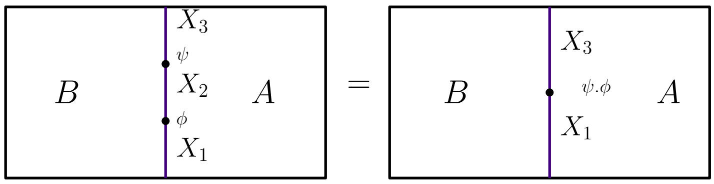
We read diagrams from bottom to top and from right the left.

A 2-morphism $\phi : X_1 \to X_2$ is depicted by a point between 1-morphisms $X_1, X_2 : A \to B$:



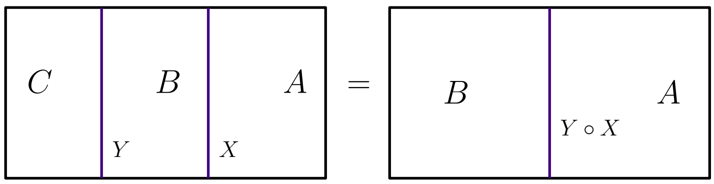
$$\hat{=} \quad (\phi : X_1 \to X_2) \tag{2.58}$$

Vertical composition of 2-morphisms $\phi : X_1 \to X_2$ and $\psi : X_2 \to X_3$ are represented by



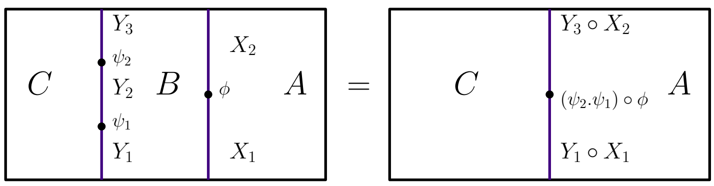
(2.59)

For $X : A \to B$ and $Y : B \to C$, horizontal composition is given by



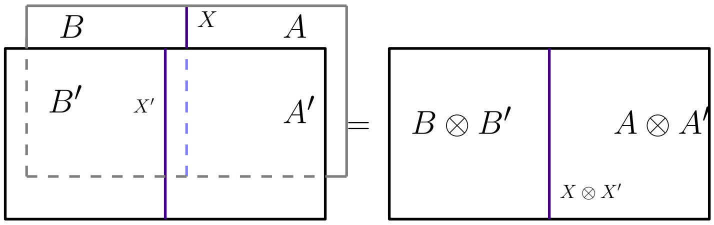
(2.60)

Our convention for horizontal composition is from right to left. For $\phi : X \to X'$ and $\psi : Y \to Y'$ with $X, X' \in Hom_{\mathcal{B}}(A, B)$ and $Y, Y' \in Hom_{\mathcal{B}}(B, C)$, horizontal composition of 1 and 2 morphisms given by



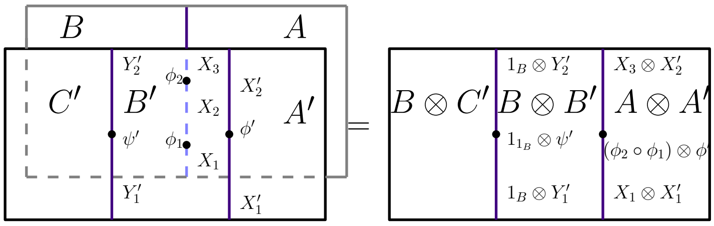
(2.61)

Monoidal structure, on the other hand, is depicted by the third direction; the convention for the composition for monoidal product is from front to back.



(2.62)

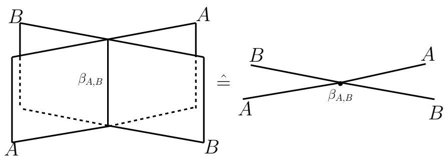
Care must be taken here. Identity maps are not shown in the graphical representation. For example, the graph below reads;



(2.63)

$$[(1_{B_B}\otimes\psi')(1_B\otimes Y_1')]\circ[(\phi_2.\phi_1\otimes\phi')(X_1\otimes X_1')] : A\otimes A'\to B\otimes C' \tag{2.64}$$

Braiding is depicted as transversal intersection of two surfaces or equivalently transversal intersection of two lines which should be thought as projection of surfaces from the top:



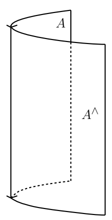
$$= \quad (\beta_{A,B} : A\otimes B\to B\otimes A) \tag{2.65}$$

2.2.2 Full Dualisability

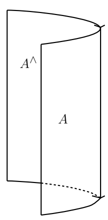
Next, we define duality in a symmetric monoidal 2-category. We will omit associators and unitors.

Definition 2.2.5. *Let* $(\mathcal{B},\otimes,\mathbb{1})$ *be a symmetric monoidal 2-category. An object* $A\in\mathcal{B}$ *is called right dualizable with duality data* $(A, A^\vee, \widetilde{ev}_A, \widetilde{coev}_A)$ *where* $A^\vee\in\mathcal{B}$ *and right evaluation and coevaluation maps are 1-morphisms in* $\mathcal{B}$ *together with invertible 2-morphisms called cusp isomorphisms* $c_1^A : (1_A\otimes coev_A)\circ(ev_A\otimes 1_A)\to 1_A$ *and* $c_2^A : (1_{A^\vee}\otimes ev_A)\circ(coev_A\otimes 1_{A^\vee})\to 1_{A^\vee}$*.*

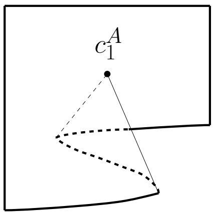
Graphically, evaluation and coevalution maps can be represented by



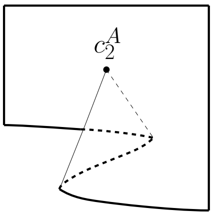
$\hat{=} \quad \widetilde{ev}_A : \mathbb{1}\to A\otimes A^\vee,$



$\hat{=} \quad \widetilde{coev}_A : \mathbb{1}\to A^\vee\otimes A$ (2.66) such that



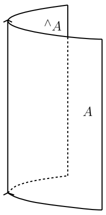
$$= \quad c_1^A : (\widetilde{ev}_A \otimes 1_A) \circ (1_A \otimes \widetilde{coev}_A) \to 1_A \tag{2.67}$$



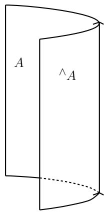
$$= \quad c_2^A : (1_{A^\vee} \otimes \widetilde{ev}_A) \circ (\widetilde{coev}_A \otimes 1_{A^\vee}) \to 1_{A^\vee} \tag{2.68}$$

with inverses $(c_1^A)^{-1}$ and $(c_2^A)^{-1}$ can be depicted by upside down graphs. These are Zorro identities but the equalities are replaced by invertable 2-morphisms c_1^A and c_2^A.

Similarly, left evaluation and coevaluation 1-morphisms



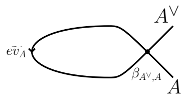
$$\hat{=} \quad ev_A : {}^\vee A \otimes A \to \mathbb{1},$$



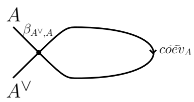
$$= \quad coev_A : \mathbb{1} \to A \otimes {}^\vee A \tag{2.69}$$

are endowed with invertible 2-morphisms realizing Zorro identities.

If $A^\vee$ is a right dual of A, then it is also left dual since 1-morphisms

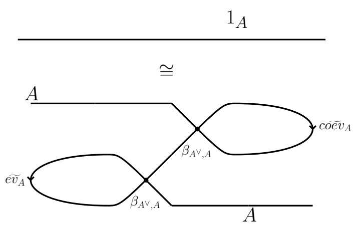


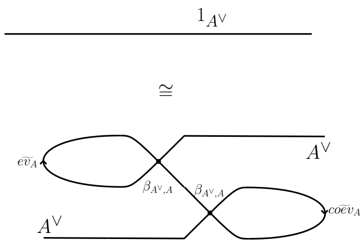
$$= \widetilde{ev} \circ \beta_{A^\vee,A} = ev_A \tag{2.70}$$



$$= \beta_{A^\vee,A} \circ \widetilde{coev}_A = coev_A \tag{2.71}$$

define left evaluation, coevaluation 1-morphisms. Zorro equivalences are



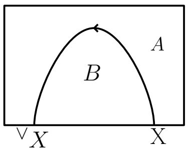


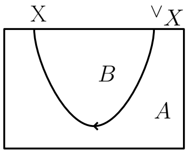
$$\tag{2.72}$$

satisfied by the naturality of the braiding and cusp isomorphisms.

Since $\mathcal{B}$ is symmetric monoidal, one can show that $A^{\vee\vee} \cong A$ and that we can choose $A^{\vee\vee} = A$.

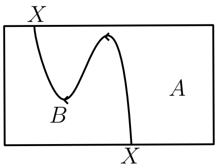
A 1-morphism $X \in Hom_{\mathcal{C}}(A, B)$ in a symmetric monoidal 2-category is said to have a left adjoint if there exists a 1-morphism ${}^{\vee}X \in Hom_{\mathcal{C}}(B, A)$ together with 2-morphisms



$= ev_X : {}^{\vee}X \circ X \to \mathbb{1}_A,$


$= coev_X : 1_B \to X \circ X^{\vee}$ (2.73)

such that



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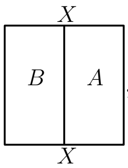

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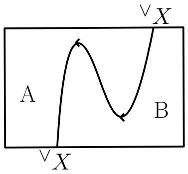

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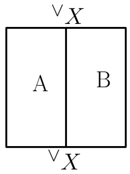
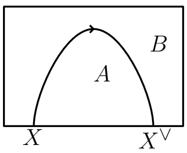
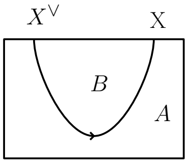


(2.74)

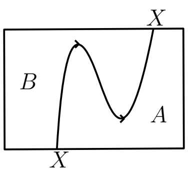
Similarly, X as above is said to have a right adjoint if there exists a 1-morphism $X^{\vee} \in Hom_{\mathcal{C}}(B, A)$ together with 2-morphisms



$= \tilde{ev}_X : X \circ X^{\vee} \to 1_B,$


$= co\tilde{e}v_X : 1_A \to X^{\vee} \circ X$ (2.75)

such that



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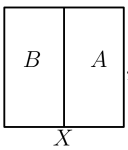

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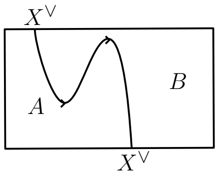

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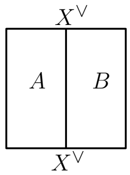
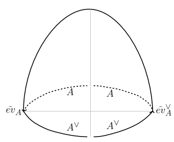
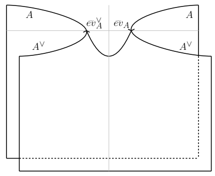


(2.76)

Definition 2.2.6. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal 2-category. An object $A \in \mathcal{C}$ called fully dualizable if it admits (right) duality data $(A, A^{\vee}, \tilde{ev}_A, \widetilde{coev}_A, c_1^A, c_2^A)$ and evaluation and coevaluation 1-morphisms also admit left and right adjoints.*

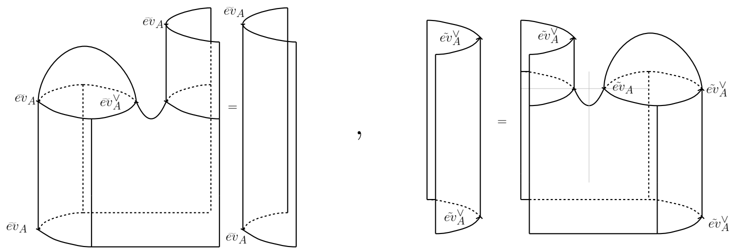
This means that $A \in \mathcal{C}$ has right duals $(A, A^{\vee}, \tilde{ev}_A, \widetilde{coev}_A)$, and the right evaluation 1-morphisms have adjoints. The right evaluation map $\tilde{ev}_A : A \otimes A^{\vee} \to \mathbb{1}$ has a right adjoint $\tilde{ev}_A^{\vee} : \mathbb{1} \to A \otimes A^{\vee}$ together with


