universität
wien

MASTERARBEIT / MASTER'S THESIS

Titel der Masterarbeit / Title of the Master's Thesis

The Birkhoff Factorization and its Applications to a Jet Determination Result for CR-Mappings

verfasst von / submitted by

Stephan Schneider, BSc

angestrebter akademischer Grad / in partial fulfilment of the requirements for the degree of

Master of Science (MSc)

Wien, 2023 / Vienna, 2023

Studienkennzahl lt. Studienblatt / degree programme code as it appears on the student record sheet:	UA 066 821
Studienrichtung lt. Studienblatt / degree programme as it appears on the student record sheet:	Masterstudium Mathematik
Betreut von / Supervisor:	Univ.-Prof. Bernhard Lamel, PhD

Zusammenfassung

Wir geben eine Einführung in die Methoden der Arbeit von Bertrand, Della Sala und Lamel [4] über ein Jet Determination Resultat für CR-Diffeomorphismen von Hyperflächen im $n-$dimensionalen komplexen Raum. Das benötigte Hintergrundwissen wird erarbeitet, dies beinhaltet die Birkhoff Faktorisierung und Resultate zu Riemann-Hilbert Problemen. Anschließend wird das Hauptresultat durch Konstruktion sogenannter k_0-stationärer Kreisscheiben bewiesen.

Abstract

We give an introduction to the methods used in the work of Bertrand, Della Sala and Lamel [4] to prove a jet determination result for CR-diffeomorphisms of hypersurfaces in $n-$dimensional complex space. After developing the required background material which includes an account of the Birkhoff factorization and results concerning Riemann-Hilbert problems we construct so called k_0-stationary disks to prove the main result.

Acknowledgements

I would like to express my gratitude to my supervisor Bernhard Lamel for his guidance and insight throughout the process of completing this thesis. Also, I want to thank my friends and family for their support in all matters outside of mathematics. Lastly, I thank Florian Fürnsinn, Florian Lang and Markus Reibnegger for helping me process the material by tolerating me talking to them about it at length.

Contents

1 Introduction

1.1 Jet Determination

A finite jet determination property is a result in the following setting:

- We have "spaces" X, Y that allow us to to talk about "differentiable" maps between them, e.g. open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$, (finitely) smooth (sub-)manifolds, (formal) varieties, etc.

- We consider a class of mappings $f : X \to Y$ that allow us to evaluate "derivatives" $f(p), f'(p), f''(p), \ldots$ at some point $p \in X$, e.g. (finitely) smooth functions, holomorphic functions, germs of the former, formal power series, etc.

Such a result can then be stated as: If $f, g : X \to Y$ are in the class of mappings and satisfy $f(p) = g(p), f'(p) = g'(p), \ldots, f^{(\ell)}(p) = g^{(\ell)}(p)$ at a point $p \in X$ for some integer ℓ, then $f = g$.
Writing $j_p^\ell(f) = (f(p), f'(p), \ldots, f^{(\ell)}(p))$ for the $\ell-$jet map at p, we can rephrase the result as injectivity of j_p^ℓ on the class of mappings.
To give a concrete example of such a theorem, consider:

Theorem 1 (Cartan)**.** *Let $U \subset \mathbb{C}^n$ be a bounded, open, connected subset and let $H : U \to U$ be a holomorphic function satisfying $H(p) = p$ and $H'(p) = \mathrm{id}$ for some $p \in U$. Then we have $H = \mathrm{id}$ on the entirety of U.*

Our setting will be that of certain families of hypersurfaces $S \subset \mathbb{C}^{n+1}$ and for mappings we take germs of finitely smooth CR-diffeomorphisms $H : S \to S$. More specifically, our result applies to perturbations of a polynomial hypersurface satsifying a Levi-nondegeneracy condition.

1.2 Overview

Chapter 3 is devoted to a proof of the jet determination result which will proceed as follows: First we construct k_0-stationary disks attached to the hypersurface S, which are a family of analytic functions on the unit disk $f : \{z \in \mathbb{C} : |z| < 1\} \to \mathbb{C}^{n+1}$ with $\mathcal{C}^{k,\alpha}$ boundary values that lie in S. Using an approximation result for CR functions called the Baouendi-Treves theorem we can show that these disks are invariant under CR-diffeomorphisms. For the polynomial hypersurface, we can give an explicit construction of a k_0-stationary disk attached to it. We will formulate a nonlinear boundary problem whose solutions are the k_0-stationary disks and transform it into a linear Riemann-Hilbert problem by means of the implicit function theorem. This allows us to get disks for the perturbed hypersurface S from the disk attached to the polynomial hypersurface. We can then use the invariance of the disks to reduce the jet determination result for CR-diffeomorphisms to a jet determination result for analytic disks. Using the Birkhoff factorization this boils down to the finite jet determination of polynomial maps.

Chapter 2 goes over some of the prerequisites to understand the material in Chapter 3. First we introduce the various Hölder spaces that will be in use. Then we give an account of the Birkhoff factorization which will make up the bulk of the chapter. This starts with the solution theory of the Hilbert problem on the unit disk. The Birkhoff factorization is then obtained from the fundamental solutions of this problem. Afterwards we apply it to the study of the linear Riemann-Hilbert problems that arise in Chapter 3 and we finish the chapter with a proof of the Baouendi-Treves theorem.

2 Background

Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in $\mathbb{C}$ and $b\Delta := \{z \in \mathbb{C} : |z| = 1\}$ its boundary.

2.1 Function Spaces

For an integer $k \geq 0$ and $0 < \alpha < 1$ let $\mathcal{C}^{k,\alpha} = \mathcal{C}^{k,\alpha}(b\Delta, \mathbb{R})$ be the Banach space of real-valued functions on $b\Delta$ of class $\mathcal{C}^{k,\alpha}$ equipped with the usual norm

$$\|v\|_{\mathcal{C}^{k,\alpha}} = \sum_{j=0}^{k} \left\|v^{(j)}\right\|_{\infty} + \sup_{\zeta \neq \eta \in b\Delta} \frac{\left|v^{(k)}(\zeta) - v^{(k)}(\eta)\right|}{|\zeta - \eta|^{\alpha}}$$

where $\left\|v^{(j)}\right\|_{\infty} = \max_{b\Delta} \left|v^{(j)}\right|$. We also need the following related spaces:

- $\mathcal{C}_e^{k,\alpha}$ resp. $\mathcal{C}_o^{k,\alpha}$, the closed subspaces of even resp. odd functions, i.e. $\mathcal{C}_e^{k,\alpha} := \{v \in \mathcal{C}^{k,\alpha} : v(-\zeta) = v(\zeta)\ \forall \zeta \in b\Delta\}$ and $\mathcal{C}_o^{k,\alpha} := \{v \in \mathcal{C}^{k,\alpha} : v(-\zeta) = -v(\zeta)\ \forall \zeta \in b\Delta\}$.

- $\mathcal{C}_{\mathbb{C}}^{k,\alpha} = \mathcal{C}^{k,\alpha} + i\mathcal{C}^{k,\alpha}$, hence $v \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$ if and only if $\operatorname{Re} v, \operatorname{Im} v \in \mathcal{C}^{k,\alpha}$.

- $\mathcal{A}^{k,\alpha} \subset \mathcal{C}_{\mathbb{C}}^{k,\alpha}$ the subspace of continuous functions $f : \overline{\Delta} \to \mathbb{C}$, holomorphic on Δ such that $f|_{b\Delta} \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$

- $\mathcal{C}_{0^m}^{k,\alpha} = ((1-\zeta)^m \mathcal{C}_{\mathbb{C}}^{k,\alpha}) \cap \mathcal{C}^{k,\alpha}$ and $\mathcal{A}_{0^m}^{k,\alpha} = (1-\zeta)^m \mathcal{A}^{k,\alpha}$ for integers $m \geq 0$, where $(1-\zeta)^m$ is the multiplication operator with $\zeta \mapsto (1-\zeta)^m$. They are not closed subspaces with the $\mathcal{C}^{k,\alpha}$ norm, instead we equip them with the norm $\|(1-\zeta)^m f\|_X := \|f\|_{\mathcal{C}^{k,\alpha}}$, $X \in \{\mathcal{C}_{0^m}^{k,\alpha}, \mathcal{A}_{0^m}^{k,\alpha}\}$ which turns them into Banach spaces.

- $\mathcal{R}_m := \{v \in \mathcal{C}_{\mathbb{C}}^{k,\alpha} : v(\zeta) = (-1)^m \zeta^{-m} \overline{v(\zeta)}\ \zeta \in b\Delta\}$ a subspace of $\mathcal{C}_{\mathbb{C}}^{k,\alpha}$ we need for technical reasons.

The connection of $\mathcal{R}_m$ with $\mathcal{C}_{0^m}^{k,\alpha}$ is given by the following lemma from [3]:

Lemma 1. *1. The map $\tau_m : \mathcal{C}_{0^m}^{k,\alpha} \to \mathcal{C}_{\mathbb{C}}^{k,\alpha}$ defined by $\tau_m((1-\zeta)^m v) = v$ is an isomorphism between $\mathcal{C}_{0^m}^{k,\alpha}$ and $\mathcal{R}_m$;*

2. if $m = 2m'$ is even, the map $v \mapsto \zeta^{m'} v$ induces an isomorphism between $\mathcal{R}_m$ and $\mathcal{R}_0 = \mathcal{C}^{k,\alpha}$;

3. *if $m = 2m' + 1$ is odd, the map $v \mapsto \zeta^{m'} v$ induces an isomorphism between $\mathcal{R}_m$ and $\mathcal{R}_1$.*

Furthermore, if m is odd the map $v(\zeta) \mapsto i\zeta^m v(\zeta^2)$ sends $\mathcal{R}_m$ isomorphically to $\mathcal{C}_o^{k,\alpha}$

Proof. 1. A function $v \in \mathcal{C}_{\mathbb{C}}^{k,\alpha}$ is in the image of τ_m if and only if $(1-\zeta)^m v \in \mathcal{C}^{k,\alpha}$, that is,

$$(1-\zeta)^m v = (1-\overline{\zeta})^m \overline{v} = (1-\zeta)^m (-1)^m \zeta^{-m} \overline{v},$$

establishing bijectivity. τ_m and its inverse are continuous because, per definition, it is an isometry.
2. Let $m = 2m'$ be even, $v \in \mathcal{R}_m, u = \zeta^{m'} v$, then

$$u = \zeta^{m'} v = \zeta^{m'} \zeta^{-2m'} \overline{v} = \zeta^{-m'} \overline{v} = \overline{u}$$

showing $u \in \mathcal{C}^{k,\alpha}$. We can multiply everything with ζ^{-m}, then the first and last equality show surjectivity.
3. Let $m = 2m' + 1$ be odd, $v \in \mathcal{R}_m, u = \zeta^{m'}$, then

$$u = \zeta^{m'} v = -\zeta^{m'} \zeta^{-2m'-1} \overline{v} = -\overline{\zeta} \zeta^{-m'} \overline{v} = -\overline{\zeta u}$$

so $u \in \mathcal{R}_1$. Again, surjectivity follows by multiplying with $\zeta^{-m'}$. For the last claim, let $v \in \mathcal{R}_m$ with m odd and set $u(\zeta) = i\zeta^m v(\zeta^2)$. Then

$$u(\zeta) = i\zeta^m v(\zeta^2) = -i\zeta^m \zeta^{-2m} \overline{v(\zeta^2)} = -i\zeta^{-m} \overline{v(\zeta^2)} = \overline{u(\zeta)}$$

and $u(-\zeta) = (-1)^m u(\zeta) = -u(\zeta)$ shows that $u \in \mathcal{C}_o^{k,\alpha}$. To show that this correspondence is an isomorphism, we give an inverse in terms of the Fourier coefficients. Write $u \in \mathcal{C}_o^{k,\alpha}$ as

$$u(\zeta) = \sum_{j \in \mathbb{Z}} a_j \zeta^{2j+1}$$

where $\overline{a_j} = a_{-j-1}$ for $j \in \mathbb{Z}$. Now define

$$v(\zeta) = -\sum_{l \in \mathbb{Z}} i a_{l+(m-1)/2} \zeta^l.$$

Since m is odd,

$$\frac{u(\zeta)}{i\zeta^m} = -\sum_{j \in \mathbb{Z}} i a_j \zeta^{2j-m+1} = -\sum_{l \in \mathbb{Z}} i a_{l+(m-1)/2} \zeta^{2l} = v(\zeta^2),$$

i.e., $u(\zeta) = i\zeta^m v(\zeta^2)$. Lastly, we show that $v \in \mathcal{R}_m$:

$$\begin{aligned}
-\zeta^{-m}\overline{v(\zeta)} = -\sum_{l \in \mathbb{Z}} i a_{l+(m-1)/2} \zeta^{-l-m} &= -\sum_{h \in \mathbb{Z}} i \overline{a}_{-h-(m+1)/2} \zeta^h \\
&= -\sum_{h \in \mathbb{Z}} i a_{h+(m-1)/2} \zeta^h = v(\zeta)
\end{aligned}$$