$= \tilde{ev}_{\tilde{ev}_A} : \tilde{ev}_A \circ \tilde{ev}_A^{\vee} \to 1_{\mathbb{k}}$ (2.77)



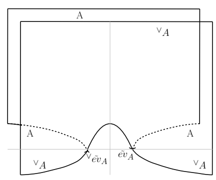
$$= \widetilde{coev}_{\widetilde{ev}_A} = 1_{A\otimes A^\vee} \to \widetilde{ev}_A^\vee \circ \widetilde{ev}_A \tag{2.78}$$

such that

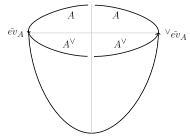


(2.79)

Simirlarly, left adjoint ${}^\vee\widetilde{ev}_A : \mathbb{1} \to A \otimes A^\vee$ together with

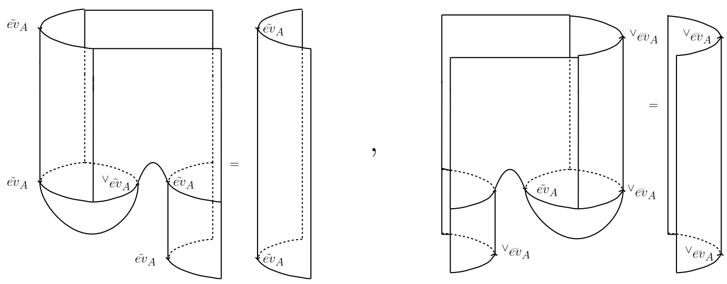


$$= ev_{\widetilde{ev}_A} = {}^\vee\widetilde{ev}_A \circ \widetilde{ev}_A \to 1_{A\otimes A^\vee} \tag{2.80}$$



$$= coev_{\widetilde{ev}_A} : \widetilde{ev}_A \circ {}^\vee\widetilde{ev}_A \to 1_{A\otimes A^\vee} \tag{2.81}$$

such that



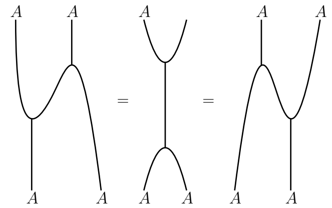
(2.82)

Similarly, $\widetilde{coev}_A$ must have left and right adjoints; four morphisms satisfying four Zorro identities as above.

2.2.3 Separable Algebras

Let $(\mathcal{C}, \otimes, \mathbb{1}, \beta)$ be a symmetric monoidal category and $(A, \mu, \eta) \in Alg(\mathcal{C})$. **The opposite algebra** denoted by A^{op} is an algebra (A, μ^{op}, η) where $\mu^{op} = \mu \circ \beta$. **The enveloping algebra** is $(A \otimes A^{op}, (\mu \otimes \mu^{op}) \circ (id \otimes \beta \otimes id), \eta \otimes \eta) \in Alg(\mathcal{C})$ and denoted by A^e.

We defined Δ-separable algebras for a Frobenius algebra by requiring that $\mu \circ \Delta = 1_A$ in Definition (2.1.6). Δ-separable algebras are, in fact, a special case of the more general notion of separable algebras. An algebra (A, μ, η) is called **separable** if the multiplication $\mu : A \otimes A \to A$ admits 'a section' $\sigma : A \to A \otimes A$ such that the Frobenius relations hold for σ:



(2.83)

Note that Frobenius relations can be interpreted as bimodule maps: $\sigma : A \to A \otimes A$ satisfies Frobenius relations if and only if it is a map of A-A bimodules, that is to say, $\sigma \in Hom_{AA}(A, A \otimes A)$. Since A-A bimodules are equivalent to left A^e-modules, σ can be defined by left A^e module map.

In $Vect_{\Bbbk}$, separable algebras can equivalently be described by split exact sequences. More precisely, a $\Bbbk$-algebra is separable if and only if the sequence $0 \to \ker(\mu) \xrightarrow{i} A^e \xrightarrow{\mu} A \to 0$ is split exact. This means that there is a section σ of its multiplication $\mu : A^e \to A$ viewed as a left A^e-module:

$$\sigma : A \to A^e \quad \text{such that} \quad \sigma(a^e.a) = a^e.\sigma(a) \tag{2.84}$$

$$\mu \circ \sigma = 1_A \quad \text{for all } a^e \in A^e. \tag{2.85}$$

Equivalently, an algebra is separable if and only if for all algebraic field extensions L of $\Bbbk$, $A \otimes L$ is semisimple. In particular, since $A \otimes \Bbbk \cong A$, the algebra A itself is semisimple [11, Prop. 1.1].

Let $A \in Alg_{\Bbbk}$. A left A-module $M \in {}_A Mod$ is **finitely generated** if M admits a finite set of generators $m_1, \ldots, m_n$ such that any $m \in M$ can be written as

$$m = \sum_{i=1}^{n} a_i.m_i \tag{2.86}$$

for some $a_i \in A$. Furthermore, M is called **projective** if there exists a set of maps $(f_1, \ldots, f_n) \in M^* = Hom_A(M, A)$ with $f_i(m) = a_i$. An element $f \in M^*$ has the property $f(a.m) = a.f(m)$. Since M^* has a natural right action: $(f.a)(m) = f(m).a$ for all $a \in A$, $M^* \in Mod_A$.

As duals of vector spaces are defined by $V^\vee = Hom_{\Bbbk}(V, \Bbbk)$, we expect M^* to be a dual of M. However, the category of A-B bimodules is not monoidal unless $A = B$ and we defined duality only in monoidal categories. Nevertheless, in the symmetric monoidal 2-category $Alg^2_{\Bbbk}$, for a left A-module $M \in {}_A Mod_{\Bbbk}$, M^* is the **left adjoint module**.

Now, take $A = A^e$, $M = A$. It follows that an algebra A is separable if and only if $A \in {}_{A^e}Mod$ is finitely generated and projective as an A^e-module. The next lemma will let us realize A^* as a left adjoint in $Alg^2_{\Bbbk}$ so that an algebra (A, μ, η) is separable if and only if $A \in {}_{A^e}Mod$ (viewed as a 1-morphism) admits a left adjoint.

Lemma 2.2.7. *Let $A \in Alg_{\Bbbk}$. A module $M \in {}_A Mod$ is finitely generated and projective if and only if*

$$\psi : M^* \otimes_A M \to End_A(M)$$

$$\psi(f \otimes m) := (x \mapsto f(x).m) \tag{2.87}$$

is an isomorphism of left A-modules.

Proof. $\Longrightarrow$: Let $(m_1, \dots, m_n) \in M$, $(f_1, \dots, f_n) \in M^*$ be bases for M and M^*. Define

$$\begin{aligned} \phi : End_A(M) &\to M^* \underset{A}{\otimes} M \\ f &\mapsto \sum_{i=1}^{n} f_i \underset{A}{\otimes} f(m_i) \end{aligned} \tag{2.88}$$

$$(\psi \circ \phi(f))(m) = (\psi(\sum_{i=1}^{n} f_i \underset{A}{\otimes} f(m_i)))(m) = \sum_{i=1}^{n} f_i(m) \underset{A}{\otimes} f(m_i) = f(m) \tag{2.89}$$

for all $m \in M$. Thus, we have $\psi \circ \phi(f) = f$.

On the other hand,

$$\begin{aligned} \phi \circ \psi(f \otimes_A m) = \phi(x \mapsto f(x).m)) \\ = \sum_{i=1}^{n} f_i \otimes_A f(m_i).p = \sum_{i=1}^{n} f_i.f(m_i) \otimes_A m = f \otimes_A m \end{aligned} \tag{2.90}$$

Thus, ψ and ϕ are inverse to each other.

$\Longleftarrow$: Given a projective basis, we define $\psi^{-1}(1_M) = \sum_i^n f_i \otimes_A m_i$ so that

$$m = \sum_{i}^{n} f_i(m).m_i = (\psi(\sum_{i,}^{n} f_i \otimes_A m_i))(m) = (\psi \circ \psi^{-1})(1_M)(m) = m \tag{2.91}$$

□

Definition 2.2.8. *Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category and $A \in Alg(\mathcal{C})$ be an algebra over $\mathcal{C}$. The algebra A is called* ***separable*** *if A viewed as a left A^e module has a left adjoint 1-morphism in $Alg^2(\mathcal{C})$.*

As we will see explicitly in the next example, A viewed as a left A^e-module has a right adjoint in $Alg^2(\mathcal{C})$ if an only if it is finite dimensional.

Example 2.2.9. *Fully dualizable objects in $Alg^2_{\Bbbk}$:*
Let $A \in Alg^2_{\Bbbk}$ be an object. Define left and right duals $A^\vee = {}^\vee A = A^{op}$ to be the opposite algebra together with right evaluation and coevaluation maps as A viewed as bimodules as follow:

$$\begin{aligned} \widetilde{ev}_A &:= {}_{\Bbbk}A_{A \otimes A^{op}} \in Alg^2_{\Bbbk}(A \otimes A^{op}, \Bbbk) \\ \widetilde{coev}_A &:= {}_{A^{op} \otimes A}A_{\Bbbk} \in Alg^2_{\Bbbk}(A^{op} \otimes A, \Bbbk) \end{aligned} \tag{2.92}$$

To show that these data are actually right evaluation and coevaluation maps, we have to construct bimodule maps as in (2.67). Labeling A's with redundant lower indices to keep *track of A's, define A_4-A_5 bimodule maps*

$$c_1^A : ({}_{\Bbbk}A_{A_1\otimes A_2^{op}} \otimes {}_{A_4}A_{A_3}) \bigotimes_{A_1\otimes A_2^{op}\otimes A_3} ({}_{A_1}A_{A_5} \otimes {}_{A_2^{op}\otimes A_3}A_{\Bbbk}) \mapsto {}_{A_4}A_{A_5}$$
$$((m_1 \otimes m_2) \otimes (m_3 \otimes m_4)) \to m_2.m_4.m_1.m_3 \qquad (2.93)$$

$$(c_1^A)^{-1} : {}_{A_4}A_{A_5} \to ({}_{\Bbbk}A_{A_1\otimes A_2^{op}} \otimes {}_{A_4}A_{A_3}) \bigotimes_{A_1\otimes A_2^{op}\otimes A_3} ({}_{A_1}A_{A_5} \otimes {}_{A_2^{op}\otimes A_3}A_{\Bbbk})$$
$$m \mapsto (1 \otimes m) \otimes (1 \otimes 1) \qquad (2.94)$$

$$\begin{aligned}(c_1^A)^{-1}.c_1^A :&(m_1 \otimes m_2) \otimes (m_3 \otimes m_4) \mapsto (1 \otimes m_2.m_4.m_1.m_3) \otimes (1 \otimes 1)\\ &= (1 \otimes m_2) \otimes (1 \otimes m_4.m_1.m_3) = (m_1.m_3 \otimes m_2) \otimes (1 \otimes m_4)\\ &= (1 \otimes m_2) \otimes (m_3 \otimes m_4.m_1) = (m_1 \otimes m_2) \otimes (m_3 \otimes m_4) \end{aligned} \qquad (2.95)$$

where we repeatedly used the property of the relative tensor product.

$$c_1^A.(c_1^A)^{-1}(m) = \phi((1 \otimes m) \otimes (1 \otimes 1)) = m \qquad (2.96)$$

Thus, it is an isomorphism. Similarly,

$$c_2^A : ({}_{A_5^{op}}A^{op} \otimes A_{A_2\otimes_{\Bbbk} A_3^{op}}) \bigotimes_{A_1^{op}\otimes A_2\otimes A_3^{op}} ({}_{A_1^{op}\otimes A_2}A_{\Bbbk} \otimes {}_{A_3^{op}}A^{op}_{A_4^{op}}) \to {}_{A_5^{op}}A^{op}_{A_4^{op}}$$
$$(m_1 \otimes m_2) \otimes (m_3 \otimes m_4) \mapsto m_4.m_2.m_3.m_4 \qquad (2.97)$$

and the inverse

$$(c_2^A)^{-1} : {}_{A_5^{op}}A^{op}_{A_4^{op}} \to ({}_{A_5^{op}}A^{op} \otimes A_{A_2\otimes_{\Bbbk} A_3^{op}}) \bigotimes_{A_1^{op}\otimes A_2\otimes A_3^{op}} ({}_{A_1^{op}\otimes A_2}A_{\Bbbk} \otimes {}_{A_3^{op}}A^{op}_{A_4^{op}})$$
$$m \mapsto (1 \otimes m) \otimes (1 \otimes 1) \qquad (2.98)$$

This completes the proof of first dualisability.

Let A further be finite dimensional as a vector space and a separable algebra. Taking $A^e = A \otimes A^{op}$, we define the right adjoint of $\widetilde{ev}_A$

$$\widetilde{ev}_A^{\vee} := {}_{A^e}(Hom_{\Bbbk}(A, \Bbbk))_{\Bbbk} \in {}_{A^e}Mod_{\Bbbk} \qquad (2.99)$$

The dual vector space $Hom_{\Bbbk}(A, \Bbbk)$ is endowed with a left A^e action defined by

$$(a_e.f)(a) = f(a.a_e) \qquad (2.100)$$

Let $(e_1, \ldots, e_n) \in A$ *and* $(f_1, \ldots, f_n) \in Hom_{\Bbbk}(A, \Bbbk)$ *be a standard orthonormal basis as vector spaces with* $f_i(e_j) = \delta_{ij}$*. An element* $a_e = a_1 \otimes a_2$ *in* A^e *acts on* $A \in Mod_{A^e}$ *by* $a.a_e = a_2.a.a_1$ *We define*

$$\begin{aligned}\widetilde{ev}_{\widetilde{ev}_A} : {}_{\Bbbk}A_{A^e} \bigotimes_{A^e} {}_{A^e}(Hom_{\Bbbk}(A, \Bbbk))_{\Bbbk} &\to \Bbbk \\ (a \otimes f) &\mapsto f(a) \\ \widetilde{coev}_{\widetilde{ev}_A} : {}_{A^e}A^e_{A^e} \to {}_{A^e}(Hom_{\Bbbk}(A, \Bbbk))_{\Bbbk} &\bigotimes_{\Bbbk} {}_{\Bbbk}A_{A^e} \\ 1 \mapsto \sum_{i=1}^{n} (f_i \otimes e_i)\end{aligned} \tag{2.101}$$

The first Zorro identity (2.79) *amounts to the composition*

$$\begin{aligned}(\widetilde{ev}_{\widetilde{ev}_A} \otimes id_{{}_{\Bbbk}A_{A^e}}) \circ (id_{{}_{\Bbbk}A_{A^e}} \otimes \widetilde{coev}_{\widetilde{ev}_A}) : {}_{\Bbbk}A_{A^e} \to {}_{\Bbbk}A_{A^e} \\ a \mapsto (a \otimes \sum_{i=1}^{n} f_i \otimes a_i) \mapsto \sum_{i=1}^{n} f_i(a).a_i = a\end{aligned} \tag{2.102}$$

and second equation of (2.79) *reads*

$$\begin{aligned}(id_{{}_{A^{eop}}A_{\Bbbk}} \otimes \widetilde{ev}_{\widetilde{ev}_A}) \circ (\widetilde{coev}_{\widetilde{ev}_A} \otimes id_{{}_{A^{eop}}A^{op}_{\Bbbk}}) : {}_{A^{eop}}(Hom_{\Bbbk}(A, \Bbbk))_{\Bbbk} \to {}_{A^{eop}}(Hom_{\Bbbk}(A, \Bbbk))_{\Bbbk} \\ f \mapsto (\sum_{i=1}^{n} f_i \otimes a_i \otimes f) \mapsto \sum_{i=1}^{n} f_i \otimes f(a_i) = f\end{aligned} \tag{2.103}$$

Similarly, we define the left adjoint of $\widetilde{ev}_A$

$$^{\vee}\widetilde{ev}_A := {}_{A^e}(Hom_{A^e}(A, A^e))_{\Bbbk} \in {}_{A^e}Mod_{\Bbbk} \tag{2.104}$$

and the left A^e *action here is defined by*

$$(a_e.f)(a) = a_e.f(a). \tag{2.105}$$

Let $a_1, \ldots, a_n \in A$ *and* $f_1, \ldots, f_n \in Hom_{A^e}(A, A^e)$ *be a projective basis as in* (2.88)*. Define bimodule maps*

$$\begin{aligned}ev_{\widetilde{ev}_A} : {}_{A^e}(Hom_{A^e}(A, A^e))_{\Bbbk} \bigotimes_{\Bbbk} {}_{\Bbbk}A_{A^e} \to {}_{A_e}A^e_{A^e} \\ (f \otimes a) \mapsto f(a)\end{aligned} \tag{2.106}$$

$$\begin{aligned}coev_{\widetilde{ev}_A} : {}_{\Bbbk}\Bbbk_{\Bbbk} \to {}_{\Bbbk}A_{A^e} \bigotimes_{A^e} {}_{A^e}(Hom_{A^e}(A, A^e))_{\Bbbk} \\ 1 \mapsto \sum_{i=1}^{n} a_i \otimes f_i\end{aligned} \tag{2.107}$$

These morphisms satisfy identities from (2.82)

$$\begin{aligned}(id_{{}_{\Bbbk}A_{A^e}} \otimes ev_{\widetilde{ev}_A}) \circ (coev_{\widetilde{ev}_A} \otimes id_{{}_{\Bbbk}A_{A^e}}) &: {}_{\Bbbk}A_{A^e} \to {}_{\Bbbk}A_{A^e} \\ a \mapsto (\sum_{i=1}^{n} a_i \otimes f_i \otimes a) &\mapsto \sum_{i=1}^{n} a_i \otimes \sum_{i=1}^{n} f_i(a) = a\end{aligned} \tag{2.108}$$

$$\begin{aligned}(coev_{\widetilde{ev}_A} \otimes id_{{}_{A^{e^{op}}}A_{\Bbbk}}) \circ (id_{{}_{A^{e^{op}}}A_{\Bbbk}} \otimes ev_{\widetilde{ev}_A}) &: {}_{A^{e^{op}}}A_{\Bbbk} \to {}_{A^{e^{op}}}A_{\Bbbk} \\ f \mapsto (f \sum_{i=1}^{n} a_i \otimes f_i) &\mapsto \sum_{i=1}^{n} f(a_i) \otimes f_i = f\end{aligned} \tag{2.109}$$

Notice that the right adjoint of the evaluation $\widetilde{ev}_A^{\vee}$ *is the dual of* $\widetilde{ev}_A$ *in* $Vect_{\Bbbk}$ *whereas the left adjoint of the evaluation is the dual of* $\widetilde{ev}_A$ *in* $Mod(A^e)$*.*

Thus, a finite dimensional separable algebra is fully dualizable in $Alg^2_{\Bbbk}$. Converse is also true; if an object $A \in Alg^2_{\Bbbk}$ is fully dualizable then it is separable and finite dimensional. For a proof we refer to [28], and in the language of ∞-categories, see [31, Theorem 4.2.6].

3 2d Extended TQFT

In this chapter, we turn from algebra to manifolds and define bordism category. First, we review n-dimensional bordism category whose objects are $(n-1)$-dimensional manifolds and morphisms are n-dimensional manifolds with boundary. Second, we allow manifolds to have a codimension 2 corner to define once extended bordism category. The classification of surfaces are then used to define generators and relations descriptions. Third, we endow manifolds with a group structure and finally define TQFTs and express the cobordism hypothesis.

3.1 Bordism

Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, and let M be a second countable, Hausdorff topological space and O be an open neighborhood of a point $m \in M$ and let $U \subset \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$. A homeomorphism $\phi : U \to O$ is called a **chart with corners** for M where n, k are positive integers such that $0 \leq k \leq n$. Taking $k = 1$ gives a **chart with boundary** and $k = 0$ gives a **chart** (without boundary without corners). Two charts $\phi_1 : U_1 \to O_1$, $\phi_2 : U_2 \to O_2$ with $O_1 \cap O_2 \neq \varnothing$ are **compatible** if $\phi_1^{-1} \circ \phi_2 : U_1 \cap U_2 \to U_2 \cap U_1$ where $U_1 \cap U_2 = \phi_1^{-1}(O_1 \cap O_2)$ is a diffeomorphism onto its image. This diffemorphism is called **a transition function**. An **atlas (with corners)** $\mathcal{A}$ for a topological space M is a family of compatible charts (with corners) $\mathcal{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \Lambda}$ that cover $M = \cup_{\alpha \in \Lambda} \phi_\alpha(U_\alpha)$. M is called a **manifold (with corners)** if it is endowed with a smooth atlas (with corners). A manifold is called **compact** if every cover has finite subcover.

Manifolds are defined as 'abstract spaces' but computations on manifolds are carried on local charts that can be glued together to form a manifold. Given a set of local spaces $\{U_\alpha\}_{\alpha \in \Lambda}$ and diffeomorphisms $\psi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$ where Λ is an index set and $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and Λ such that

$$\psi_{\alpha\beta}(U_{\alpha\beta} \cap U_{\alpha\gamma}) = U_{\gamma\alpha} \cap U_{\gamma\beta}, \tag{3.1}$$

and

$$\psi_{\alpha\beta} = \psi_{\gamma\beta} \circ \psi_{\alpha\gamma} \qquad \text{on} \qquad U_{\alpha\beta} \cap U_{\alpha\gamma} \tag{3.2}$$

there is a unique manifold with an atlas whose transition functions are diffeomorphic to $\psi_{\alpha\beta}$. This is the co-cycle condition. The proof can be found in any text book on manifolds.

We would like to define a notion of composition of manifolds with boundary with corners for categorification. First, we focus on manifolds with boundary. For two

n-dimensional manifolds M_1, M_2 with a common boundary Σ as a closed $(n-1)$-dimensional manifold, we can use the local charts to construct the glued manifold $M_1 \cup_\Sigma M_2 = M_1 \sqcup M_2 / \sim_\Sigma$. Let $\theta_i : \Sigma \to M_i$ be embeddings. The equivalence relation for $m, m' \in M_1 \sqcup M_2$ is the following: $m \sim m'$ iff $m = \theta_1(\sigma)$ and $m' = \theta_2(\sigma)$ for some $\sigma \in \Sigma$. In the case of topological manifolds, $M_1 \sqcup_\Sigma M_2$ is well-defined and unique up to homeomorphism. The embeddings are canonical. However, in the smooth setting, even though the resulting manifold is unique up to diffeomorphism, the diffeomorphism class is not unique. In other words, $M_1 \cup_\Sigma M_2$ can be endowed with non-diffeomorphic smooth structures.

To overcome this problem, we need the data of embeddings of open neighborhood of the boundary. A **collar** is a locally diffeomorphic embedding $\theta : [0,1) \times \Sigma \to M$. Given collars $\theta_1 : (-1,0] \times \Sigma \to M_1$, $\theta_2 : (-1,0] \times \Sigma \to M_2$ for the boundary Σ, the glued manifold has a unique smooth structure up to diffeomorphism. Different choices of collars induce diffeomorphic smooth structures on the glued manifold. Though the diffeomorphism is non-canonical, this is sufficient to define bordism category by taking diffeomorphism classes.