□

2.2 Birkhoff Factorization

Let $N > 0$ be an integer and $G : b\Delta \to GL_N(\mathbb{C})$ a $\mathcal{C}^{k,\alpha}$ map into the general linear group over $\mathbb{C}^N$. A Birkhoff factorization of $-\overline{G^{-1}}G$ is given by $\mathcal{C}^{k,\alpha}$ maps $B^+ : \overline{\Delta} \to GL_N(\mathbb{C})$ and $B^- : (\mathbb{C} \cup \infty) \setminus \Delta \to GL_N(\mathbb{C})$, holomorphic on Δ and $\mathbb{C} \setminus \overline{\Delta}$ respectively with

$$-\overline{G(\zeta)^{-1}}G(\zeta) = B^+(\zeta) \begin{pmatrix} \zeta^{\kappa_1} & & & (0) \\ & \zeta^{\kappa_2} & & \\ & & \ddots & \\ (0) & & & \zeta^{\kappa_N} \end{pmatrix} B^-(\zeta) \quad \forall \zeta \in b\Delta.$$

The integers $\kappa_1, \ldots, \kappa_N$ are called the partial indices of $-\overline{G^{-1}}G$ and their sum $\kappa := \sum_{j=1}^N \kappa_j$ is called the Maslov index of $-\overline{G^{-1}}G$. The Maslov index is equal to the winding number of the map $\zeta \mapsto \det\left(-\overline{G(\zeta)^{-1}}G(\zeta)\right)$ around the origin and is thus even. It is also possible to find a $\mathcal{C}^{k,\alpha}$ map $\Theta : \overline{\Delta} \to GL_N(\mathbb{C})$, holomorphic on Δ, such that $B^+ := \Theta$ and $B^- := \overline{\Theta^{-1}}$.
The proof of this result requires a bit of work and is not essential to understanding the rest of the material here, it is included mainly for the sake of completeness.
As the main ingredient of the proof we will need so called fundamental systems of homogeneous Hilbert problems, we construct these following the exposition by Vekua in [6]. We start with some terminology. Abbreviate $\Delta^- := (\mathbb{C} \cup \{\infty\}) \setminus \overline{\Delta}$. A sectionally holomorphic function φ is a function $\varphi : (\mathbb{C} \cup \{\infty\}) \setminus b\Delta \to \mathbb{C}$ which

1. is holomorphic except possibly at ∞ and
2. has one sided limits on $b\Delta$, i.e., the limits

$$\varphi^+(\zeta) = \lim_{\substack{z \to \zeta \\ z \in \Delta}} \varphi(z) \quad \varphi^-(\zeta) = \lim_{\substack{z \to \zeta \\ z \in \Delta^-}} \varphi(z)$$

exist for all $\zeta \in b\Delta$. Note that in this case φ^+ and φ^- provide continuous extensions of $\varphi|_\Delta$ and $\varphi|_{\Delta^-}$, respectively, to the boundary $b\Delta$.

A sectionally holomorphic function φ has finite degree at infinity if $\varphi(z)/|z|^m \to 0$ as $z \to \infty$ for some $m > 0$. In this case we have the following expansion for sufficiently large z :

$$\varphi(z) = a_k z^k + a_{k-1} z^{k-1} + \ldots$$

for some integer k, where we impose $a_k \neq 0$ so φ cannot vanish identically in a neighborhood of infinity. For $k > 0$, φ has a pole of order k at infinity, for $k < 0$ φ has a zero of order $|k|$ at infinity. For $k = 0$ $\varphi(z)$ has a finite non-zero value at infinity in which case we will sometimes say that φ has a order zero pole (or zero) at infinity.
If $\varphi(z) = \gamma(z) + O(1/z)$ in a neighborhood of infinity for some polynomial γ, then γ will be called the principal part of φ at infinity. For negative degrees, $k < 0$, $\gamma = 0$.

We will call $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ a sectionally holomorphic vector if all its components $\varphi_1, \ldots, \varphi_N$ are sectionally holomorphic. If all the components of φ have finite degree at infinity, then we say that $\boldsymbol{\varphi}$ has finite degree at infinity and we set its degree k as the maximum degree of its components. If the components $\varphi_1, \ldots, \varphi_N$ have principal parts $\gamma_1, \ldots, \gamma_N$, respectively, at infinity then $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N)$ will be called the principal part of φ at infinity. Again, if the degree k is negative, $k < 0$, then $\boldsymbol{\gamma} = 0$.

2.2.1 The Sokhotski-Plemelj Formula

Let $v \in \mathcal{C}^{k,\alpha}$, then

$$\varphi(z) = \frac{1}{2\pi i} \int_{b\Delta} \frac{v(\zeta)}{\zeta - z} d\zeta$$

defines a holomorphic function on $\Delta \cup \Delta^-$ that vanishes at ∞. The Sokhotski-Plemelj formulae give the one sided limits:

$$\varphi^+(\zeta) = \frac{1}{2} v(\zeta) + \frac{1}{2\pi i} \int_{b\Delta} \frac{v(\xi)}{\xi - \zeta} d\xi \quad (\zeta \in b\Delta) \tag{SP$^+$}$$

$$\varphi^-(\zeta) = -\frac{1}{2} v(\zeta) + \frac{1}{2\pi i} \int_{b\Delta} \frac{v(\xi)}{\xi - \zeta} d\xi \quad (\zeta \in b\Delta) \tag{SP$^-$}$$

where the integrals are understood as Cauchy principal value integrals. Thus, φ is a sectionally holomorphic function vanishing at ∞.

Proof. Set $\varepsilon(z) := (1 - |z|)^\beta$ for $z \in \Delta$, where $\beta \in (0, 1)$ will be determined later. Define the contours $C_z := B_{\varepsilon(z)}(z/|z|) \cap b\Delta$ and $L_z := b\Delta \setminus C_z$, and split the integral for φ as

$$\varphi(z) = \frac{1}{2\pi i} \int_{C_z} \frac{v(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{L_z} \frac{v(\zeta)}{\zeta - z} d\zeta. \tag{2.1}$$

For the first integral, we estimate

$$\begin{aligned} \left| \int_{C_z} \frac{v(\zeta) - v(\frac{z}{|z|})}{\zeta - z} d\zeta \right| &\lesssim \frac{1}{1 - |z|} \int_{C_z} \left| \zeta - \frac{z}{|z|} \right|^\alpha |d\zeta| = \frac{1}{1 - |z|} \int_{-2 \arcsin \varepsilon(z)}^{2 \arcsin \varepsilon(z)} \left| e^{i\theta} - 1 \right|^\alpha d\theta = \\ &= \frac{1}{1 - |z|} \int_{-2 \arcsin \varepsilon(z)}^{2 \arcsin \varepsilon(z)} \left| 2 \sin\left(\frac{\theta}{2}\right) \right|^\alpha d\theta \le \frac{1}{1 - |z|} \int_{-2 \arcsin \varepsilon(z)}^{2 \arcsin \varepsilon(z)} |\theta|^\alpha d\theta \lesssim \\ &\lesssim \frac{\arcsin \varepsilon(z)^{\alpha+1}}{1 - |z|} \lesssim \frac{\varepsilon(z)^{\alpha+1}}{1 - |z|} \to 0 \end{aligned}$$

as $|z| \to 1$, provided we choose $\beta \in (1/(1 + \alpha), 1)$. Consequently, the first integral can be replaced by the following easier to evaluate expression when taking the limit:

$$\int_{C_z} \frac{v(\frac{z}{|z|})}{\zeta - z} d\zeta.$$

Denote by a_z and b_Z the starting point and the endpoint of the arc C_z respectively. Writing $[z_1, z_2]$ for the path $[0,1] \ni t \mapsto (1-t)z_1 + tz_2$ we can proceed using Cauchy's integral theorem

$$\int_{C_z} \frac{1}{\zeta - z} d\zeta = \underbrace{\int_{C_z + [b_z, 0] + [0, a_z]} \frac{1}{\zeta - z} d\zeta}_{=2\pi i} + \int_{[0,b_z]} \frac{1}{\zeta - z} d\zeta - \int_{[0,a_z]} \frac{1}{\zeta - z} d\zeta.$$

Thus, we need to show that the difference of the last two integrals tends to $-i\pi$. We use coordinates such that $b_z = e^{i\theta}, z = r, a_z = e^{-i\theta}$:

$$\int_{[0,a_z]} \frac{1}{\zeta - z} d\zeta - \int_{[0,b_z]} \frac{1}{\zeta - z} d\zeta = \int_0^1 \frac{1}{e^{-i\theta} t - r} e^{-i\theta} dt - \int_0^1 \frac{1}{e^{i\theta} t - r} e^{i\theta} dt$$
$$= \int_0^1 \frac{2ir\sin(\theta)}{t^2 - 2r\cos(\theta)t + r^2} dt = 2i \arctan\left(\frac{1 - r\cos(\theta)}{r\sin(\theta)}\right) + i\pi - i\pi\theta.$$

Since $\theta = 2\arcsin(\varepsilon(z)/2) \simeq \varepsilon(z)$ goes to 0 as $z \to 1$ we only need to find the limit of the expression inside the arctangent. $\theta/\sin(\theta) \to 1$ as $z \to 1$ this simplifies further to finding the limit of $(1 - r\cos(\theta))/(r\theta)$. To this end, we calculate

$$\frac{1 - r\cos(\theta)}{r\theta} = \frac{1 - r + r\theta^2/2 + O(\theta^4)}{r\theta} = \frac{1-r}{r\theta} + \theta/2 + O(\theta^3) \simeq \frac{1-r}{(1-r)^\beta} \to 0,$$

as $z \to 1$. We have now dealt with the integral over C_z in (2.1). For the integral over L_z, we estimate

$$\left| \int_{L_z} v(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z/|z|} \right) d\zeta \right| \lesssim \int_{L_z} \frac{|z - z/|z||}{|\zeta - z||\zeta - z/|z||} |d\zeta|$$
$$\lesssim |1 - |z|| \int_{L_z} \frac{1}{|\zeta - z/|z||^2} |d\zeta| \lesssim \varepsilon(z)^{1/\beta} \int_{\varepsilon(z)}^{\pi} \frac{1}{\theta^2} d\theta \lesssim \varepsilon(z)^{1/\beta - 1} \xrightarrow{|z| \to 1} 0,$$

and

$$\left| \int_{L_z} \frac{v(\zeta)}{\zeta - z/|z|} d\zeta - \int_{b\Delta} \frac{v(\zeta)}{\zeta - z/|z|} d\zeta \right| = \lim_{r \to 0} \left| \int_{A_{r,z}} \frac{v(\zeta)}{\zeta - z/|z|} d\zeta \right|$$
$$= \lim_{r \to 0} \left| \int_{A_{r,z}} \frac{v(\zeta) - v(z/|z|)}{\zeta - z/|z|} + \underbrace{\int_{A_{r,z}} \frac{v(z/|z|)}{\zeta - z/|z|} d\zeta}_{=0} \right| \lesssim \lim_{r \to 0} \int_{A_{r,z}} |\zeta - z/|z||^{\alpha - 1} |d\zeta|$$
$$\lesssim \varepsilon(z)^\alpha \xrightarrow{|z| \to 1} 0,$$

here $A_{r,z}$ denotes $C_z \setminus B_r(z/|z|)$. Combining these estimates, we see that

$$\lim_{z \to \zeta_0} \int_{L_z} \frac{v(\zeta)}{\zeta - z} d\zeta = \lim_{z \to \zeta_0} \int_{b\Delta} \frac{v(\zeta)}{\zeta - z/|z|} d\zeta$$

and the formula follows from the continuity in ζ of the principal value integral in (SP$^+$). In fact, we actually have Hölder continuity here, which we will prove as a separate result. $\square$

Theorem 2 (Plemelj-Privalov). *Let $v \in \mathcal{C}^{k,\alpha}$, the one-sided limits $\varphi^{\pm} : b\Delta \to \mathbb{C}$ of the sectionally holomorphic function*

$$\varphi(z) = \frac{1}{2\pi i} \int_{b\Delta} \frac{v(\zeta)}{\zeta - z} d\zeta$$

defined on $\Delta \cup \Delta^-$ are themselves Hölder continuous, i.e. $\varphi^{\pm} \in \mathcal{C}^{k,\alpha}$. Moreover, the mapping $v \mapsto \varphi^{\pm}$ is continuous as a map from $\mathcal{C}^{k,\alpha}$ to itself.