Definition 3.1.1. *An n-dimensional bordism category $Bord_{n,n-1}$ is a symmetric monoidal category with the following data:*

1. *Objects are closed $(n-1)$-dimensional manifolds.*
2. *For objects $E_1, E_2 \in Bord_{n,n-1}$, a morphism (called bordism) $E_1 \to E_2$ is a tuple (M, θ_1, θ_2) where M is a diffeomorphism class of a compact manifold with boundary together with decomposition into incoming and outgoing boundaries $\partial M = \partial_{in} M \sqcup \partial_{out} M$ and collars; $\theta_1 : [0,1) \times E_1 \to M$ and $\theta_2 : (-1,0] \times E_2 \to M$ such that $\theta(\{0\} \times E_1) = \partial_{in} M$ and $\theta_2(\{0\} \times E_2) = \partial_{out} M$.*
3. *Composition is gluing along the common boundary with the collars,*
4. *Unit morphism for $E \in Bord_{n,n-1}$ is the cylinder $[0,1] \times E$,*
5. *Monoidal product is the disjoint union,*
6. *Monoidal unit is the empty set $\varnothing$,*
7. *Associators for $E_1, E_2, E_3 \in Bord_{n,n-1}$ are canonical isomorphisms $(E_1 \sqcup E_2) \sqcup E_3 \cong E_1 \sqcup (E_2 \sqcup E_3)$ and unitors for $E \in Bord_{n,n-1}$ are $\varnothing \sqcup E \cong E \cong E \sqcup \varnothing$.*

Let M be a manifold with boundary equipped with an atlas $\mathcal{A} = \{U_\alpha, \phi_\alpha\}_{\alpha \in \Lambda}$. A set of diffeomorphisms

$$\begin{aligned} \partial\phi_\alpha : U_\alpha \times \mathbb{R}^n &\to TO_\alpha \\ (x, v) &\to (\phi_\alpha(x), \partial\phi_\alpha(v)|_x) \end{aligned} \tag{3.3}$$

forms a set of compatible charts with transition functions

$$\partial\phi_i^{-1} \circ \partial\phi_j : (U_i \times \mathbb{R}^n) \cap (U_j \times \mathbb{R}^n) \to (U_j \times \mathbb{R}^n) \cap (U_i \times \mathbb{R}^n) \tag{3.4}$$

These charts satisfy the cocyle condition and thus, there exists a unique manifold TM with such charts.

The projection $\pi : TM \to M$, locally defined by $\pi \circ \partial\phi(x, v) = \phi(x)$ is smooth and surjective, $T_mM = \pi^{-1}(m) \cong \mathbb{R}^n$ is an isomorphism of vector spaces.

An **oriented atlas** on M is a collection of compatible charts $\{U_\alpha, \phi_\alpha\}$ such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ have positive Jacobian determinant whenever their domains intersect nontrivially, i.e. $det(\partial\phi_\alpha \circ \partial\phi_\beta^{-1}) > 0$. Here, the positive Jacobian determinant ensures that the transition functions between coordinate systems preserve the chosen orientation. An orientation on a manifold is a manifold equipped with an oriented atlas. In other words, an orientation on a manifold M is 'a consistent choice of assignment of positive direction at each point of the manifold'. This choice is made in the tangent space T_mM at any point $m \in M$.

A manifold is **orientable** if it admits an atlas that is oriented. Equivalently, an n-dimensional manifold is orientable if and only if it possesses a nowhere-vanishing n-form (i.e., a non-zero n-form at each point). If the manifold M has a boundary ∂M, then a chart $U \to \mathbb{R}^{n-1} \times \mathbb{R}_+$ of the boundary whose restriction to the interior of M is oriented is an oriented chart of the boundary. Therefore, the orientation on M induces an orientation on the boundary.

Objects and bordisms of the symmetric monoidal category $Bord_{n,n-1}$ can be endowed with orientation to form $Bord^{or}_{n,n-1}$. Objects of $Bord^{or}_{n,n-1}$ are $(n-1)$-dimensional closed oriented manifolds and morphisms are oriented diffeomorphism classes of n-dimensional compact manifolds together with collars that respect orientation.

2-dimensional oriented connected closed manifolds are classified up to diffeomorphism by the number of genus. If we allow manifolds to have a boundary, the classification includes number of connected incoming-boundary components and number of outgoing-boundary components.

Since any 1-dimensional closed manifold is diffeomorphic to a finite disjoint union of circles, the objects of $Bord^{or}_{2,1}$ is generated by S^1 under disjoint union and bordisms are generated as a symmetric monoidal category by four morphisms under disjoint union and composition [19, Section 2] such that relations [20, Section 1.4] satisfy. These relations give precisely the relations of commutative Frobenius algebras.

In the case of manifolds with corners, it is more difficult to construct a well-defined composition. The aim now is to construct a symmetric monoidal 2-category $Bord_{n,n-1,n-2}$. Roughly speaking, objects of this category are closed $(n-2)$-dimensional manifolds, 1-morphisms are $(n-1)$-dimensional manifolds with boundary and 2-morphisms are compact n-dimensional manifolds with corners. We must be able to compose 1-morphisms horizontally and compose 2-morphisms horizontally and vertically. So far, we defined gluing only up to non-canonical diffeomorphism and taking diffeomorphism classes as composition of 1-morphisms as we did for $Bord_{n,n-1}$ will lead us to fail on constructing a well-defined 2-morphism.

This is explicitly constructed in [28, Chapter 3]. Here, we give the rough idea and give reference to this paper for details and focus on dimension 2.

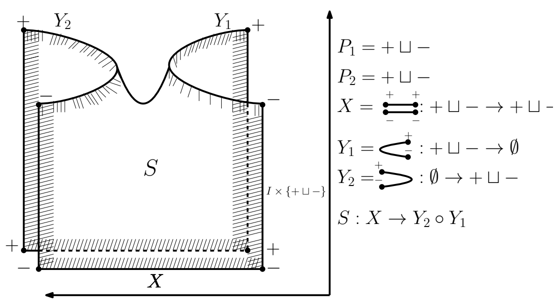
We should be able to glue n-dimensional manfiolds with corners along their common

$(n-1)$-dimensional *faces*. Let S be a manifold with corners and (ϕ, U) be a chart at $s \in S$. The **codimension** at s is the number of zero coordinates of $\phi^{-1}(x) \in \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$. This is independent of the choice of chart. A **face** of S is the topological closure of all codimension 1 points of S. We restrict our attention to manifolds with corners such that every point $s \in S$ belongs to -codimension at s-many connected components of faces. In the 2 dimensional example (3.5), corners have codimension 2 and they belong to two connected components, e.g. left top corner with label $+$ belongs to two connected faces, Y_2 and $+ \times I$ where I is the closed interval and $+$ denotes the point with positive orientation.

We further assume that a manifold with corner S is equipped with an ordered set of n-faces $(\partial_0 S, \dots, \partial_{n-1} S)$ such that their union covers boundary and arbitrary intersection of any two $\partial_i S$ and $\partial_j S$ is also a face.

[28, Prop. 3.1] proves that given two manifolds with corners S_1 and S_2 with a common face X together with *collars*, there is a canonical smooth structure on the glued manifold $S_1 \cup_X S_2$ compatible with the smooth structures of S_1 and S_2. We use this proposition to construct horizontal and vertical compositions.

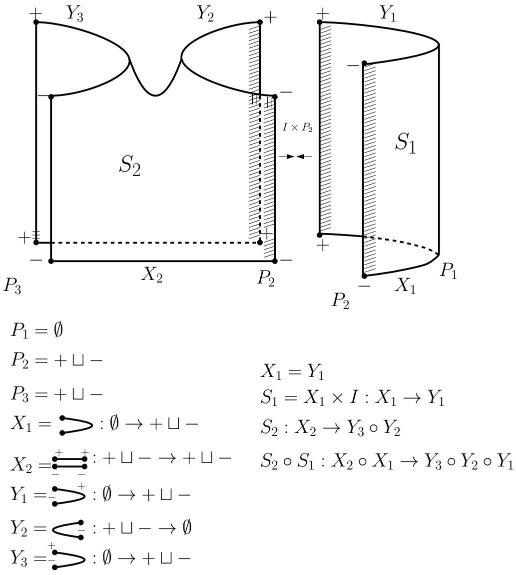
Let P_1, P_2 be closed $(n-2)$-dimensional manifolds. An $(n-1)$-dimensional manifold X with boundary with collars $\partial X \cong \partial_{in} X \sqcup \partial_{out} X \cong P_1 \sqcup P_2$ is called a **1-bordism** from P_1 to P_2. In the 2-dimensional example (3.5), $X = I \times (+ \sqcup -)$ is a unit 1-bordism for $(+ \sqcup -)$ and Y_1, Y_2 are 1-bordisms.



(3.5)

Let P_1, P_2, P_3 be closed $(n-2)$-dimensional manifolds and let $X_1 : P_1 \to P_2$ and $X_2 : P_2 \to P_3$ be 1-bordisms with chosen collars as above. This choice is necessary since we are not taking diffeomorphism classes. We use the axiom of choice to glue 1-bordisms. The resulting manifold with boundary $X_2 \cup_{P_2} X_1$ has a canonical smooth structure and boundary, thus again a 1-bordism from P_1 to P_3. We will call this **horizontal composition** of 1-bordisms. In the 2-dimensional example (3.5), two 1 bordisms Y_1, Y_2 are horizontally composed along common face $\emptyset$ whereas in (3.6), Y_1, Y_2

are composed along $+ \sqcup -$ to form S^1.



(3.6)

For two closed $(n-2)$-dimensional manifolds P_1, P_2, and two 1-bordisms X_1 and X_2 from P_1 to P_2, we define a **2-bordism** $S : X_1 \to X_2$ to be a compact n-dimensional manifold with boundary with corners together with decompositions

$$\partial_0 S = \partial_{0,in} S \sqcup \partial_{0,out} S \xrightarrow{g} X_1 \sqcup X_2$$
$$\partial_1 S = \partial_{1,in} S \sqcup \partial_{1,out} \xrightarrow{f} P_1 \times [0,1] \sqcup P_2 \times [0,1] \tag{3.7}$$

such that

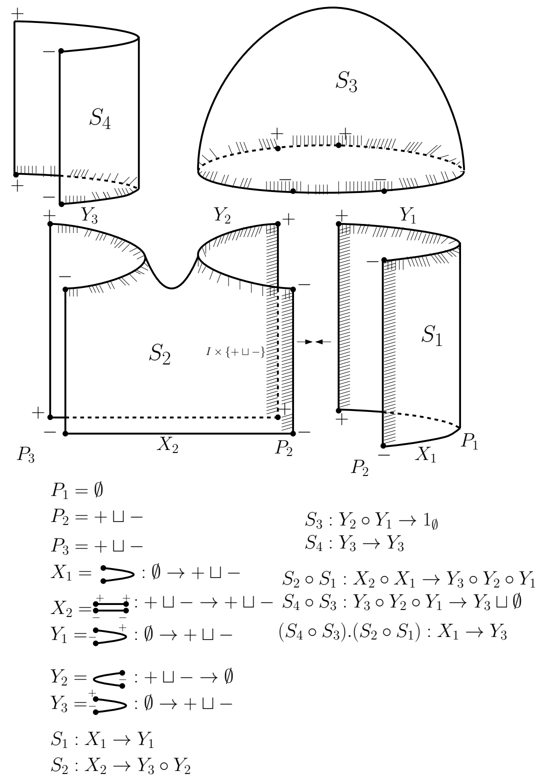
$$f^{-1} g : \partial_{in} X_1 \sqcup \partial_{out} X_1 \to P_1 \times \{0\} \sqcup P_2 \times \{0\}$$
$$f^{-1} g : \partial_{in} X_2 \sqcup \partial_{out} X_2 \to P_1 \times \{1\} \sqcup P_2 \times \{1\} \tag{3.8}$$

and that these isomorphisms coincide with the boundary isomorphisms of X_1 and X_2. In (3.5), the collars are shown by shaded areas on the faces of a 2-bordism S where $n = 2$.

Let P_1, P_2, P_3 be $(n-2)$-dimensional closed manifolds, $X_1, X_2 : P_1 \to P_2$ and $Y_1, Y_2 : P_2 \to P_3$ be 1-bordisms and let $S : (X_1 : P_1 \to P_2) \to (X_2 : P_1 \to P_2)$ and $S' : (Y_1 : P_2 \to P_3) \to (Y_2 : P_2 \to P_3)$ be 2-bordisms. Both S_1 and S_2 have a common face $P_2 \times I$ and the collars for P_2 in S_1 and S_2 induce collars for $P_2 \times I$ [28, Lemma 3.7]. Thus, they can be glued among this face. The glued manifold with corners $S_2 \cup_{P_2 \times I} S_1 : Y_1 \cup_{P_2} X_1 \to Y_2 \cup_{P_2} X_2$ is again a 2-bordism. This is the **the horizontal composition of 2-bordisms**. (3.6) is a 2-dimensional example of horizontal composition of 1-bordisms.

Let P_1, P_2 be closed $(n-2)$-dimensional manifolds, X_1, X_2, X_3 be $(n-1)$-dimensional 1-bordisms from P_1 to P_2 and $S_1 : X_1 \to X_2$ and $S_2 : X_2 \to X_3$ be 2-bordisms. Similarly, we glue manifolds with corners along the common face to form the **vertical**

composition $S_2 \cup_{X_2} S_1 : X_1 \to X_3$. (3.9) depicts a 2-dimensional example of horizontal and vertical compositions.



(3.9)

We are now ready to define the symmetric monoidal 2-category $Bord_{n,n-1,n-2}$.

Definition 3.1.2. *The symmetric monoidal 2-category $Bord_{n,n-1,n-2}$ consists of*

1. *Objects are $(n-2)$-dimensional closed manifolds.*

2. *For every pair of objects $P_1, P_2 \in Bord_{n,n-1,n-2}$, a 1-morphism is a 1-bordism $X : P_1 \to P_2$, that is a compact 1-dimensional manifold with boundary with collars.*

3. *For every pair of objects $P_1, P_2 \in Bord_{n,n-1,n-2}$ and for every pair of 1-bordisms $X, Y \in Bord_{n,n-1,n-2}(P_1, P_2)$, a 2-morphism is an equivalence class of a 2-bordism $S : X \to Y$, that is, a diffeomorphism class of a compact 2-dimensional manifold with boundary with corners together with collars.*

4. *For every triple of objects $P_1, P_2, P_3 \in Bord_{n,n-1,n-2}$, and $X : P_1 \to P_2$, $Y : P_2 \to P_3$ the horizontal composition of 1-bordisms is $Y \circ X = Y \cup_{P_2} X : P_1 \to P_3$.*

5. *For every object $P \in Bord_{n,n-1,n-2}$, an identity 1-morphism of the horizontal composition is the cylinder over P: $I \times P : P \to P$.*

6. *For every triple of objects $P_1, P_2, P_3 \in Bord_{n,n-1,n-2}$ and 1-bordisms $X_1, Y_1 : P_1 \to P_2$, $Y_1, Y_2 : P_2 \to P_3$, and 2-bordisms $S_1 : X_1 \to Y_1$, $S_2 : X_2 \to Y_2$ the horizontal composition of 2-bordisms is $S_2 \circ S_1 = S_2 \cup_{P_2 \times I} S_1 : Y_1 \circ X_1 \to Y_2 \circ X_2$.*

7. *For every 1-bordism X, an identity 2-bordism of the horizontal composition is $X \times I$.*

8. *For every pair of objects $P_1, P_2 \in Bord_{2,1,0}$ and every triple of 1-bordisms $X_1, X_2, X_3 : P_1 \to P_2$, the vertical composition of 2-bordisms $S_1 : X_1 \to X_2$ and $S_2 : X_2 \to X_3$ is $S_2.S_1 = S_2 \cup_{X_2} S_1 : X_1 \to X_3$.*

9. *The monoidal product is the disjoint union and the monoidal unit the empty set.*

10. *For every pair of objects $P_1, P_2 \in Bord_{n,n-1,n-2}$, 1-morphism components of the symmetric braiding are $\beta_{P_1,P_2} : P_1 \sqcup P_2 \to P_2 \sqcup P_1$ whereas 2-morphism components are $\beta_{X_1,X_2} : X_1 \sqcup X_2 \to X_2 \sqcup X_1$ for all 1-bordisms X_1, X_2.*

An **orientation** on a manifold with corners M involves defining a consistent choice of positive directions not just on the smooth parts but also at the corners of the manifold.

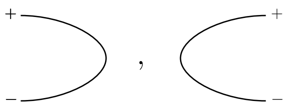
Consider an n-dimensional manifold with corners M embedded in $\mathbb{R}^N$. At each point p of M, the orientation determines a notion of positive directions for the tangent space T_pM. For smooth points within the interior of M, the orientation behaves similarly to manifolds as discussed previously. At these points, the orientation is defined using charts and transition functions that preserve the orientation in a manner consistent with manifolds. At the corners, there is not a simple notion of tangent space like in smooth points. Hence, the definition of orientation at corners involves considering admissible coordinate systems that respect the corners. We refer to [18, Chapter 7] for precise definition.

Theorem 3.1.3. *The symmetric monoidal 2-category $Bord^{or}_{n,n-1,n-2}$ is generated as a symmetric monoidal 2-category by the following generators:*

objects

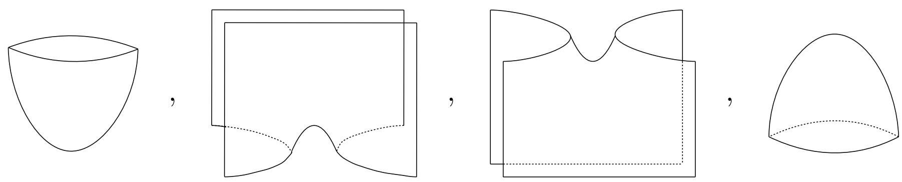
$$\bullet^+ \quad , \quad \bullet^- \tag{3.10}$$

1-bordisms



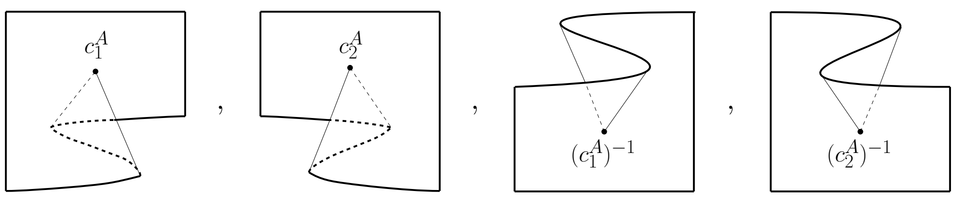
(3.11)

2-bordisms



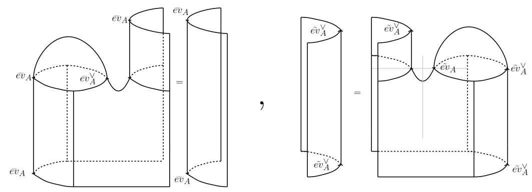
(3.12)

and

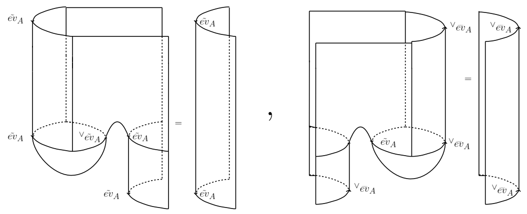


(3.13)

subject to 2*D Morse relations, the swallowtail relations, the cusp flip relations and the cusp inversion relations [28, Figue 13]. The* 2*D Morse relations are*



(3.14)



(3.15)

This means that any 0-dimensional closed oriented manifold is arbitrary disjoint union of points with positive and negative orientations.

Any oriented 1-dimensional manifold with boundary is under horizontal composition of 1-bordisms and disjoint union *generated* by (3.11) together with identity 1-bordisms and the braiding.

Any oriented 2-dimensional closed monifold with boundary with corners is generated by the above 2-bordisms under horizontal and vertical compositions of 2-bordisms and the disjoint union. However, we quotient relations (such as (3.14), (3.15) that correspond to Zorro identities) that are double-counted by the generators under such operations. These generators and relations show that $Bord^{or}_{n,n-1}$ is generated by a fully dualizable object.

3.2 Group Structures

We defined the orientation as a consistent choice of direction for the tangent space at each point on the manifold. We will now reformulate orientation to motivate equipping bordism category with G-structure for a Lie group G.

Definition 3.2.1. *A fiber bundle or locally trivial fibration is a tuple* (E, π, M, F)

1. $\pi : E \to M$ *is a smooth surjection,* M, E *called base and total spaces are manifolds and* F *is a manifold called fiber.*

2. *The bundle is locally trivial; for every point* $m \in M$*, and for any open neighborhood* O *with* $m \in O$*, there exists a diffeomorphism*

$$\begin{aligned} h : & O \times F \to \pi^{-1}(O) \qquad \text{such that} \\ & \pi \circ h(m, f) = m, \end{aligned} \tag{3.16}$$

We can construct an atlas for the total space E from an atlas $\{\phi_\alpha, U_\alpha\}_{\alpha\in\Lambda}$ of M and a family of local trivializations $h_\alpha : O_\alpha \times F \to \pi^{-1}(O_\alpha)$. Abusing the notation, we call $\{h_\alpha, U_\alpha \times F\}_{\alpha\in\Lambda}$ **a bundle atlas** for E.

In particular, fixing $m \in M$, the local trivialization induces a diffeomorphism $h_\alpha(m, .) : F \to \pi^{-1}(m)$. Hence, one may conceptualize fiber bundles as the act of affixing a smoothly varying manifold $\pi^{-1}(m) \cong F$ to each point $m \in M$ to form a new manifold E whose charts are compatible with that of M.

If $F = V$ is taken to be a $\mathbb{k}$-vector space, for $m \in M$, the bundle map induces $h_\alpha(m, .) : V \to \pi^{-1}(m)$ an isomorphism of vector spaces, then we have **a vector bundle** (E, π, M, V). We see that the tangent bundle $\pi : TM \to M$ is a vector bundle over an n-dimensional manifold M with fibers $F = \mathbb{R}^n$ since $T_pM \cong \mathbb{R}^n$ for all $p \in M$ and charts $\{\partial\phi_\alpha, U_\alpha \times \mathbb{R}^n\}_{\alpha\in\Lambda}$.

Definition 3.2.2. *Let (E, π, M, F) be a fiber bundle and $f : N \to M$ be a smooth map. The pullback bundle of (E, π, M, F) along f is a tuple (f^*E, π', N, F) where*

$$f^*E = \{(n, x) \in N \times E | f(n) = \pi(x)\} \tag{3.17}$$

$$\pi' : f^*(E) \to N$$

$$(n, x) \mapsto n \tag{3.18}$$

The transition functions $\psi_{\alpha\beta} =: (U_\alpha \cap U_\beta) \times F \to (U_\beta \cap U_\alpha) \times F$ satisfy the cocycle condition, endowing E with a smooth manifold structure. Thus, (f^*E, π', N, F) is a fiber bundle [21, Thm. 2.1.6].