Proof. Case 1: $v \in \mathcal{C}^{\alpha}$
As a first step we will manipulate the principal value integral into an actual integral:

$$\int_{b\Delta} \frac{v(\xi)}{\xi - \zeta} d\xi = \lim_{r \to 0} \int_{b\Delta \setminus B_r(\zeta)} \frac{v(\xi)}{\xi - \zeta} d\xi = \lim_{r \to 0} \int_{b\Delta \setminus B_r(\zeta)} \frac{v(\xi) - v(\zeta)}{\xi - \zeta} d\xi + \int_{b\Delta \setminus B_r(\zeta)} \frac{v(\zeta)}{\xi - \zeta} d\xi =$$
$$= \int_{b\Delta} \frac{v(\xi) - v(\zeta)}{\xi - \zeta} d\xi + i\pi v(\zeta)$$

Here, we used dominated convergence for the first integral and compute the second as in the previous proof. Thus we need to analyze integrals over "difference quotients". We split the domain of integration:

$$\left| \int_{b\Delta} \frac{v(\xi) - v(\zeta)}{\xi - \zeta} d\xi - \int_{b\Delta} \frac{v(\xi) - v(\zeta_0)}{\xi - \zeta_0} d\xi \right| \le \left| \int_{B_r(\zeta_0)} \frac{v(\xi) - v(\zeta)}{\xi - \zeta} d\xi - \int_{B_r(\zeta_0)} \frac{v(\xi) - v(\zeta_0)}{\xi - \zeta_0} d\xi \right| +$$
$$+ \left| \int_{b\Delta \setminus B_r(\zeta_0)} \frac{v(\xi) - v(\zeta)}{\xi - \zeta} d\xi - \int_{b\Delta \setminus B_r(\zeta_0)} \frac{v(\xi) - v(\zeta_0)}{\xi - \zeta_0} d\xi \right|$$

where $r = 2|\zeta - \zeta_0|$.
For the first summand we use the estimate

$$\left| \int_{B_r(\zeta_0)} \frac{v(\xi) - v(\zeta)}{\xi - \zeta} d\xi \right| \le \|v\|_{\mathcal{C}^{\alpha}} \int_{B_r(\zeta_0)} |\xi - \zeta|^{\alpha - 1} |d\xi| \le 6 \|v\|_{\mathcal{C}^{\alpha}} |\zeta - \zeta_0|^{\alpha}$$

which also holds for the other integral over $B_r(\zeta_0)$ and the triangle inequality to bound it by $12\|v\|_{\mathcal{C}^{\alpha}}|\zeta - \zeta_0|$.
We split the second summand further:

$$\left| \int_{b\Delta \setminus B_r(\zeta_0)} \frac{v(\xi) - v(\zeta)}{\xi - \zeta} - \frac{v(\xi) - v(\zeta_0)}{\xi - \zeta_0} d\xi \right| \le \left| \int_{b\Delta \setminus B_r(\zeta_0)} \frac{v(\xi) - v(\zeta)}{\xi - \zeta_0} - \frac{v(\xi) - v(\zeta_0)}{\xi - \zeta_0} d\xi \right| +$$
$$+ \left| \int_{b\Delta \setminus B_r(\zeta_0)} (v(\xi) - v(\zeta)) \left(\frac{1}{\xi - \zeta} - \frac{1}{\xi - \zeta_0} \right) d\xi \right|.$$

Here we can evaluate as before

$$\left| \int_{b\Delta \setminus B_r(\zeta_0)} \frac{v(\zeta) - v(\zeta_0)}{\xi - \zeta_0} d\xi \right| \le |v(\zeta) - v(\zeta_0)| \left| \int_{b\Delta \setminus B_r(\zeta_0)} \frac{1}{\xi - \zeta_0} d\xi \right| \le \pi \|v\|_{\mathcal{C}^{\alpha}} |\zeta - \zeta_0|^{\alpha}.$$

Lastly, we can combine the bound $|\xi-\zeta_0|/|\xi-\zeta| \leq 2$ for $\xi \in b\Delta \setminus B_r(\zeta_0)$ and the previous estimate:

$$\begin{aligned}\left|\int_{b\Delta\setminus B_r(\zeta_0)}(v(\xi)-v(\zeta))\frac{\zeta-\zeta_0}{(\xi-\zeta)(\xi-\zeta_0)}d\xi\right| &\leq \frac{8}{\pi}\|v\|_{\mathcal{C}^\alpha}|\zeta-\zeta_0|\int_{|\zeta-\zeta_0|}^{\pi}\theta^{\alpha-2}d\theta = \\ &= \frac{8}{\pi}\|v\|_{\mathcal{C}^\alpha}|\zeta-\zeta_0|\left(\frac{|\zeta-\zeta_0|^{\alpha-1}-\pi^{\alpha-1}}{\alpha-1}\right).\end{aligned} \tag{2.2}$$

With this, the first case is completed.
For the general case $v \in \mathcal{C}^{k,\alpha}$ we need to differentiate under the integral sign. This is easier with a change of variables:

$$\begin{aligned}\partial_\zeta\int_{b\Delta}\frac{v(\xi)}{\xi-\zeta}d\xi = \partial_\zeta\int_{b\Delta}\frac{v(\zeta\xi)}{\xi-1}d\xi = \partial_\zeta\left(\int_{b\Delta}\frac{v(\zeta\xi)-v(\zeta)}{\xi-1}d\xi + i\pi v(\zeta)\right) = \\ = \int_{b\Delta}\frac{\xi v'(\zeta\xi)-v'(\zeta)}{\xi-1}d\xi + i\pi v'(\zeta) = \int_{b\Delta}\frac{\xi v'(\zeta\xi)}{\xi-1}d\xi.\end{aligned}$$

Here, $(\xi v'(\zeta\xi)-v'(\zeta))/(\xi-1)$ is integrable because

$$\left|\xi v'(\zeta\xi)-v'(\zeta)\right| \leq |\xi-1|\left|v'(\zeta\xi)\right| + \left|v'(\zeta\xi)-v'(\zeta)\right| \lesssim |\xi-1|\left|v'(\zeta\xi)\right| + |\zeta\xi-\zeta|^\alpha.$$

We continue

$$\int_{b\Delta}\frac{\xi v'(\zeta\xi)}{\xi-1}d\xi = \int_{b\Delta}v'(\zeta\xi)d\xi + \int_{b\Delta}\frac{v'(\zeta\xi)}{\xi-1}d\xi = \int_{b\Delta}\frac{v'(\xi)}{\xi-\zeta}d\xi,$$

so we can apply the reasoning of Case 1 to v' by replacing the $\mathcal{C}^\alpha$ norms with $\|v''\|_\infty$ if $k>1$. Only (2.2) looks slightly different, here we evaluate the integral as $\log(\pi)-\log(|\zeta-\zeta_0|)$ which is still controlled by the $|\zeta-\zeta_0|$ factor. Using induction on k, the theorem follows. □

2.2.2 Boundary values of holomorphic functions

We will now develop conditions for a function on $b\Delta$ to be the boundary value of a holomorphic function on Δ or Δ^-.
Let $\varphi : \Delta \cup b\Delta \to \mathbb{C}$ be continuous, holomorphic on the interior and denote by $\varphi^+(\zeta)$ the boundary values for $\zeta \in b\Delta$. The Cauchy integral formula gives:

$$\varphi(z) = \frac{1}{2\pi i}\int_{b\Delta}\frac{\varphi^+(\zeta)}{\zeta-z}d\zeta \quad (z \in \Delta) \tag{2.3}$$

$$0 = \frac{1}{2\pi i}\int_{b\Delta}\frac{\varphi^+(\zeta)}{\zeta-z}d\zeta \quad (z \in \Delta^-). \tag{2.4}$$

Then (2.4) is a necessary condition for a continuous function φ^+ on $b\Delta$ to be the boundary value of a continuous function on $\Delta \cup b\Delta$, holomorphic on Δ.

This condition is actually sufficient. We will prove this in the case of Hölder continuous boundary values, which is enough for our purposes. So let $v \in \mathcal{C}^{k,\alpha}$ be the prospective boundary value and let

$$\varphi(z) = \frac{1}{2\pi i} \int_{b\Delta} \frac{v(\zeta)}{\zeta - z} d\zeta \quad (z \in \mathbb{C} \setminus b\Delta).$$

Taking limits in Δ^-, applying the Sokhotski-Plemelj formula (SP$^-$) to v and using (2.4) we get

$$0 = \lim_{\substack{z \to \zeta \\ z \in \Delta^-}} \varphi(z) = -\frac{1}{2} v(\zeta) + \frac{1}{2\pi i} \int_{b\Delta} \frac{v(\xi)}{\xi - \zeta} d\xi \quad (\zeta \in b\Delta).$$

Adding v to both sides,

$$v(\zeta) = \frac{1}{2} v(\zeta) + \frac{1}{2\pi i} \int_{b\Delta} \frac{v(\xi)}{\xi - \zeta} d\xi \quad (\zeta \in b\Delta),$$

we recognize the boundary value φ^+ on the right hand side, by the other Sokhotski-Plemelj formula (SP$^+$). Thus we have $v = \varphi^+$ so v is indeed the boundary value of a holomorphic function.
In this proof, we transformed the necessary condition (2.4) into the sufficient condition

$$0 = -\frac{1}{2} \varphi^+(\zeta) + \frac{1}{2\pi i} \int_{b\Delta} \frac{\varphi^+(\xi)}{\xi - \zeta} d\xi \quad (\zeta \in b\Delta). \tag{2.5}$$

Thus the conditions (2.4) and (2.5) are both necessary and sufficient, and equivalent.
Analogously, let $\varphi : b\Delta \cup \Delta^- \to \mathbb{C}$ be a continuous holomorphic in the interior with finite degree at infinity and denote by $\varphi^-(\zeta)$ the boundary values for $\zeta \in b\Delta$. Let γ be its principal part, by expanding into power series and using the residue theorem to evaluate the integral $\int_{b\Delta} \zeta^k / (\zeta - z) d\zeta$ we get:

$$-\varphi(z) = \frac{1}{2\pi i} \int_{b\Delta} \frac{\varphi^-(\zeta)}{\zeta - z} d\zeta - \gamma(z) \quad (z \in \Delta^-) \tag{2.6}$$

$$0 = \frac{1}{2\pi i} \int_{b\Delta} \frac{\varphi^-(\zeta)}{\zeta - z} d\zeta - \gamma(z) \quad (z \in \Delta). \tag{2.7}$$

(2.7) is a necessary and sufficient condition for a continuous function on $b\Delta$ to be the boundary value of a continuous function on $b\Delta \cup \Delta^-$, holomorphic on Δ^-. The proof is the same as before if we again assume Hölder continuity. We also have that (2.7) is equivalent to

$$0 = \frac{1}{2} \varphi^-(\zeta) + \frac{1}{2\pi i} \int_{b\Delta} \frac{\varphi^-(\xi)}{\xi - \zeta} d\xi - \gamma(\zeta) \quad (\zeta \in b\Delta). \tag{2.8}$$

All the arguments in this subsection also apply to the case of sectionally holomorphic vectors $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$.

2.2.3 The Homogeneous Hilbert Problem

The homogeneous Hilbert Problem for N unknown functions is stated as:
Find the sectionally holomorphic vector $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ which has finite degree at infinity and satisfies the boundary condition

$$\boldsymbol{\varphi}^+(\zeta) = G(\zeta)\boldsymbol{\varphi}^-(\zeta) \quad (\zeta \in b\Delta), \tag{I}$$

where $G : b\Delta \to GL_N(\mathbb{C})$ is Hölder continuous. A solution of (I) is always understood to not be the trivial solution $\boldsymbol{\varphi} = 0$. For any solutions $\boldsymbol{\varphi}_1 \ldots, \boldsymbol{\varphi}_k$ and polynomials $p_1, \ldots, p_k$ we get another solution of (I):

$$\boldsymbol{\varphi} = \sum_{j=1}^{k} p_j \boldsymbol{\varphi}_j$$

We will later show that the boundary value of any solution of (I) is Hölder continuous, which is a consequence of G being Hölder continuous. For now, we will mean by solutions of (I) only those where both $\boldsymbol{\varphi}^+$ and $\boldsymbol{\varphi}^-$ are Hölder continuous (note: since G is Hölder continuous it is enough to assume that either one is continuous, continuity of the other follows from (I)).
First we will seek solutions with prescribed principal part $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N)$ for given polynomials $\gamma_1, \ldots, \gamma_N$. We can reformulate problem (I) as follows: Find a Hölder continuous vector $\boldsymbol{\varphi}^-$ on $b\Delta$ satisfying the conditions

1. $\boldsymbol{\varphi}^-$ is the boundary value of a holomorphic vector on Δ^-, right continuous on $b\Delta$ with principal part $\boldsymbol{\gamma}$ at infinity.