Definition 3.2.3. *A tuple (P, π, M, G) is called a principal G-bundle if*

1. *G is a Lie group.*
2. *$\pi : P \to M$ is a smooth surjection between manifolds.*
3. *There exists a (right) G-action on the total space, that is, a smooth map $P \times G \to P$, $(p, g) \mapsto p.g$ with $(p.g).h = p.(h.g)$ $\forall g, h \in G$, and $p.e = p$, $\forall p \in P$ such that it is*

- *simply transitive on fibers; $\forall m \in M$ and $\forall x, y \in \pi^{-1}(m)$, $\exists! g \in G$ such that $y.g = x$,*
- *free on fibers; $\forall m \in M$, $\forall x \in \pi^{-1}(m)$ $x.g = x \implies g = e$.*

4. *There exists a G-equivariant bundle atlas $\{h_\alpha, O_\alpha \times G\}_{\alpha\in\Lambda}$, that is, a set of diffeomorphisms*

$$h_\alpha : O_\alpha \times G \to \pi^{-1}(O_\alpha) \text{ such that}$$

$$pr_1 \circ h_\alpha^{-1} = \pi$$

$$h_\alpha(x.g) = h_\alpha(x).g \qquad \forall g \in G,\ \forall x \in O_i. \tag{3.19}$$

where $\{\phi_\alpha, U_\alpha\}$ is an atlas for M.

Equivalently, a principal G-bundle can be defined by projection from the total space to *the orbit space*. The orbit space of a G-action at $p \in P$ is

$$G_p = \{x \in P |\ p.g = x \text{ for some } g \in G\} \tag{3.20}$$

This is an equivalence relation on P. Therefore, for a principal G-bundle (P, π_2, M, G), we can define a quotient of P by the G-action: $\pi_1 : P \to P/G$ by $p \mapsto G_p = [p]$. Since π_1 is surjective, define a map $f : P/G \to M$ by $f \circ \pi_1(p) = \pi_2(p)$ so that the diagram

$$\begin{array}{ccc} P & & \\ {\scriptstyle \pi_1}\downarrow & \searrow^{\pi_2} & \\ P/G & \xrightarrow[f]{} & M \end{array} \tag{3.21}$$

commutes. This is well-defined since π_2 is transitive on the fibers $\pi_2(g.p) = \pi_2(p)$. It is injective since $\pi_2(p) = \pi_2(q)$ implies that p, q are on the same fiber and it is surjective since any $m \in M$, any $p \in \pi_2^{-1}(x)$ is mapped to m. Therefore, f is a bijection. We can now declare f to be a diffeomorphism, inducing a manifold structure on P/G from M.

Definition 3.2.4. *Fix a manifold M and let (P, π, M, G) be a principal G-bundle and let (P', π', M, H) be a principal H-bundle over M. A bundle morphism is a tuple (f, λ) where $f : P \to P'$ is a smooth map, that restricts to identity on M, i.e. $\pi' \circ f = \pi$ and $\lambda : G \to H$ is a Lie group homomorphism such that*

$$f(p.g) = f(p).\lambda(g) \qquad \forall g \in G, \forall p \in P \tag{3.22}$$

If $G \subset H$ is a Lie subgroup and λ is the inclusion, then f is called λ-reduction whereas $H \subset G$ and λ is projection, then f is called λ-lift. If $G = H$, $\lambda = id_G$ and f is a diffeomorphism, then f is called a bundle isomorphism.

Let $(f, id_G) : (P, \pi, M, G) \to (P', \pi', M, G)$ be a bundle morphism between principal G-bundles over M. Then, $f : P \to P'$ is injective since $f(p_1) = f(p_2)$ implies that $\widetilde{\pi} \circ f(p_1) = \widetilde{\pi} \circ f(p_2)$ and $\pi(p_1) = \pi(p_2)$. Since $p_1, p_2 \in \pi^{-1}(m)$ are on the same fiber for some $m \in M$, there exists a unique $g \in G$ with $p_2 = p_1.g$ because the G-action is transitive on the fibers. $f(p_1) = f(p_1.g) = f(p_1).g$ implies $g = e$. Since the G-action is free, $p_1 = p_2$. f is surjective since any point $p \in \pi^{-1}(m)$ is mapped to a point in the fiber $\widetilde{\pi}^{-1}(m)$ and by transitivity, there exists a unique $g \in G$ with $f(p).g = p'$ for any $p' \in P'$. Thus, $f(p.g) = p'$. Therefore, (f, id_G) is a bundle isomorphism.

Let M be a manifold and let G be a Lie Group. **The groupoid of pricincipal G-bundles over** M $Bun_G(M)$ is a category whose objects are principal G-bundles and morphisms are bundle isomorphisms $(f, id_G) : (P, \pi, M, G) \to (P', \pi', M, G)$.

Let (P, π, M, G) be a principal G-bundle and $f : N \to M$ be a smooth map. The pullback bundle (f^*P, π', N, G) is endowed with G-equivariant charts and therefore, it is a principal G-bundle.

Recall from homotopy theory that for any Lie group G, there exists its universal covering space EG, that is a contractible covering space for G. EG is unique up to homotopy and therefore called the universal cover[14, Section 1.3]. The projection $\pi : EG \to BG$ forms a principal G-bundle where $BG = EG/G$ is called **the classifying space**. Furthermore, classifying spaces are *functorial* in the sense that a Lie group homomorphism $f : G \to H$ induces a smooth map $Bf : BG \to BH$. BG and BH realized as manifolds, Bf induces pullback so that the diagram

$$\begin{array}{ccc} EG & \xleftarrow{(Bf)^*} & EH \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ BG & \xrightarrow[Bf]{} & BH \end{array} \tag{3.23}$$

commutes up to homotopy. This means that $(Bf)^* \circ \pi_1 \circ Bf \simeq Bf$ are homotopy equivalent. Two continuous maps $f_0, f_1 : X \to Y$ between topological spaces are called **homotopic** if there is a continuous map $H : X \times [0,1] \to Y$ with $H(.,0) = f_0$ and $H(.,1) = f_1$.

Similarly, for any manifold M, any smooth map $f : M \to BG$ induces a principal G-bundle over M, $\pi_1 : f^*(EG) \to M$ up to homotopy. Moreover, given a Lie group G and a manifold M, homotopy classes of maps $f : M \to BG$ are in bijection with the set of isomorphism classes of principal G-bundles over M; $[M, BG] \cong |Bun_G(M)|$.

$$\begin{array}{ccc} P & \xleftarrow{f^*} & EG \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow[f]{} & BG = EG/G \end{array} \tag{3.24}$$

In particular, taking $G = GL(\mathbb{R}^n)$ and taking a homotopy class of a smooth map $c_M : M \to BGL_n$ defines a principal GL_n bundle over M by the commutative diagram

$$\begin{array}{ccc} P & \xleftarrow{c_M^*} & EGL_n \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ M & \xrightarrow[c_M]{} & BGL_n \end{array} \tag{3.25}$$

Let M be an n-dimensional manifold and consider now the set $\mathcal{B}(T_pM) = \{e = (e_1, \ldots, e_n) |\ e \text{ is an ordered basis for } T_pM\}$. This is linearly isomorphic to the set of invertible linear maps from T_pM to itself and therefore under the isomorphism $T_pM \cong \mathbb{R}^n$, we have an isomorphism of vector spaces $\mathcal{B}(T_pM) \cong GL(\mathbb{R}^n)$. Define **the frame bundle**

$$\mathcal{B}(TM) = \bigsqcup_{p \in M} \mathcal{B}(T_pM) \tag{3.26}$$

$$\pi : \mathcal{B}(TM) \to M$$
$$(p, e) \mapsto p \tag{3.27}$$

$GL(\mathbb{R}^n)$ acts on $\mathcal{B}(TM)$ by

$$e.A = (e_1.A, \ldots e_n.A) \tag{3.28}$$

This action is simply transitive and free on fibers. Furthermore, one can construct a bundle chart for $\mathcal{B}(TM)$. Therefore, $\mathcal{B}(TM)$ is a principal $GL(\mathbb{R}^n)$ bundle over M. Since $Bun_G(M)$ is a groupoid, we have the commutative diagram:

$$\begin{array}{ccccc} B(TM) & \xleftarrow{f^*} & P & \xleftarrow{c_M^*} & EGL_n \\ \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow[f]{} & M & \xrightarrow[c_M]{} & BGL_n \end{array} \tag{3.29}$$

where f is a bundle isomorphism. Given a map $c_M : M \to BGL_n$, **a $GL(\mathbb{R}^n)$-structure** on an n-dimensional manifold M is the bundle $\pi : \mathcal{B}(TM) \to M$ constructed by the above diagram.

Now let $\rho : G \to GL(\mathbb{R}^n)$ be a representation of G on $\mathbb{R}^n$ and let $c_{M,G} : M \to BG$ be a smooth map such that $c_M \simeq c_{M,G} \circ B\rho$ are homotopic maps. Combining the diagrams above, we have a commutative diagram:

$$\begin{array}{ccccccc} B(TM) & \xleftarrow{f^*} & P & \xleftarrow{c_{M,G}^*} & EG & \xleftarrow{(B\rho)^*} & EGL_n \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_3} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} \\ M & \xrightarrow[f]{} & M & \xrightarrow[c_{M,G}]{} & BG & \xrightarrow[B\rho]{} & BGL_n \end{array} \quad (M \xrightarrow{c_M} BGL_n) \tag{3.30}$$

A G-structure on an n-dimensional manifold M is the principal G-bundle $\pi : \mathcal{B}(TM) \to M$ constructed by composition of pullbacks by the above diagram.

Now we will define orientation in this language. First, we note that $GL(\mathbb{R}^n) = \{A \in M_n(\mathbb{R}) | det(A) \neq 0\}$ and the determinant function $det : M_n(\mathbb{R}) \to \mathbb{R}$ is smooth, and $det^{-1}(0)$ spans an $(n-1)$-dimensional hyperplane in $\mathbb{R}^{n^2}$. Thus, $GL(\mathbb{R}^n) = GL^+(\mathbb{R}^n) \sqcup GL^-(\mathbb{R}^n)$ has two diffeomoirphic simply connected components $GL^+(\mathbb{R}^n) \cong GL^-(\mathbb{R}^n)$. The isomorphism is the reflection with respect to this hyperplane. An orientation on an n-dimensional manifold M is a $GL(\mathbb{R}^n)$-structure along the inclusion $i : GL^+(\mathbb{R}^n) \to GL(\mathbb{R}^n)$.

Since $O(n) \simeq GL(\mathbb{R}^n)$ are homotopy equivalent, an orientation can equivalently be defined by the inclusion $i : SO(n) \to O(n)$ and a smooth map $c_M : M \to BO(n)$ together with $c_{M,O(n)} : M \to BSO(n)$ such that $c_M \simeq c_{M,G} \circ Bi$.

3.3 Cobordism Hypothesis

We are now ready to define TQFTs. Dualisablity imposes strong finiteness condition on TQFTs and allows us to prove classification theorems. Exploiting classification result of surfaces, we prove cobordism hypothesis in dimension 2 and give the general cobordism hypothesis heuristically.

Definition 3.3.1. *i) Let $\mathcal{C}$ be a symmetric monoidal category and let G be a Lie group together with a representation $\rho : G \to O(n)$. An n-dimensional closed TQFT with a*

G-structure with target $\mathcal{C}$ is a symmetric monoidal functor

$$Z_G : Bord^G_{n,n-1} \to \mathcal{C} \tag{3.31}$$

ii) The category of TQFTs denoted by $Fun(Bord^G_{n,n-1}, \mathcal{C})$ is a functor category whose objects are symmetric monoidal functors Z_G, morphisms are monoidal natural transformations.

In fact, $Fun(Bord^G_{n,n-1}, \mathcal{C})$ is a groupoid since every monoidal transformation $\eta : F \to T$ for $F, T \in Fun(Bord^G_{n,n-1}, \mathcal{C})$ is invertable because the domain is *rigid*.