2. the vector $\boldsymbol{\varphi}^+$ defined by

$$\boldsymbol{\varphi}^+(\zeta) = G(\zeta)\boldsymbol{\varphi}^-(\zeta) \quad (\zeta \in b\Delta)$$

   is the boundary value of a holomorphic vector on Δ, continuous from the left on $b\Delta$.

In the last section we have seen that these conditions are equivalent to

$$\frac{1}{2}\boldsymbol{\varphi}^-(\zeta) + \frac{1}{2\pi i}\int_{b\Delta} \frac{\boldsymbol{\varphi}^-(\xi)}{\xi - \zeta} d\xi = \boldsymbol{\gamma}(\zeta) \quad (\zeta \in b\Delta) \tag{2.9}$$

$$-\frac{1}{2}G(\zeta)\boldsymbol{\varphi}^-(\zeta) + \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\xi)\boldsymbol{\varphi}^-(\xi)}{\xi - \zeta} d\xi = 0 \quad (\zeta \in b\Delta). \tag{2.10}$$

Multiplying (2.10) with $G(\zeta)^{-1}$ from the left we get

$$-\frac{1}{2}\boldsymbol{\varphi}^-(\zeta) + \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\zeta)^{-1}G(\xi)\boldsymbol{\varphi}^-(\xi)}{\xi - \zeta} d\xi = 0 \quad (\zeta \in b\Delta) \tag{2.11}$$

which we can subtract from (2.9) to get

$$\boldsymbol{\varphi}^-(\zeta) - \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\zeta)^{-1}G(\xi) - I}{\xi - \zeta}\boldsymbol{\varphi}^-(\xi)d\xi = \boldsymbol{\gamma}(\zeta) \quad (\zeta \in b\Delta) \tag{IF}$$

where I is the identity matrix.
(IF) is a Fredholm equation of the second kind so we want to apply Fredholm theory in the L^2 setting. As a first step we show that the integral operator is even $L^1 \to L^\infty$-bounded, this essentially follows from the Young inequality since the operator is a convolution "up to a bounded function":

$$\begin{aligned}\left|\int_{b\Delta} \frac{G(\zeta)^{-1}G(\xi) - I}{\xi - \zeta}\boldsymbol{\varphi}(\xi)d\xi\right| &\leq \int_{b\Delta} \left\|G^{-1}\right\|_\infty \frac{|G(\xi) - G(\zeta)|}{|\xi - \zeta|^\alpha}|\xi - \zeta|^{\alpha-1}|\boldsymbol{\varphi}(\xi)||d\xi| \leq \\ &\leq \left\|G^{-1}\right\|_\infty \|G\|_{\mathcal{C}^\alpha}\left\||\cdot|^{\alpha-1} * |\boldsymbol{\varphi}|\right\|_{L^\infty(b\Delta)} \lesssim \left\|G^{-1}\right\|_\infty \|G\|_{\mathcal{C}^\alpha}\left\||\cdot|^{\alpha-1} * |\boldsymbol{\varphi}|\right\|_{L^1(b\Delta)} \leq \\ &\leq \left\|G^{-1}\right\|_\infty \|G\|_{\mathcal{C}^\alpha}\left\||\cdot|^{\alpha-1}\right\|_{L^1(b\Delta)}\|\boldsymbol{\varphi}\|_{L^1(b\Delta)}\end{aligned}$$

Next, we show compactness of the operator by expressing it as the limit of Hilbert-Schmidt operators. To this end, consider the regularized kernels

$$g_\varepsilon(\zeta, \xi) := \frac{1}{2\pi i}\frac{G(\zeta)^{-1}G(\xi) - I}{\xi - \zeta + \varepsilon\frac{\zeta - \xi}{|\zeta - \xi|}} \quad (\zeta, \xi \in b\Delta).$$

Since g_ε is continuous, we get $g_\varepsilon \in L^2(b\Delta \times b\Delta)$ and the corresponding integral operator

$$K_\varepsilon\boldsymbol{\varphi}(\zeta) := \int_{b\Delta} g_\varepsilon(\zeta, \xi)\boldsymbol{\varphi}(\xi)d\xi \quad (\zeta, \xi \in b\Delta)$$

is Hilbert-Schmidt and thus compact. As before, using the Young inequality, we can get the estimate

$$\|(K_0 - K_\varepsilon)\boldsymbol{\varphi}\|_{L^2(b\Delta)} \lesssim \left\|G^{-1}\right\|_\infty \|G\|_{\mathcal{C}^\alpha}\left\||\cdot|^{\alpha-1} - (|\cdot| + \varepsilon)^{\alpha-1}\right\|_{L^1(b\Delta)}\|\boldsymbol{\varphi}\|_{L^2(b\Delta)}.$$

Consequently $K_\varepsilon \to K_0$ as $\varepsilon \to 0$ by dominated convergence, so K_0 is also compact.
To ensure that L^2 is the right setting, we need to prove that any L^2 solution to (IF) is also $\mathcal{C}^{k,\alpha}$. Since $\boldsymbol{\gamma} \in \mathcal{C}^{k,\alpha}$, we can assume $\boldsymbol{\gamma} = 0$ for this purpose. Observe that

$$\boldsymbol{\varphi}(\zeta) = \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\zeta)^{-1}G(\xi) - I}{\xi - \zeta}\boldsymbol{\varphi}(\xi)d\xi = \frac{G(\zeta)^{-1}}{2\pi i}\int_{b\Delta} \frac{G(\xi) - G(\zeta)}{\xi - \zeta}\boldsymbol{\varphi}(\xi)d\xi.$$

Since G^{-1} is $\mathcal{C}^{k,\alpha}$, we only need to show that the integral is as well.
Case $k = 0$. $\boldsymbol{\varphi} \in L^\infty(b\Delta)$ so this follows exactly as in the proof of Privalov's theorem.
Case $k = 1$. We calculate

$$\begin{aligned}\partial_\zeta \int_{b\Delta} \frac{G(\xi) - G(\zeta)}{\xi - \zeta}\boldsymbol{\varphi}(\xi)d\xi &= \int_{b\Delta} \frac{-G'(\zeta)}{\xi - \zeta} + \frac{G(\xi) - G(\zeta)}{(\xi - \zeta)^2}\boldsymbol{\varphi}(\xi)d\xi \\ &= \int_{b\Delta} \frac{\frac{1}{\xi - \zeta}\int_\zeta^\xi G'(\eta) - G'(\zeta)d\eta}{\xi - \zeta}\boldsymbol{\varphi}(\xi)d\xi.\end{aligned}$$

Using the estimate $\left|\frac{1}{\xi-\zeta}\int_\zeta^\xi G'(\eta)-G'(\xi)d\eta\right| \lesssim |\xi-\zeta|^\alpha$ we can conclude integrability and are therefore justified in differentiating inside the integral. Moreover, it also follows that $\boldsymbol{\varphi}'$ is bounded. With this, we can now employ the substitution $\xi \to \zeta\xi$:

$$\partial_\zeta \int_{b\Delta} \frac{G(\zeta\xi)-G(\zeta)}{\xi-1}\boldsymbol{\varphi}(\zeta\xi)d\xi$$
$$= \int_{b\Delta} \frac{\xi G'(\zeta\xi)-G'(\zeta)}{\xi-1}\boldsymbol{\varphi}(\zeta\xi)d\xi + \int_{b\Delta} \frac{G(\zeta\xi)-G(\zeta)}{\xi-1}\xi\boldsymbol{\varphi}'(\zeta\xi)d\xi =$$
$$\int_{b\Delta} G'(\zeta\xi)\boldsymbol{\varphi}(\zeta\xi)+G(\zeta\xi)\boldsymbol{\varphi}'(\zeta\xi)d\xi + \int_{b\Delta} \frac{G'(\zeta\xi)-G'(\zeta)}{\xi-1}\boldsymbol{\varphi}(\zeta\xi)d\xi$$
$$+ \int_{b\Delta} \frac{G(\zeta\xi)-G(\zeta)}{\xi-1}\boldsymbol{\varphi}'(\zeta\xi)d\xi.$$

After changing back $\zeta\xi \to \xi$, we get a multiple of $1/\zeta$ for the first integral, the other two can be shown to be $\mathcal{C}^\alpha$ as in the proof of Privalov's theorem. Now we proceed using induction. Assume that we have already shown that $\boldsymbol{\varphi} \in \mathcal{C}^{k-1,\alpha}(b\Delta)$ and that the $k-1-$th derivative of the integral is of the form

$$\partial_\zeta^{k-1} \int_{b\Delta} \frac{G(\xi)-G(\zeta)}{\xi-\zeta}\boldsymbol{\varphi}(\xi)d\xi = P_{k-1}(1/\zeta)+\sum_{l=0} \binom{k-1}{l} \int_{b\Delta} \frac{G^{(l)}(\xi)-G^{(l)}(\zeta)}{\xi-\zeta}\boldsymbol{\varphi}^{(k-1-l)}(\xi)d\xi$$

for some polynomial P_l. But then each integral in the sum can be differentiated as in the $k=1$ case, we pick up some linear terms in $1/\zeta$ which can be combined with the derivative of P_{k-1}. With this, we have established the $\mathcal{C}^{k,\alpha}$ regularity of solutions to (IF). Now we investigate two questions:

1. When can we solve (IF)?
2. Does each solution of (IF) produce a solution of the original problem (I)?

We start with the second one. Let $\boldsymbol{\varphi}^-$ be a solution of (IF). This will be a solution of (I) if it satisfies the conditions (2.9) and (2.10). We reformulate these conditions by introducing a new sectionally holomorphic vector $\boldsymbol{\psi}$ defined by

$$\boldsymbol{\psi}(z) = \frac{1}{2\pi i}\int_{b\Delta} \frac{\boldsymbol{\varphi}^-(\zeta)}{\zeta-z}d\zeta - \gamma(z) \quad (z \in \Delta) \tag{2.12}$$
$$\boldsymbol{\psi}(z) = \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\zeta)\boldsymbol{\varphi}^-(\zeta)}{\zeta-z}d\zeta \quad (z \in \Delta^-). \tag{2.13}$$

Then, $\boldsymbol{\psi}$ vanishes at infinity. Conditions (2.9) and (2.10) are now equivalent to

$$\boldsymbol{\psi}^+(\zeta) = 0,\ \boldsymbol{\psi}^-(\zeta) = 0 \quad (\zeta \in b\Delta) \tag{2.14}$$

which is further equivalent to $\boldsymbol{\psi} = 0$.
We can rewrite (IF) in terms of $\boldsymbol{\psi}$ as follows:

$$\boldsymbol{\psi}^+(\zeta) = G(\zeta)^{-1}\boldsymbol{\psi}^-(\zeta) \quad (\zeta \in b\Delta) \tag{II}$$

Following Plemelj, we call the problem (II) the accompanying problem of (I). With the above arguments, we have just shown:

Lemma 2. *If the accompanying problem (II) has no non trivial solutions vanishing at infinity, then every solution of the Fredholm integral equation (IF) yields a solution of the original problem (I)*

The methods for the construction of the Fredholm integral equation (IF) can be used to obtain a similar integral equation for (II). We will however, find a Fredholm equation for $\boldsymbol{\psi}^+$ instead of $\boldsymbol{\psi}^-$. We start by writing down (II) in the equivalent form

$$\boldsymbol{\psi}^-(\zeta) = G(\zeta)\boldsymbol{\psi}^+(\zeta) \quad (\zeta \in b\Delta).$$

Now using the same arguments as before (note: we search for solutions vanishing at infinity, so $\boldsymbol{\gamma} = 0$), the Fredholm integral equation for problem (II) is

$$\boldsymbol{\psi}^+(\zeta) + \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\zeta)^{-1}G(\xi) - I}{\xi - \zeta}\boldsymbol{\psi}^+(\xi)d\xi = 0 \quad (\zeta \in b\Delta). \tag{2.15}$$

Next we look at the solvability of (IF). For this we introduce the adjoint equation

$$\boldsymbol{\psi}'^+(\zeta) + \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\zeta)^t(G(\xi)^t)^{-1} - I}{\xi - \zeta}\boldsymbol{\psi}'^+(\xi)d\xi = 0 \quad (\zeta \in b\Delta). \tag{II$'$F}$$

This is again connected to a Hilbert problem:

$$\boldsymbol{\varphi}'^+(\zeta) = (G(\zeta)^t)^{-1}\boldsymbol{\varphi}'^-(\zeta) \quad (\zeta \in b\Delta) \tag{I$'$}$$

which we will call the associate problem to (I). There is also the problem

$$\boldsymbol{\psi}'^+(\zeta) = G(\zeta)^t\boldsymbol{\psi}'^-(\zeta) \quad (\zeta \in b\Delta) \tag{II$'$}$$

accompanying the associate problem (I′). For (I′) we can again construct a Fredholm integral equation

$$\boldsymbol{\varphi}'^-(\zeta) - \frac{1}{2\pi i}\int_{b\Delta} \frac{G(\zeta)^t(G(\xi)^t)^{-1} - I}{\xi - \zeta}\boldsymbol{\varphi}'^-(\xi)d\xi = 0 \quad (\zeta \in b\Delta) \tag{I$'$F}$$

for solutions vanishing at infinity. The Fredholm integral equation for (II′) is given by (II′F). In summary, the adjoint integral equation to (IF) is the integral equation associated to the problem (II′) which is the accompanying problem of the associate one.