Example 3.3.2. *Taking $n = 1$, $\mathcal{C} = Vect_{\Bbbk}$, $G = SO(1)$ with inclusion $i : SO(1) \to O(1)$*

$$\begin{aligned} Fun(Bord^{so}_{1,0}, Vect_{\Bbbk}) &\cong (vect_{\Bbbk})^{\times} \\ Z^{so} &\mapsto Z^{so}(+) \end{aligned} \tag{3.32}$$

where $(vect_{\Bbbk})$ is the category of finite dimensional vector spaces and linear isomorphisms. This equivalence is the result of duality condition of topological field theories that results in finiteness on the algebraic part. 1-dimensional bordism category is generated by two points $\{+, -\}$ and 1-dimensional bordisms are generated by $\widetilde{ev}_+, \widetilde{coev}_+$ such that relations that correspond to Zorro moves satisfy.

In fact, for any symmetric monoidal category $\mathcal{C}$, we can define a category $DuDa(\mathcal{C})$ whose objects are chosen duality data and morphisms are morphisms of $\mathcal{C}$ compatible with the duality data. In other words, a morphism $(A, A^\vee, \widetilde{ev}_A, \widetilde{coev}_A) \to (B, B^\vee, \widetilde{ev}_B, \widetilde{coev}_B)$ is a morphism $\phi : A \to B$ in $\mathcal{C}$, together with $\psi : A^\vee \to B^\vee$. Zorro identities imply that a moprhism (ϕ, ψ) in $DuDa(\mathcal{C})$ is invertible with inverse $(\psi^\vee, \phi^\vee)$ making it a groupoid. Defining $DuDa(\mathcal{C})$ in this way, we choose a duality data for a dualizable object $A \in \mathcal{C}$ and use the property of the duality data to see that every morphism is invertible. We can go the other direction and consider a full subcategory $\mathcal{C}^{fd}$ whose objects are dualisabile objects of $\mathcal{C}$ (without a choice of a duality data) and morphisms are in $\mathcal{C}$. Now define $(\mathcal{C}^{fd})^\times$ as a *maximal subgroupoid* of $\mathcal{C}^{fd}$ where morphisms are isomorphisms in $\mathcal{C}$. It is straightforward to see that there is an equivalence of categories $DuDa(\mathcal{C}) \cong (\mathcal{C}^{fd})^\times$. This corresponds to the 1-dimensional cobordism hypothesis:

Theorem 3.3.3 (1-dimensional cobordism hypothesis)**.**

$$\begin{aligned} Fun(Bord^{so}_{1,0}, \mathcal{C}) &\cong (\mathcal{C}^{fd})^\times \\ Z^{so} &\mapsto Z^{so}(+) \end{aligned} \tag{3.33}$$

Example 3.3.4. *Taking $n = 2$, $G = SO(2)$, $\mathcal{C} = Vect_{\Bbbk}$*

$$Fun(Bord^{so}_{2,1}, Vect_{\Bbbk}) \cong comFrobAlg(Vect_{\Bbbk}) \tag{3.34}$$

$$Z^{so} \mapsto Z^{so}(S^1) \tag{3.35}$$

An example of a class of 2-dimensional closed TQFTs is ***state sum models****. Algebraically, they classify TQFTs that are formed from a semisimple algebra A by taking its* *center:* $Z(S^1) = Z(A)$. *Semisimple algebras can be endowed with a Frobenius structure, and their center form commutative Frobenius algebras.*

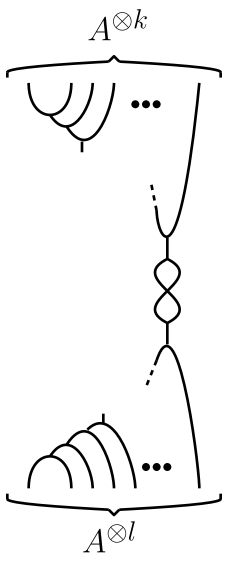
Equivalently, state sum models can be constructed by methods of algebraic topology in four steps:

1. *Let* $M : \Sigma_1 \to \Sigma_2$ *be a bordism in* $Bord^{or}_{2,1}$. *Take an oriented triangulation of* M. *Note that any two triangulations of a manifold are related to each other by arbitrarily many implementation of 2 Pachner moves [23, Theorem 5.9], [26].*

2. *Take the Poincare dual of triangulation, interpret resulting diagrams as string diagrams in* $Vect_{\Bbbk}$ *(or any symmetric monoidal category) where a string represents an object* $A \in Vect_{\Bbbk}$. *The intersection of three lines interpreted as multiplication or comultiplication endows objects* $A \in Vect_{\Bbbk}$ *with algebra and coalgebra structure.*

3. *Imposing triangulation invariance (invariance under Pachner moves) in the interior of* M *in the Poincare dual picture endows* A *with a* Δ*-separable symmetric Frobenius algebra structure* $(A, \mu, \eta, \Delta, \epsilon)$.

4. *The result still depends on how many triangles are on the boundary* Σ_i. *To overcome this problem, we first observe that the Poincare dual of any triangulation of* $S^1 \times [0,1]$ *is of the form* $P_{k,l} : A^{\otimes l} \to A^{\otimes k}$ *for* $k, l \in \mathbb{Z}_+$*:*

$P_{k,l}=$  (3.36)

We notice that the colimit of such maps $colim(P_{k,l}) = im(P_{1,1})$ *which in* $Vect_{\Bbbk}$ *corresponds to taking the center of the* Δ*-separable Frobenius algebra [1, Section 2.7]. Now we declare* $Z(S^1) = Z(A)$. *Morphisms are morphisms of* $ssFrob_{\Bbbk}$.

A special case of state-sum models also corresponds to physically relavant Dijkgraaf-Witten models where algebraic input is a finite group G where the Δ-separable symmetric algebra is taken to be the group algebra $A = \Bbbk[G]$. The center of a group algebra corresponds to *class functions* $G \to \Bbbk$ which are constant on conjugacy classes and composition of two such functions is the *convolution product.* The bijection between class functions with convolution product and the center of a group algebra is well-known (for example see [12]).

It is useful to realize gauge theoretic aspects of Dijkgraaf-Witten models by principal G-bundles. Consider the groupoid $Bun_G(M)$ where objects are principal G-bundles and morphisms are bundle isomorphisms. Under $|Bun_G(M)| \cong [M, BG]$, isomorphism classes of objects of $Bun_G(M)$ can be represented by homotopy classes of continuous maps $\phi : M \to BG$. Note that given a representation of $G \to GL_n$, ϕ corresponds to *a field*; a section of the frame bundle $\pi : B(TM) \to M$.

The embeddings of collars

$$\begin{array}{ccccc} & & M & & \\ & \overset{i_1}{\nearrow} & & \overset{i_2}{\nwarrow} & \\ \Sigma_1 & & & & \Sigma_2 \end{array} \tag{3.37}$$

induce maps by pre-composition

$$\begin{array}{ccccc} & & [Bun_G(M)] & & \\ & \overset{r_1}{\swarrow} & & \overset{r_2}{\searrow} & \\ [Bun_G(\Sigma_1)] & & & & [Bun_G(\Sigma_2)] \end{array} \tag{3.38}$$

on the isomorphism classes of principal G-bundles. Now, consider the set of functions on a principal G-bundle over Σ, $\bar{C}_\Sigma = \{[Bun_G(\Sigma)] \to \Bbbk\}$. This is a $\Bbbk$-vector space under point-wise addition and multiplication with $\Bbbk$. The maps r_1, r_2 induce pullback and pushforward maps

$$\begin{aligned} r_1^* : \bar{C}_{\Sigma_1} &\to \bar{C}_M \\ \xi &\mapsto \xi \circ r_1 \end{aligned} \tag{3.39}$$

$$\begin{aligned} r_{2*} : \bar{C}_M &\to \bar{C}_{\Sigma_2} \\ \xi &\mapsto \left([\phi] \mapsto \sum_{[\Phi] \in r_2^{-1}([\phi])} \frac{|Aut(\phi)|}{|Aut(\Phi)|} \xi(\Phi) \right) \end{aligned} \tag{3.40}$$

which are clearly linear.

Now we can define Dijkgraaf-Witten TQFT as follows:

$$\begin{aligned} Z_G : Bord_{2,1}^{so} &\to Vect_\Bbbk \\ \Sigma_i &\mapsto \bar{C}_{\Sigma_i} = \{[Bun_G(\Sigma_i)] \to \Bbbk\} \\ M &\mapsto r_{2*} \circ {r_1}^* : Z_G(\Sigma_1) \to Z_G(\Sigma_2) \end{aligned} \tag{3.41}$$

To see that Z_G is a functor, let $M_1 : \Sigma_1 \to \Sigma_0$, $M_2 : \Sigma_0 \to \Sigma_2$ be bordisms and define projections $p_i : [Bun_G(M_1 \cup_{\Sigma_0} M_2)] \to [Bun_G(M_i]$ for $i \in \{1, 2\}$ and consider the induced pullback and pushforward maps

$$\begin{array}{ccccccc} & & & \bar{C}_{M_1 \cup_{\Sigma_0} M_2} & & & \\ & & \overset{p_1^*}{\nearrow} & & \overset{p_{2*}}{\searrow} & & \\ & \bar{C}_{M_1} & & & & \bar{C}_{M_2} & \\ \overset{r_1^*}{\nearrow} & & \overset{r_{0*}}{\searrow} & & \overset{r_0^*}{\nearrow} & & \overset{r_{2*}}{\searrow} \\ \bar{C}_{\Sigma_1} & & & \bar{C}_{\Sigma_0} & & & \bar{C}_{\Sigma_2} \end{array} \tag{3.42}$$

The fact that this diagram commutes [12, Section 4], is the result of *locality* of principal G-bundles. The pushforward and pullback maps depend only on the local neighborhood of the boundaries. Since $(r_1 \circ p_1)^* = p_1^* \circ r_1^*$ and $(r_2 \circ p_2)^* = r_{2*} \circ p_{2*}$, Z_G is a functor. It is also symmetric monoidal functor since there are natural isomorphisms $\bar{C}_{\Sigma_1 \sqcup \Sigma_2} \cong \bar{C}_{\Sigma_1} \otimes \bar{C}_{\Sigma_2}$.

The significance of state sum models lies in the simplicity of its algebraic description and its role on connecting algebraic topological, geometric-group theoretic and theoretical physicists' approaches.

Definition 3.3.5. *A 2-dimensional extended TQFT with a G-structure is a symmetric monoidal 2-functor from the symmetric monoidal 2-category of bordisms with G-structure to a chosen symmetric monoidal 2-category $\mathcal{B}$:*

$$Z : Bord^G_{n,n-1,n-2} \to \mathcal{B} \tag{3.43}$$

Let $(ssFrob^2_{\mathbb{k}})^\times$ be *the maximal subgroupoid* of separable symmetric Frobenius algebras in $Alg^2_{\mathbb{k}}$.