Lemma 3. *If none of the accompanying and associate problems of (I) have non trivial solutions vanishing at infinity, then the Fredholm equation (IF) is solvable for any polynomial right hand side and every solution of this equation produces a solution of the original problem.*

By Lemma 2 we only need to see that, assuming the above conditions, (IF) is solvable for any polynomial $\boldsymbol{\gamma}$. (IF) is a Fredholm equation so this is the case if we have

$$\int_{b\Delta} \gamma_1\psi_1'^+ + \ldots + \gamma_N\psi_N'^+ = 0 \tag{2.16}$$

for every solution ψ'^+ of (II′F). But by the assumed conditions each solution of this problem is a solution of problem (II′) and is thus the boundary value of a holomorphic function. Thus the integral (2.16) vanishes by Cauchy's theorem.

The conditions of the previous lemma might seem restrictive but we will show that the general case can always be reduced to this one. Moreover, we will construct general solutions from those bounded at infinity. Therefore we will temporarily assume the conditions of the previous lemma and seek solutions bounded at infinity.

By lemma 3 all such solutions come from the integral equation (IF). Now we construct a basis of solutions: Let $\boldsymbol{\varphi}_1^-, \ldots, \boldsymbol{\varphi}_N^-$ be solutions of (IF) with $\boldsymbol{\gamma}$ equal to $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ respectively. Denote by $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_N$ be the corresponding solutions of the Hilbert problem. They have the property

$$\varphi_{k,l}(\infty) = \delta_{k,l} \tag{2.17}$$

Then it is clear that any (bounded) solution of (IF) can be written as

$$\boldsymbol{\varphi}^-(\zeta) = \gamma_1 \boldsymbol{\varphi}_1^-(\zeta) + \ldots + \gamma_N \boldsymbol{\varphi}_N^-(\zeta) + \gamma_{N+1} \boldsymbol{\varphi}_{N+1}^-(\zeta) + \ldots + \gamma_m \boldsymbol{\varphi}_m^-(\zeta) \quad (\zeta \in b\Delta) \tag{2.18}$$

where $\boldsymbol{\varphi}_{N+1}^-, \ldots, \boldsymbol{\varphi}_m^-$ are a basis of solution for the homogeneous problem and $\gamma_{N+1}, \ldots, \gamma_m$ are constants. Such a solution then induces a solution $\boldsymbol{\varphi} = \gamma_1 \boldsymbol{\varphi}_1 + \ldots + \gamma_m \boldsymbol{\varphi}_m$ of the homogeneous Hilbert problem (I).

We now consider the general case, where the conditions of Lemma 3 are not necessarily fulfilled. To start off, we make an observation that will be crucial in the construction of the fundamental system of solutions:

Remark 1. *There is a bound $s \geq 0$, such that the order of the zero at infinity of any solution of problem (I) does not exceed s.*

Proof. Let the homogeneous equation corresponding to (IF) have at most s linearly independent solutions (such an s exists because (IF) is a Fredholm equation) and let $\boldsymbol{\varphi}(z)$ be any solution of (I). If $\boldsymbol{\varphi}$ has a zero of order k at infinity, then $\boldsymbol{\varphi}(z), z\boldsymbol{\varphi}(z), \ldots, z^{k-1}\boldsymbol{\varphi}(z)$ are all solutions of (I) that vanish at infinity. Therefore they all represent linearly independent solutions of the homogeneous equation (IF) so $k \leq s$. □

Applying this to the accompanying problem (II) and the associate problem (I′), we can find an integer $r \geq 0$ such that neither problem has a solution with a zero of order greater than r at infinity.

Next, we want to characterize all solutions of (I) of degree less than r at infinity where r is as above. Let $\boldsymbol{\varphi}$ be one of these solutions. Then $\dot{\boldsymbol{\varphi}}$ defined by

$$\dot{\boldsymbol{\varphi}}(z) = \boldsymbol{\varphi}(z) \quad (z \in \Delta) \tag{2.19}$$

$$\dot{\boldsymbol{\varphi}}(z) = \frac{\boldsymbol{\varphi}(z)}{(z-a)^r} \quad (z \in \Delta^-) \tag{2.20}$$

where $a \in \Delta$ is arbitrary, solves the Hilbert problem

$$\dot{\boldsymbol{\varphi}}^+(\zeta) = (\zeta - a)^r G(\zeta)\dot{\boldsymbol{\varphi}}^-(\zeta) \quad (\zeta \in b\Delta) \tag{İ}$$

and remains bounded at infinity. We can see that finding solutions to (I) with degree less than r at infinity is reduced to finding solutions of (İ) which remain bounded at infinity.
We now show that this problem satisfies the conditions of 2 and 3. The accompanying problem to (İ) is

$$\dot{\boldsymbol{\psi}}^+(\zeta) = (\zeta - a)^{-r} G(\zeta)^{-1}\dot{\boldsymbol{\psi}}^-(\zeta) \quad (\zeta \in b\Delta). \tag{İI}$$

Any solution to this, vanishing at infinity, would induce a solution $\boldsymbol{\psi}$ to problem (II)

$$\begin{aligned} \boldsymbol{\psi}(z) &= \dot{\boldsymbol{\psi}}(z) \quad (z \in \Delta) \\ \boldsymbol{\psi}(z) &= (z - a)^{-r}\dot{\boldsymbol{\psi}}(z) \quad (z \in \Delta^-) \end{aligned}$$

which has a zero of order greater than r at infinity, a contradiction to the definition of r. Analogously, we can prove that the problem associated to (İ) has no solutions which vanish at infinity. Thus we can, as before, characterize solutions that vanish at infinity of (İ) and transfer these to solutions of the original problem, we formulate this in the following theorem:

Theorem 3. *For the problem* (I) *and $r \geq 0$ sufficiently large, each solution of degree less than r at infinity is of the form:*

$$\boldsymbol{\varphi}(z) = \gamma_1\boldsymbol{\varphi}_1(z) + \ldots + \gamma_N\boldsymbol{\varphi}_N(z) + \gamma_{N+1}\boldsymbol{\varphi}_{N+1}(z) + \ldots + \gamma_m\boldsymbol{\varphi}_m(z) \quad (z \in \mathbb{C} \setminus b\Delta) \tag{2.21}$$

with constants $\gamma_1, \ldots, \gamma_m$ and linearly independent particular solutions $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m$. The first N of these solutions have the property:

$$\lim_{z\to\infty} z^{-r}\varphi_{j,k}(z) = \delta_{j,k} \quad j, k = 1, \ldots, N \tag{2.22}$$

and those remaining (possibly none if $N = m$) have degree less than r at infintiy.

2.2.4 Fundamental Systems

Let r be sufficiently large such that theorem 3 applies, we assume r is fixed from now on.
The first N solutions are linearly independent over the polynomials, i.e. if there are polynomials $q_1, \ldots, q_N$ such that

$$q_1\boldsymbol{\varphi}_1 + \ldots + q_N\boldsymbol{\varphi}_N = 0 \tag{2.23}$$

then $q_1 = \ldots = q_N = 0$. This follows from the fact that the determinant $\det(\boldsymbol{\varphi}_1|\ldots|\boldsymbol{\varphi}_N)$ is not identically 0 which is immediate by 2.22.

Next, we construct a special system of solutions:
Since the order of a zero at infinity of a solution is bounded above, there is a solution of the form 2.21 which has minimal degree at infinity among all solutions. Denote this degree by $(-\kappa_1)$ and let χ_1 be one solution having degree $(-\kappa_1)$.
Now consider the solutions 2.21 which do not lie in the span of $\boldsymbol{\chi}_1$ over the polynomials, i.e. those which cannot be written as $p_1\boldsymbol{\chi}_1$ for any polynomial p_1. Let $(-\kappa_2)$ the lowest degree of such solutions, then $\kappa_1 \geq \kappa_2$. Let $\boldsymbol{\chi}_2$ be a solution of degree $(-\kappa_2)$. Let $(-\kappa_3)$ be the lowest degree of solutions not of the form $p_1\boldsymbol{\chi}_1 + p_2\boldsymbol{\chi}_2$ with polynomials p_1, p_2 and let $\boldsymbol{\chi}_3$ be a solution of degree $(-\kappa_3)$.
We claim that we can continue this process to construct N solutions $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N$, to this end, assume that we have already constructed k solutions $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_k$ this way with $k < N$. Then there will exist solutions not of the form $p_1\boldsymbol{\chi}_1 + \ldots + p_k\boldsymbol{\chi}_k$ for polynomials $p_1, \ldots, p_k$. Because if all solutions were of this form, then the solutions $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_N$ would lie in the span of $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_k$ over the polynomials, contradicting their linear independence. Hence, this process can be continued to construct N solutions $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N$ with respective degrees $-\kappa_1, -\kappa_2, \ldots, -\kappa_N$ where $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_N$.
We will show later that this process can not be continued further. Moreover each solution of (I) can be represented as $\sum_{k=1}^{N} p_k\boldsymbol{\chi}_k$ for some polynomials $p_1, \ldots, p_N$.
For now, we only prove this for solutions $\boldsymbol{\chi}$ of degree strictly less than $(-\kappa_k)$, for such a solution we can find polynomials $p_1, \ldots, p_{k-1}$ with $\boldsymbol{\chi}(z) = p_1(z)\boldsymbol{\chi}_1(z) + \ldots + p_{k-1}(z)\boldsymbol{\chi}_{k-1}(z)$.
Assume towards a contradiction that this is not true. Then $\boldsymbol{\chi}(z)$ lies outside the span of $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_{k-1}$ so by the above construction the degree of $\boldsymbol{\chi}_k$ would be less than the degree of $\boldsymbol{\chi}$ by minimality of $\boldsymbol{\chi}_k$. This contradicts our assumption on $\boldsymbol{\chi}$.
Next we prove the following important property of the fundamental system of solutions: Any linear combination

$$\boldsymbol{\chi}(z) = a_1\boldsymbol{\chi}_1(z) + \ldots + a_N\boldsymbol{\chi}_N(z)$$

with $a_1, \ldots, a_N \in \mathbb{C}$ not all equal to zero cannot vanish anywhere in $\mathbb{C}$.
Suppose $\boldsymbol{\chi}(c) = 0$ for some $c \notin b\Delta$. Then $\boldsymbol{\chi}(z) = (z - c)\boldsymbol{\varphi}(z)$ for some sectionally holomorphic $\boldsymbol{\varphi}$ that is a solution to (I). Let a_k the last nonzero coefficient, then the degree of $\boldsymbol{\varphi}$ is less than the degree of $\boldsymbol{\chi}_k$ and we can write it as

$$\boldsymbol{\varphi}(z) = p_1(z)\boldsymbol{\chi}_1(z) + \ldots + p_{k-1}(z)\boldsymbol{\chi}_{k-1}(z)$$

for some polynomials $p_1, \ldots, p_{k-1}$. But this contradicts the linear independence of $\chi_1, \ldots, \chi_k$ over the polynomials.
Now consider the case when $c \in b\Delta$. Vanishing of $\boldsymbol{\chi}$ on $b\Delta$ means $\boldsymbol{\chi}^+(c) = 0$ or $\boldsymbol{\chi}^-(c) = 0$, both are equivalent since $\boldsymbol{\chi}^+(c) = G(c)\boldsymbol{\chi}^-(c)$. We can argue as before if we show that we again have $\boldsymbol{\chi}(z) = (z - c)\boldsymbol{\varphi}(z)$ with $\boldsymbol{\varphi}$ being a solution of (I). Accordingly, define the vector

$$\boldsymbol{\varphi}(z) := \frac{\boldsymbol{\chi}(z)}{z - c} \quad (z \in \mathbb{C} \setminus b\Delta).$$

Then, $\boldsymbol{\varphi}^{\pm} \in L^1(b\Delta) \cap \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(b\Delta \setminus \{c\})$ since $\boldsymbol{\chi}$ vanishes at least to order α at c and since $\zeta \mapsto 1/(\zeta - c)$ is $\mathcal{C}^{k,\alpha}$ on any closed arc that does not contain c. Hence the Sokhotski-Plemelj formulae hold for $\boldsymbol{\varphi}$ except at the point c, so we can deduce that $\boldsymbol{\varphi}$ also solves (IF). But then we know that $\boldsymbol{\varphi} \in \mathcal{C}^{k,\alpha}(b\Delta)$ and we can argue as in the case $c \notin b\Delta$.
As a corollary we obtain: Property 1. The determinant

$$\delta(z) := \det(\chi_{k,j}(z))$$

does not vanish on $\mathbb{C}$.
We can also characterize the behaviour of Δ at ∞ :
Property 2. Write

$$\boldsymbol{\chi}^0_k(z) = z^{\kappa_k}\boldsymbol{\chi}_k(z)$$

for $k = 1, \ldots, N$, then the determinant

$$\delta^0(z) = \det\big(\chi^0_1(z)|\ldots|\chi^0_N(z)\big)$$

has a finite nonzero value at ∞.

Proof. By construction, δ^0 has finite value at infinity, since the degree of $\boldsymbol{\chi}_k$ at infinity is exactly $(-\kappa_k)$. Now assume that $\delta^0(\infty) = 0$, then there are $a_1, \ldots, a_N \in \mathbb{C}$ not all zero such that

$$a_1 z_1^{\kappa}\boldsymbol{\chi}_1(z) + \ldots + a_N z_N^{\kappa}\boldsymbol{\chi}_N(z) = O(1/z)$$

Let a_k be the last nonzero coefficient then

$$\boldsymbol{\chi}(z) := a_1 z^{\kappa_1 - \kappa_k}\boldsymbol{\chi}_1(z) + \ldots + a_k\boldsymbol{\chi}_k(z) = O(z^{-\kappa_k - 1})$$

Consequently, $\boldsymbol{\chi}$ has degree less than $(-\kappa_k)$ at infinity and would thus be representable as $\sum_{j=1}^{k-1} p_j\boldsymbol{\chi}_j$ contradicting linear independence of $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_k$ over the polynomials. $\square$

Property 2 has the following consequence: For a sum

$$p_1(z)\boldsymbol{\chi}_1(z) + \ldots + p_N(z)\boldsymbol{\chi}_N(z)$$

with polynomials $p_1, \ldots, p_N$ with repsective degrees $m_1, \ldots, m_N$ we can we always know its degree. Indeed, since $\delta^0(\infty) \neq 0$ this is just the maximum of $m_1 - \kappa_1, \ldots, m_N - \kappa_N$. In other words the terms of highest degree degree cannot cancel.
From now on, any N solutions of (I) $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N$ with properties 1 and 2 will be called a fundamental system of solutions. The $N \times N$ matrix $X(z) = (\boldsymbol{\chi}_1(z)|\ldots|\boldsymbol{\chi}_N(z))$ will be called the fundamental matrix of the homogeneous Hilbert problem.
By construction the fundamental matrix satisfies

$$X^+(\zeta) = G(\zeta)X^-(\zeta) \quad (\zeta \in b\Delta)$$

or equivalently

$$G(\zeta) = X^+(\zeta)(X^-(\zeta))^{-1} \quad (\zeta \in b\Delta).$$

We will use this identity to prove

Theorem 4. *Any solution* $\boldsymbol{\varphi}$ *of* (I) *with finite degree at infinity can be written as*

$$\boldsymbol{\varphi}(z) = p_1(z)\boldsymbol{\chi}_1(z) + \ldots + p_N(z)\boldsymbol{\chi}_N(z)$$

where $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N$ *is a fundamental system of solutions and* $p_1, \ldots, p_N$ *are polynomials.*

Note: With this theorem we can now abandon the requirement that $\boldsymbol{\varphi}$ must be Hölder continuous.

Proof. Substituting $G(\zeta) = X^+(\zeta)(X^-(\zeta))^{-1}$ $(\zeta \in b\Delta)$ into (I) we get

$$(X^+(\zeta))^{-1}\boldsymbol{\varphi}^+(\zeta) = (X^-(\zeta))^{-1}\boldsymbol{\varphi}^-(\zeta) \quad (\zeta \in b\Delta)$$

This implies that $(X(z))^{-1}\boldsymbol{\varphi}(z)$ is holomorphic on the entirety of $\mathbb{C}$. Combining this with the assumption that $\boldsymbol{\varphi}$ has finite degree at infinity we see that $(X(z))^{-1}\boldsymbol{\varphi}(z)$ is polynomial, i.e.

$$(X(z))^{-1}\boldsymbol{\varphi}(z) = \boldsymbol{p}(z) \quad (z \in \mathbb{C})$$

for some polynomial $\boldsymbol{p}$. Consequently,

$$\boldsymbol{\varphi}(z) = X(z)\boldsymbol{p}(z)$$

which was to be proven. □

For this proof it was actually only necessary to assume Property 1 for the matrix X. Property 2 gives us more control over the polynomials in the expression for $\boldsymbol{\varphi}$. Specifically, a solution $\boldsymbol{\varphi}$ of degree not greater than some given integer k at infinity can be written as

$$\boldsymbol{\varphi}(z) = p_1(z)\boldsymbol{\chi}_1(z) + \ldots + p_N(z)\boldsymbol{\chi}_N(z)$$

where the p_j have degree not greater than $k+\kappa_j$. Here we have that $p_j = 0$ if $k+\kappa_j < 0$. The solution will have degree exactly k if the degree of one of the p_j is $k+\kappa_j$. If $k+\kappa_j < 0$ for all $j = 1, \ldots, N$ then the problem has no non-trivial solutions with degree at infinity not greater than k.

Next, we will investigate the relation between different fundamental systems of solutions. First, we will prove that the integers $\kappa_1, \ldots, \kappa_N$ are the same for any fundamental system. Let $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_N$ and $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_N$ be two fundamental systems and let $-\kappa_1, \ldots, -\kappa_N$ and $-\lambda_1, \ldots, -\lambda_N$ be the respective degrees of the solutions at infinity. We assume $\kappa_1 \geq \ldots \geq \kappa_N, \lambda_1 \geq \ldots \geq \lambda_N$ and will prove $\kappa_1 = \lambda_1, \ldots, \kappa_N = \lambda_N$.

Since the $\boldsymbol{\chi}_j$ form a fundamental system, we can express $\boldsymbol{\zeta}_k$ as

$$\boldsymbol{\zeta}_k = p_{k,1}\boldsymbol{\chi}_1 + \ldots + p_{k,N}\boldsymbol{\chi}_N \tag{2.24}$$

for polynomials $p_{k,1}, \ldots, p_{k,N}$. Similarly, we can express the $\boldsymbol{\chi}_j$ in terms of $\boldsymbol{\zeta}_k$.

The degrees of the fundamental systems are ordered as $\kappa_1 = \kappa_2 = \ldots = \kappa_k > \kappa_{k+1}$ and $\lambda_1 = \lambda_2 = \ldots = \lambda_l > \lambda_{l+1}$. We now prove $\kappa_1 = \lambda_1$ and $k = l$. Comparing the degrees of

both sides of (2.24) we see that $-\lambda_1 \geq -\kappa_1$ because the degree of the right hand side at infinity cannot be less than $-\kappa_1$. By interchanging the roles of the $\boldsymbol{\chi}_j$ and the $\boldsymbol{\zeta}_k$ it follows that $-\kappa_1 \geq -\lambda_1$, so $\kappa_1 = \lambda_1$.
Comparing no the degrees of both sides of (2.24), we obtain relations

$$\boldsymbol{\zeta}_j = p_{j,1}\boldsymbol{\chi}_1 + \ldots + p_{j,N}\boldsymbol{\chi}_N$$

for $j = 1, \ldots, l$ and polynomials $p_{j,1}, \ldots, p_{j,N}$ (in this case the polynomials are actually constants). If $l > k$ then these relations contradict the linear independence of the $\boldsymbol{\zeta}_j$ over the polynomials. Completely analogously, we can rule out $l > k$ by switching the roles of the fundamental systems. Hence, $l = k$.
We now have: $\kappa_1 = \lambda_1, \ldots, \kappa_k = \lambda_k$, $\kappa_k > \kappa_{k+1} = \ldots = \kappa_{k+r} > \kappa_{k+r+1}$, $\lambda_k > \lambda_{k+1} = \ldots = \lambda_{k+s} > \lambda_{k+s+1}$. Our next objective is proving $\kappa_{k+1} = \lambda_{k+1}$ and $r = s$. Again we start by comparing degrees of both sides of (2.24) for $\boldsymbol{\zeta}_{k+1}, \ldots, \boldsymbol{\zeta}_{k+s}$ to see that the right hand side will contain only $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_{k+r}$. Assuming that one of these right hand sides only contains $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_k$ would contradict linear independence of the $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_{k+s}$ (the κ_{k+j} has some coefficients for the $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_k$ which can be expressed in terms of the coefficients of the $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_k$). From this, we can conclude $-\lambda_{k+1} \geq -\kappa_{k+1}$ which implies, by symmetry, $\lambda_{k+1} = \kappa_{k+1}$.
As above, assuming now that $s > r$ would contradict linear independence of $\boldsymbol{\chi}_1, \ldots, \boldsymbol{\chi}_{k+s}$ and it follows again that $s = r$. The rest of the proof proceeds by induction.
We will now outline a construction of all fundamental systems starting from a given one. Assume that the κ_j are ordered as

$$\kappa_1 = \kappa_2 = \ldots = \kappa_{k_1} > \kappa_{k_1+1} = \ldots = \kappa_{k_1+k_2} > \ldots > \kappa_{k_1+\ldots+k_r+1} = \ldots = \kappa_n$$

The same arguments as in the previous proof allow us to better describe the degrees of the polynomials in (2.24): Let

$$\boldsymbol{\zeta}_j = \boldsymbol{\chi}_1 p_{j,1} + \ldots + \boldsymbol{\chi}_{k_1} p_{j,k_1} + \ldots + \boldsymbol{\chi}_{k_1+k_2} p_{j,k_1+k_2} + \ldots + \boldsymbol{\chi}_n p_{j,n},$$

then the polynomials $p_{j,k}$ are constants for $j, k = 1, 2, \ldots, k_1$; $p_{j,k} = 0$ for $j = 1, \ldots, k_1$, $k = k_1 + 1, \ldots, n$; $p_{j,k}$ have degree not greater than $\kappa_{k_1} - \kappa_{k_2}$ for $j = k_1 + 1, \ldots, k_1 + k_2$, $k = 1, \ldots, k_1$; $p_{j,k}$ are constant for $j, k = k_1 + 1 \ldots, k_1 + k_2$; etc.
In summary, the $p_{j,k}$ are polynomials of degree not greater than $\kappa_k - \kappa_j$, where $p_{j,k} = 0$ if $\kappa_k - \kappa_j < 0$. We can package these statements into matrix form: Let $P_{l,m}$, $l, m = 1, \ldots, r$ be the matrix with entries $p_{j,k}$ for $j = k_1 + \ldots + k_{l-1} + 1, \ldots, k_1 + \ldots + k_l$ and $k = k_1 + \ldots + k_{m-1} + 1, \ldots, k_1 + \ldots + k_m$ and set

$$H = (p_{j,k}) = \begin{pmatrix} P_{1,1} & 0 & \ldots & 0 \\ P_{2,1} & P_{2,2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{r,1} & P_{r,2} & \ldots & P_{r,r} \end{pmatrix}.$$