Theorem 3.3.6. *2-dimensional fully extended oriented TQFTs with target $Alg^2_{\mathbb{k}}$ are classified by separable symmetric Frobenius algebras. In other words, there is an equivalence of symmetric monoidal 2-categories:*

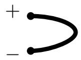
$$\begin{aligned} Fun(Bord^{so}_{2,1,0}, Alg^2_{\mathbb{k}}) &\cong (ssFrob^2_{\mathbb{k}})^\times \\ Z^{so}_* &\mapsto Z^{so}_*(+) \end{aligned} \tag{3.44}$$

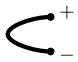
Proof. Only a sketch of the proof is presented here. By [28, Theorem 2.78], it is sufficient to show the equivalence on the generators of $Bord^{so}_{2,1,0}$ and check the relations. Clearly, the generators of $Bord^{so}_{2,1,0}$ and the $2D$ Morse relations enforce $Z^{so}_*(+)$ to be a fully dualizable object in $Alg^2_{\mathbb{k}}$. We showed in Example (2.2.9) that an algebra A is fully dualizable if and only if it is finite dimensional as a vector space and separable. The swallowtail and the cusp inversion relations imply that 1-morphisms must be a *Morita context*, that is, an adjoint equivalence in $Alg^2_{\mathbb{k}}$. The cusp flip relations by [28, Lemma 3.74], endows A with a separable symmetric Frobenius algebra. □

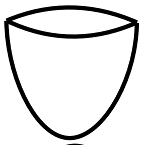
To summarize, combining Example (2.2.9) and the previous theorem, we can compute the image of any bordism of $Bord_{2,1,0}$ by

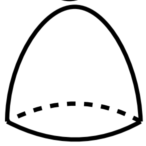

$\longmapsto A$

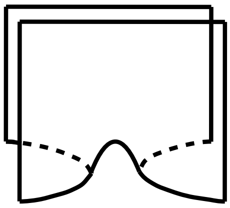

$\longmapsto A^{op}$

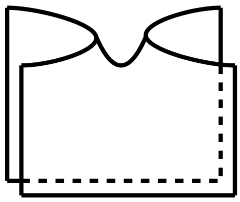

$\longmapsto {}_{A^{e^{op}}}A_k$

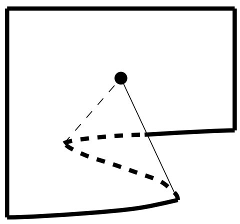

$\longmapsto {}_kA_{A^e}$

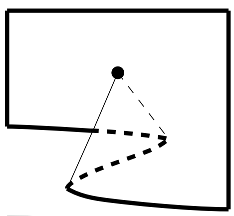

$\longmapsto {}_kk_k \to {}_kA_{A^e} \otimes_{A^e} {}_{A^e}Hom_{A^e}(A, A^e))_k$

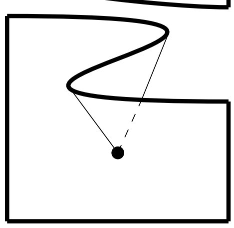

$\longmapsto {}_kA_{A^e} \otimes_{A^e} {}_{A^e}Hom_{A^e}(A, A^e))_k \to {}_kk_k$

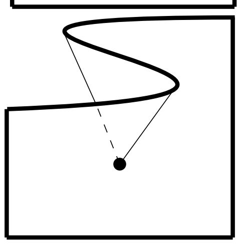

$\longmapsto {}_{A^e}(Hom_{A^e}(A, A^e))_k \otimes_k ({}_kA_{A^e} \to {}_{A^e}A^e_{A^e}$


$\longmapsto {}_{A^e}A^e_{A^e} \to {}_{A^e}(Hom_{A^e}(A, A^e))_k \otimes_k {}_kA_{A^e}$


$\longmapsto ({}_kA_{A^e} \otimes {}_AA_A) \otimes_{A^e \otimes A} ({}_AA_A \otimes {}_{A^{e^{op}}}A_k) \to {}_AA_A$


$\longmapsto ({}_{A^{op}}A^{op} \otimes A_{A^e}) \otimes_{A^{op} \otimes A^e} ({}_{A^{e^{op}}}A_k \otimes {}_{A^{op}}A^{op}_{A^{op}}) \to {}_{A^{op}}A^{op}_{A^{op}}$


$\longmapsto {}_{A^{op}}A^{op}_{A^{op}} \to ({}_{A^{op}}A^{op} \otimes A_{A^e}) \otimes_{A^{op} \otimes A^e} ({}_{A^{e^{op}}}A_k \otimes {}_{A^{op}}A^{op}_{A^{op}})$


$\longmapsto {}_AA_A \to ({}_kA_{A^e} \otimes {}_AA_A) \otimes_{A^e \otimes A} ({}_AA_A \otimes {}_{A^{e^{op}}}A_k)$

As an example, $Z^{so}_*(S^1) = A \otimes_{A^e} A \cong Z(A)$.

$$\begin{aligned} Z^{so}_*(S^2) : \Bbbk &\to A \otimes_{A^e} Hom_{A^e}(A, A^e) \to \Bbbk \\ \lambda &\mapsto \sum_{i=1} \lambda a_i \otimes f_i \mapsto \sum_{i=1} \lambda f_i(a_i) \end{aligned} \tag{3.45}$$

as expected.

The equivalence $DuDa(\mathcal{C}) \cong (\mathcal{C}^{fd})^\times$ extends to symmetric monoidal 2-categories with a detail. We showed that left duality data can be constructed from the braiding and the right duality data. This is sufficient but one needs to make sure that this data is coherent in the sense that cusp isomorphisms c_1^A, c_2^A satisfy the *swallowtail identities*; (see [27, Figure 1-2] for details and further references). Then, we have an equivalence of symmetric monoidal 2-categories

$$\begin{aligned} DuDa^{coh}(\mathcal{B}) &\cong (\mathcal{B}^d)^\times \\ (A, A^\vee, \widetilde{ev}_A, \widetilde{coev}_A, c_1^A, c_2^A) &\mapsto A \end{aligned} \tag{3.46}$$

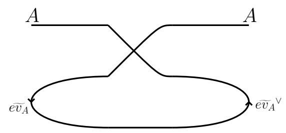
where $(\mathcal{B}^d)^\times$ is *a maximal 2-subgrupoid* of dualizable objects in $\mathcal{B}$.

Next, we extend this equivalence to fully dualizable objects. Note that left and right adjoints of the right evaluation map are not necessarily isomorphic to each other. However, this is the case for TQFT's.

Define a 1-morphism $S_A \in Hom_{\mathcal{C}}(A, A)$, called **the Serre automorphism**;

$$S_A = (1_A \otimes \widetilde{ev}_A) \circ (\beta_{A,A} \otimes 1_{A^\vee}) \circ (1_A \otimes \widetilde{ev}_A^\vee) \tag{3.47}$$

$$S_A = \tag{3.48}$$

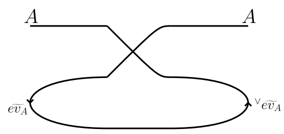


(3.49)

Note that the Serre automorphism is (weakly) invertible with

$$S_A^{-1} = (\widetilde{ev}_A \otimes 1_A) \circ (1_A \otimes 1) \circ (1_A \otimes {}^\vee\widetilde{ev}_A) \tag{3.50}$$

$$S_A^{-1} = \tag{3.51}$$



so that there are natural isomorphisms $\phi : S_A \circ S_A^{-1} \to 1_A$, $\psi : S_A^{-1} \circ S_A \to 1_A$.

The Serre automorphism is said to be **trivializable** if there is an invertible 2-morphism $\lambda_A : S_A \to 1_A$. Then,

$$\begin{aligned} {}^\vee\widetilde{ev}_A &\cong coev_A \cong \widetilde{ev}_A^\vee \\ {}^\vee\widetilde{coev}_A &\cong ev_A \cong \widetilde{coev}_A^\vee \end{aligned} \tag{3.52}$$

See [3, Equation 2.16] for details and further references. *Trivialization of the Serre automorphism* corresponds to *homotopy fixed points* of $SO(2)$ action [15, Corollary 4.10]. Following the remarks in [27], there is an equivalence

$$FuDuDa^{coh}(\mathcal{B}) \cong (\mathcal{B}^{fd})^{\times}$$
$$(A, \mathcal{A}^{\vee}, \widetilde{ev}_A, \widetilde{coev}_A, c_A^1, c_A^2, ev_{\widetilde{ev}_A}, coev_{\widetilde{ev}_A}, \widetilde{ev}_{\widetilde{ev}_a}, \widetilde{coev}_{\widetilde{ev}_A}, S_A, S_A^{-1}, \phi, \psi) \mapsto A \tag{3.53}$$

where $FuDuDa^{coh}(\mathcal{B})$ stands for *coherent fully dualizable objects* in $\mathcal{B}$. A fully dualizable object $A \in \mathcal{B}$ is said to be coherent if the Serre automorphism is trivializable.

Theorem 3.3.7. *There is an equivalence of symmetric monoidal 2-categories:*

$$Fun(Bord^G_{2,1,0}, \mathcal{B}) \cong (\mathcal{B}^{fd})^{\times}$$
$$Z \mapsto Z(+) \tag{3.54}$$

Proof. [28, section 3.6]. □

This result works even more generally. We will now give a sketch of the cobordism hypothesis. Having discussed once extended bordism category $Bord_{n,n-1,n-2}$, we can ask if we can further extend this category. There are two directions to go. We can go lower in dimension; this leads us to the fully extended bordism category. Objects of the fully extended bordism category are 0-dimensional compact manifolds, 1-bordisms are 1-dimensional manifolds with boundary between 0-dimensional points, 2-bordisms are manifolds with boundary with corners realizing 1-bordisms as boundary, 3-bordisms are manifolds with corners between 2-bordisms and so on till n-bordisms between $(n-1)$-bordisms. As we have seen that the once extended bordism category has 2 compositions; horizontal and vertical besides the symmetric monoidal structure, fully extended n-dimensional bordism category has n compositions. Compositions have to satisfy associativity and unit properties. A k-bordism for $0 \leq k \leq n$, has to preserve boundary and corners of lower dimensions. Furthermore, coherence conditions must be implemented for each k-bordism for lower dimensional bordisms. This data gets exponentially more complicated when the dimension of n increases.

It is also useful to go higher in dimension. For n-bordisms X_1, X_2 in $Bord_{n,\ldots,0}$, an $(n+1)$-bordism is a diffeomorphism of manifolds with boundary with corners $\phi : X_1 \to X_2$. For two such differomorphisms ϕ_0, ϕ_1, an $(n+2)$-bordism is a homotopy equivalence. $(n+3)$-bordisms are homotopy equivalences between homotopies and this can be iterated all the way to infinity. The diffeomorphisms and homotopies must be compatible with all lower dimensional data. In this way, one can construct an (∞, n)-category. Symmetric monoidal structure is similarly disjoint union compatible with the (∞, n)-structure. A symmetric monoidal (∞, n)-functor between (∞, n)-categories is a symmetric monoidal functor that preserves (∞, n)-structures. For a detailed construction, we refer to [25].

Now, we are ready to define fully extended field theories.

Definition 3.3.8. *Let $\mathcal{C}$ be an (∞, n)-category. An n-dimensional fully extended topological field theory with target $\mathcal{C}$ is a symmetric monoidal functor*

$$F : Bord_{\infty,n} \to \mathcal{C} \tag{3.55}$$

We defined dual objects in a 1-category as an object with evaluation and coevaluation maps satisfying the so-called Zorro identities. In the 2-categorical setting, fully dualizable objects are also defined similarly with an object together with evaluation and coevaluation 1-morphisms but the Zorro identities are replaced with invertible 2-morphisms and evaluation and coevaluation 1-morhisms are required to have adjoints with Zorro identities.

In the (∞, n)-setting, this argument can be iterated. An object is fully dualizable if there exists a dual object with evaluation and coevaluation 1-morphisms and k-morphisms of evaluation and coevaluation maps have adjoint k-morphisms such that there exists invertable $(k+1)$-morphisms for Zorro equivalences.

Theorem 3.3.9. *(Cobordism hypothesis [25]) Let $\mathcal{C}$ be a symmetric monoidal (∞, n) category. There is an equivalence between (∞, n)-categories.*

$$Fun(Bord^G_{\infty,n}, \mathcal{C}) \cong [(\mathcal{C}^{fd})^{\times}]^{hG} \tag{3.56}$$

where $(\mathcal{C}^{fd})^{\times}]^{hG}$ denotes *the homotopy fixed points of maximal (∞, n)-subgroupoid of fully dualizable objects in $\mathcal{C}$.*

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