Then the blocks $P_{l,m}$ have degree at most $\kappa_m - \kappa_l$. We can now write

$$Z(z) = X(z)H(z)^t \tag{FM}$$

where Z is the matrix with entries $\zeta_{j,k}$, $j,k = 1,\ldots,N$. The determinant of Z does not vanish, thus $\det H(z) \neq 0$. Furthermore, since H has block form,

$$\det H = \det P_{1,1} \ldots \det P_{r,r},$$

each of these blocks has constant entries so $\det H$ is constant as well.
Altogether, we see that any fundamental system ζ is related to X by $\zeta(z) = X(z)H(z)^t$. Here H is a lower blocktriangular matrix with polynomial entries where: the diagonal blocks are square, the entries of each block $P_{l,m}$ have degree less than $\kappa_m - \kappa_l$, and $\det H$ is a nonzero constant.
The integers $\kappa_1, \ldots, \kappa_N$ will be called the component indices of the homogeneous Hilbert problem and their sum

$$\kappa = \kappa_1 + \ldots + \kappa_N$$

the total index or simply the index of the problem.
The total index can be directly calculated from the matrix G of the Hilbert problem. Writing, as before, $\delta(z) = \det(X(z))$ we get, by using $X^+(\zeta) = G(\zeta)X^-(\zeta)$

$$\delta^+(\zeta) = \det(G(\zeta))\delta^-(\zeta) \quad (\zeta \in b\delta)$$

Taking logarithmic derivatives here, we conclude the following identity of winding numbers around 0 :

$$\mathrm{Ind}_{\delta^+(b\Delta)}(0) = \mathrm{Ind}_{\det(G)(b\Delta)}(0) + \mathrm{Ind}_{\delta^-(b\Delta)}(0)$$

Since δ is holomorphic and nonvanishing on Δ, by the argument principle, $\mathrm{Ind}_{\delta^+(b\Delta)}(0) = 0$. Since $\delta(z) = \delta^0(z)/z^\kappa$, δ^0 is holomorphic on Δ^- and nonzero at ∞ we get $\mathrm{Ind}_{\delta^-(b\Delta)}(0) = -\kappa$. Combining these observations, we conclude

$$\kappa = \mathrm{Ind}_{\det(G)(b\Delta)}(0). \tag{Ind}$$

2.2.5 Proof of the Birkhoff factorization

Lemma 4 (Special fundamental matrix)**.** *For a Hölder continuous matrix with $G^{-1} = \overline{G}$, the homogeneous Hilbert problem*

$$\varphi^+ = G\varphi^-$$

has a fundamental matrix Ω such that $\Omega_* = \Omega\Lambda$, where Ω_* denotes the Schwarz reflection of Ω about the circle and $\Lambda(z) = \mathrm{diag}(z^{\kappa_1}, \ldots, z^{\kappa_N})$

Proof. Start with any fundamental matrix $X(z) = (\boldsymbol{\chi}_1|\ldots|\boldsymbol{\chi}_N)$ then we see that $X_*\Lambda^{-1}$ is also a fundamental matrix and hence is connected to X by an identity of the form $X_*\Lambda^{-1} = XH^t$. Here H is a matrix as in (FM), all its entries are polynomials $p_{k,j}$ of degree not exceeding $\kappa_j - \kappa_k$ and $\det H$ is constant and nonzero. Conversely any such matrix H produces a fundamental matrix XH^t. We will now find a diagonal matrix

$\delta = \text{diag}(\delta_1, \dots, \delta_N)$ where the δ_k are nonzero constants such that $\Omega = (X\delta)_*\Lambda^{-1} + X\delta$. It is readily apparent that all matrices of this form satisfy the required identity

$$((X\delta)_*\Lambda^{-1} + X\delta)_* = (X\delta)\Lambda + (X\delta)_* = ((X\delta)_*\Lambda^{-1} + X\delta)\Lambda,$$

we only need to find δ such that we get a fundamental matrix. Using (FM) again and setting $\gamma = \overline{\delta}\delta^{-1}$ we get

$$(X\delta)_*\Lambda^{-1} + X\delta = X(\gamma H + I)^t\delta$$

Hence a matrix of this form gives a new fundamental matrix if we choose δ such that $\det(\gamma H(z) + I) \neq 0$ (it is already constant since the diagonal blocks of H are constant), this is clearly possible. $\square$

We are finally ready to prove the Birkhoff factorization:
Let $B(\zeta) = G(\zeta)\overline{G(\zeta)^{-1}}$ ($\zeta \in b\Delta$) and observe that $\overline{B(\zeta)} = B(\zeta)^{-1}$ ($\zeta \in b\Delta$ and that B is of class $\mathcal{C}^\alpha$ We now consider the homogeneous Hilbert problem associated to B, i.e. solutions of

$$\boldsymbol{\Psi}^+(\zeta) = B(\zeta)\boldsymbol{\Psi}^-(\zeta) \quad (\zeta \in b\Delta)$$

such that $\boldsymbol{\Psi}^+ : \overline{\Delta} \to \mathbb{C}$ and $\boldsymbol{\Psi}^- : \mathbb{C} \setminus \Delta$ are continuous and holomorphic on the interior with $\boldsymbol{\Psi}^-$ having only a pole at infinity. Suppose $(\boldsymbol{\Psi}^+, \boldsymbol{\Psi}^-)$ is a solution of such a Hilbert problem, then the reflections

$$\begin{aligned} \boldsymbol{\Psi}^+_*(\zeta) &= \overline{\boldsymbol{\Psi}^-(1/\overline{\zeta})} \quad \zeta \in \overline{\Delta} \setminus \{0\} \\ \boldsymbol{\Psi}^-_*(\zeta) &= \overline{\boldsymbol{\Psi}^+(1/\overline{\zeta})} \quad \zeta \in (\mathbb{C} \cup \{\infty\}) \setminus \Delta \end{aligned}$$

are again holomorphic on the interior. Furthermore, by taking conjugates in $\boldsymbol{\Psi}^+ = B\boldsymbol{\Psi}^-$ we get $\overline{\boldsymbol{\Psi}^+} = B^{-1}\overline{\boldsymbol{\Psi}^-}$ which implies

$$\boldsymbol{\Psi}^+_*(\zeta) = B(\zeta)\boldsymbol{\Psi}^-_*(\zeta) \quad \zeta \in b\Delta$$

so $(\boldsymbol{\Psi}^+_*, \boldsymbol{\Psi}^-_*)$ is also a solution of the generalized Hilbert Problem as it is not necessarily holomorphic at 0. Using this property we obtain the matrix Θ as a special fundamental matrix of the problem: Namely, according to Lemma 4 there is a fundamental matrix Ω of the problem satisfying $\Omega^+_*(\zeta) = \Omega^+(\zeta)\Lambda(\zeta)$. Setting $\Theta = \Omega^+$, we verify

$$\Omega^+(\zeta)\Lambda(\zeta) = \Omega^+_*(\zeta) = B(\zeta)\Omega^-_*(\zeta) = B(\zeta)\overline{\Omega^+(\zeta)} \quad (\zeta \in b\Delta)$$

2.3 Riemann-Hilbert problems

In order to solve the type of Riemann-Hilbert problem that arises in our setting we make use of the following theorem by Florian Bertrand and Giuseppe Della Sala [3]

Theorem 5. *Let $k, m \geq 0$ be integers and let $0 < \alpha < 1$. Consider the following operator*

$$L : \left(\mathcal{A}^{k,\alpha}_{0^m}\right)^N \to \left(\mathcal{C}^{k,\alpha}_{0^m}\right)^N$$

given by

$$L(\boldsymbol{f}) = 2\,\mathrm{Re}(\overline{G}\boldsymbol{f})$$

with smooth $G : b\Delta \to GL(N, \mathbb{C})$. Denote by $\kappa_1, \ldots, \kappa_N$ and κ the partial indices and the Maslov index of $-\overline{G}^{-1}G$. Then

1. *The map L is surjective if and only if $\kappa_j \geq m - 1$ for all $j = 1, \ldots, N$.*
2. *Assume L is surjective. Then the kernel of L is finite dimensional with real dimension $\kappa + N - Nm$.*

This theorem is an extension of results by Globevnik [5] who proved the nonsingular analogue:

Theorem 6. *Let $k \geq 0$ be an integer and let $0 < \alpha < 1$. Consider the following operator*

$$L : \left(\mathcal{A}^{k,\alpha}\right)^N \to \left(\mathcal{C}^{k,\alpha}\right)^N$$

given by

$$L(\boldsymbol{f}) = 2\,\mathrm{Re}(\overline{G}\boldsymbol{f})$$

with smooth $G : b\Delta \to GL(N, \mathbb{C})$. Denote by $\kappa_1, \ldots, \kappa_N$ and κ the partial indices and the Maslov index of $-\overline{G}^{-1}G$. Then

1. *The map L is surjective if and only if $\kappa_j \geq -1$ for all $j = 1, \ldots, N$.*
2. *Assume L is surjective. Then the kernel of L is finite dimensional with real dimension $\kappa + N$.*

Proof. For the first step in the proof we simplify the problem from arbitrary smooth matrices G to special block diagonal matrices: Start with the Birkhoff factorization $-\overline{G^{-1}}G = \Theta\Lambda\overline{\Theta^{-1}}$ where $\Theta : \overline{\Delta} \to GL(N, \mathbb{C})$ is smooth and holomorphic on Δ and $\Lambda(\zeta) = \mathrm{diag}(\zeta^{\kappa_1}, \ldots, \zeta^{\kappa_N})$. Here the κ_j are ordered such that the first $2s$ partial indices are odd and all the others even. Now consider the block matrix $M(\zeta) = P_1(\zeta) \oplus \ldots P_s(\zeta) \oplus \mathrm{diag}(\zeta^{-\kappa_{2s+1}/2}, \ldots, \zeta^{-\kappa_N/2})$ with blocks

$$P_j(\zeta) = \begin{pmatrix} 1+\zeta & -i(1-\zeta) \\ i(1-\zeta) & 1+\zeta \end{pmatrix} \begin{pmatrix} \zeta^{-\frac{\kappa_{2j-1}+1}{2}} & 0 \\ 0 & \zeta^{-\frac{\kappa_{2j}+1}{2}} \end{pmatrix}.$$

and set $V = M(i\Theta)^{-1}\overline{G^{-1}}$. M is chosen such that we get three properties: $\overline{M}M^{-1} = \Lambda$, the entries of $V(\zeta)$ are real for all $\zeta \in b\Delta$ and, most importantly, $V\overline{G} = M(-i\Theta)^{-1}$.

The first is straightforward matrix multiplication, the third is by construction. For the second, we calculate

$$\overline{V} = \overline{M}i\overline{\Theta^{-1}}G^{-1} = i\overline{M}\Lambda^{-1}\Theta^{-1}\overline{G^{-1}} = M(-i\Theta)^{-1}\overline{G^{-1}} = V.$$

Since multiplication by V and $(-i\Theta)^{-1}$ give automorphisms of $(\mathcal{C}^{k,\alpha})^N$ and $(\mathcal{A}^{k,\alpha})^N$ respectively, the third property allows us to replace $\overline{G}$ with M without changing whether L is surjective or what the dimension of its kernel is. Now the operator given by $\boldsymbol{f} \mapsto 2\,\mathrm{Re}(M\boldsymbol{f})$ is surjective if and only if

1. for $j = 1, \ldots, s$ the operator $L_j : (\mathcal{A}^{k,\alpha})^2 \to (\mathcal{C}^{k,\alpha})^2$ given by $L_j(\boldsymbol{f}) = 2\,\mathrm{Re}(P_j\boldsymbol{f})$ is surjective.
2. for $j = 2s+1, \ldots, N$ the operator $L_j : \mathcal{A}^{k,\alpha} \to \mathcal{C}^{k,\alpha}$ given by $L_j(\boldsymbol{f}) = 2\,\mathrm{Re}(\zeta^{-m_j}\boldsymbol{f})$ is surjective, where $\kappa_j = 2m_j$.

We start with 2. For $\varphi \in \mathcal{C}^{k,\alpha}$ denote by $\tilde{\varphi}$ its harmonic conjugate and set $w = \varphi + i\tilde{\varphi}$. w is in $\mathcal{A}^{k,\alpha}$ and satisfies $\varphi(\zeta) = \mathrm{Re}(\zeta^{-l}(\zeta^l w(\zeta)))$ $(\zeta \in b\Delta)$ for all integers l. For $l \geq 0$ we have $\zeta \mapsto \zeta^l w(\zeta) \in \mathcal{A}^{k,\alpha}$ and thus L_J is surjective for $m_j \geq 0$.
If $m_j < 0$, we have $\int_{b\Delta} \zeta^{-1-m_j} w(\zeta)\mathrm{d}\zeta = 0$ for all $w \in \mathcal{A}^{k,\alpha}$. This implies $\int_0^{2\pi} L_j(w)(\theta)\mathrm{d}\theta = 0$ for $w \in \mathcal{A}^{k,\alpha}$ and hence L_j is not surjective in this case, for example $1 \notin L_j(\mathcal{A}^{k,\alpha})$.
For case 1 set $\kappa_j = 2m_j - 1$. We investigate equations of the form $\mathrm{Re}(P_j(\zeta^2)w(\zeta^2)) = \varphi(\zeta^2)$ $(\zeta \in b\Delta)$ for $\varphi \in (\mathcal{C}^{k,\alpha})^2$ and $w \in (\mathcal{A}^{k,\alpha})^2$, i.e.

$$\mathrm{Re}\left(\begin{pmatrix} 1+\zeta^2 & -i(1-\zeta^2) \\ i(1-\zeta^2) & 1+\zeta^2 \end{pmatrix}\begin{pmatrix} \zeta^{-2l}w_1(\zeta^2) \\ \zeta^{-2r}w_2(\zeta^2) \end{pmatrix}\right) = \begin{pmatrix} \varphi_1(\zeta^2) \\ \varphi_2(\zeta^2) \end{pmatrix} \ (\zeta \in b\Delta)$$

for integers l, r. We can factor out ζ in both columns of $P_j(\zeta^2)$ to see that this is equivalent to

$$2\,\mathrm{Re}\left(\begin{pmatrix} \zeta\zeta^{-2l}w_1(\zeta^2) \\ \zeta\zeta^{-2r}w_2(\zeta^2) \end{pmatrix}\right) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} \varphi_1(\zeta^2) \\ \varphi_2(\zeta^2) \end{pmatrix} \ (\zeta \in b\Delta) \tag{2.25}$$

where $\zeta = e^{i\theta}$. The point here is that now the right hand side is an odd function, recall that a function φ on $b\Delta$ is odd if $\varphi(-\zeta) = -\varphi(\zeta)$ $(\zeta \in b\Delta)$. Furthermore, by looking at the power series expansion, we see that $\varphi \in \mathcal{A}^{k,\alpha}$ is odd if only if $\varphi(\zeta) = \zeta\psi(\zeta^2)$ with $\psi \in \mathcal{A}^{k,\alpha}$ and $\varphi \in \mathcal{C}^{k,\alpha}$ odd implies $\tilde{\varphi}$ odd and unique not just up to constants since the value at 0 has to be 0.
Let $l = r = 0$. We have just established that for odd $\varphi \in \mathcal{C}^{k,\alpha}$ there is a unique odd function in $\mathcal{A}^{k,\alpha}$, namely $\varphi + i\tilde{\varphi}$ such that φ is its real part. Let $\varphi \in \mathcal{C}^{k,\alpha}$ be 1/2 times the right hand side of (2.25), thus, evoking the notation of the previous paragraph, the unique solution of the equation is given by the $\psi \in (\mathcal{A}^{k,\alpha})$ associated to $\varphi + i\tilde{\varphi}$.
For $l, r \leq 0$ we can reduce to the case $l = r = 0$ by the same trick as before by using $\zeta w(\zeta^2) = \zeta\zeta^{2a}(\zeta^{-2a}w(\zeta^2))$ and $\zeta^{-2a}w(\zeta^2) \in \mathcal{A}^{k,\alpha}$ for any integer $a \leq 0$ and any $w \in \mathcal{A}^{k,\alpha}$.
Let $l \geq 1$. Choose a $g \in \mathcal{A}^{k,\alpha}$ with $g(0) \neq 0$ and decompose it into real and imaginary part as $g = \varphi_1 - i\varphi_2$. We are led to the equation

$$\mathrm{Re}(\zeta g(\zeta^2)) = \varphi_1(\zeta^2)\cos(\theta) + \varphi_2(\zeta^2)\sin(\theta) \ (\zeta \in b\Delta)$$

where $\zeta = e^{i\theta}$. By the previous arguments $\zeta \mapsto \zeta g(\zeta^2)$ is the only function in $\mathcal{A}^{k,\alpha}$ whose real part equals the right hand side. We are now faced with the task of expressing $\zeta g(\zeta^2)$ as $\zeta\zeta^{2j}h(\zeta^2)$ for some $h \in \mathcal{A}^{k,\alpha}$ which is rendered impossible by the multiplicities of the zero at 0 being 1 and $1+2j$ respectively. We conclude that we do not have surjectivity in this case. Argue along these lines for $r \geq 1$ to finish the proof of the first claim. The second claim will be shown in the next proof. □

Proof of Theorem 5. By the same matrix factorization argument we have that the operator is surjective if and only if

1. for $j = 1, \ldots, s$ the operator $L_j : (\mathcal{A}^{k,\alpha}_{0^m})^2 \to (\mathcal{C}^{k,\alpha}_{0^m})^2$ given by $L_j(\boldsymbol{f}) = 2\,\mathrm{Re}(P_j \boldsymbol{f})$ is surjective.
2. for $j = 2s+1, \ldots, N$ the operator $L_j : \mathcal{A}^{k,\alpha}_{0^m} \to \mathcal{C}^{k,\alpha}_{0^m}$ given by $L_j(\boldsymbol{f}) = 2\,\mathrm{Re}(\zeta^{-m_j}\boldsymbol{f})$ is surjective, where $\kappa_j = 2m_j$

Case 2 is again easier: Take $\varphi = (1-\zeta)^m v \in \mathcal{C}^{k,\alpha}_{0^m}$ where $v \in \mathcal{R}_m$. We need to study the equation

$$\zeta^{-r} f + \zeta^r \overline{f} = \varphi \quad f \in \mathcal{A}^{k,\alpha}_{0^m} \tag{2.26}$$

We can express f as $(1-\zeta)^m g$ with $g \in \mathcal{A}^{k,\alpha}$ and cancel the $(1-\zeta)^m$ factors to get

$$\zeta^{-r} g + (-1)^m \zeta^{r-m}\overline{g} = v$$

Split into two subcases: For $m = 2m'$ even, we can multiply by $\zeta^{m'}$ to get

$$\zeta^{-(r-m')} g + \zeta^{r-m'}\overline{g} = \zeta^{m'} v$$

By Lemma 1 the right hand side is now in $\mathcal{C}^{k,\alpha}$ so we can argue as in the proof of Theorem 6.
For $m = 2m'+1$ odd, we again multiply by $\zeta^{m'}$ to get

$$\zeta^{-(r-m')} g - \zeta^{r-m'}\overline{\zeta g} = \zeta^{m'} v$$

Applying Lemma 1,we see that the right hand side $\zeta^{m'} v =: u$ sits in $\mathcal{R}_1$. Take the orthogonal decomposition $u = u' + u''$ where $u' = \mathcal{P}(u) \in \mathcal{A}^{k,\alpha}$ is the Szegő projection. Using $u = -\overline{\zeta u}$ and $u'' = \overline{i\zeta(\overline{i\zeta u})'}$ we have $u'' = -\overline{\zeta u'}$ (see Chapter 4 of [2]). If $r - m' \geq 0$ then $g = \zeta^{r-m'}u' \in \mathcal{A}^{k,\alpha}$ and satisfies (2.26). If $r - m' < 0$ then

$$\int \zeta^{-(r-m')} g \mathrm{d}\theta = \int \zeta^{r-m'-1}\overline{g}\mathrm{d}\theta = 0$$

so $1 - \overline{\zeta}$ for example is not in range.
Now for the two dimensional Case 1. The subcase of $m = 2m'$ can be handled in the same way as in the proof of Theorem 6 by multiplying with $\zeta^{m'}$ we get surjectivity if and only if both $l - m' \geq 0$ and $r - m' \geq 0$.

Lastly, if $m = 2m' + 1$ odd we have by multiplication with $\zeta^{m'}$

$$\zeta^{m'} P\boldsymbol{g} - \zeta^{-m'-1}\overline{P\boldsymbol{g}} = \zeta^{m'} v$$

We substitute $\zeta = \xi^2$ and multiply by ξ to get

$$\xi^m P(\xi^2)\boldsymbol{g}(\xi^2) - \xi^{-m}\overline{P(\xi^2)\boldsymbol{g}(\xi^2)} = \xi^m \boldsymbol{v}(\xi^2)$$

Multiply by i and use Lemma 1

$$2\,\mathrm{Re}(\xi^m P(\xi^2) i g(\xi^2)) = i\xi^m \boldsymbol{v}(\xi^2)$$

similar to before we get

$$4\,\mathrm{Re}\left(\begin{pmatrix} i\xi^{-(2r_1-m-1)} g_1(\xi^2) \\ i\xi^{-(2r_2-m-1)} g_2(\xi^2) \end{pmatrix}\right) = i \begin{pmatrix} \mathrm{Re}\,\xi & \mathrm{Im}\,\xi \\ -\mathrm{Im}\,\xi & \mathrm{Re}\,\xi \end{pmatrix} \xi^m \boldsymbol{v}(\xi^2)$$

According to Lemma 1, $\xi \mapsto i\xi^m \boldsymbol{v}(\xi^2)$ is odd so the right hand side is now in $(\mathcal{C}_e^{k,\alpha})^2$ and the problem reduces to a pair of one-dimensional problems

$$\xi^{-(2r_j-m-1)} g_j(\xi^2) + \xi^{2r_j-m-1}\overline{g}_j(\xi^2) = u_j(\xi)$$

with $u_j \in \mathcal{C}^{k,\alpha}$ even. Setting $u_j(\xi) = u'_j(\xi^2)$ with $u_j \in \mathcal{C}^{k,\alpha}$

$$\zeta^{-(2r_j-m-1)/2)} g_j(\zeta) + \zeta^{(2r_j-m-1)/2}\overline{g}_j(\zeta) = u'_j(\zeta)$$

we have reduced it to the one-dimensional case considered before. Surjectivity here is equivalent to $2r_j - m - 1 \geq 0$ and since m is odd, we are done.
Now for the second part of the theorem concerning the (real) dimension of the kernel in the surjective case. Assume

$$2\,\mathrm{Re}(M\boldsymbol{f}) = 0$$

on $b\Delta$ for some $\boldsymbol{f} \in (\mathcal{A}_{0^m}^{k,\alpha})^N$. Multiplying with M^{-1} and using the construction of M gives

$$\boldsymbol{f} = -M^{-1}\overline{M\boldsymbol{f}} = -\,\mathrm{diag}(\zeta^{\kappa_1}, \ldots, \zeta^{\kappa_N})\overline{\boldsymbol{f}}$$

Thus the determination of the kernel reduces to the one dimensional problem

$$f + \zeta^l \overline{f} = 0$$

for $f = (1-\zeta)^m g \in \mathcal{A}_{0^m}^{k,\alpha}$ and $l \geq m - 1$. By factoring out $(-\overline{\zeta})^m$ we get

$$g + (-1)^m \zeta^{l-m}\overline{g} = 0$$

We see that solutions have the form $g(\zeta) = \sum_{r=0}^{l-m} a_r \zeta^r$ with conditions $a_r + (-1)^m \overline{a_{l-m-r}} = 0$. We have $2(l - m + 1)$ (real) degrees of freedom for these coefficients and $m - l + 1$ conditions so the space of solutions has real dimension $l - m + 1$. □

2.4 Approximation of CR Functions

We need an approximation result, which can be found in [1], we reproduce the proof here.

Defiition 1. *Let M be a manifold and $\mathcal{V}$ an $n-$dimensional subbundle of its complexified tangent bundle $\mathbb{C}TM$. $\mathcal{V}$ is called integrable if at each point $p \in M$ there is an open neighbourhood Ω of p and $Z_1, \dots, Z_m \in \mathcal{C}^\infty(\Omega, \mathbb{C})$, where $m = \dim_{\mathbb{R}} M - n$ such that*

- *the differentials $dZ_1, \dots, dZ_m$ are linearly independent over $\mathbb{C}$*
- *For any section $L \in \Gamma(M, \mathcal{V})$ we have*

$$LZ_j = 0 \quad \forall j = 1, \dots, m.$$

In this case, we call $(M, \mathcal{V})$ an integrable structure.

Theorem 7 (Baouendi-Treves)**.** *Let $(M, \mathcal{V})$ be an integrable structure, $p_0 \in M$, and $Z = (Z_1, \dots, Z_m)$ a family of basic solutions near p_0. Then there exists a compact neighbourhood K of p_0 in M such that for any continuous solution h in M, there is a sequence of holomorphic polynomials $P_\nu(z)$ in m complex variables with the property that*

$$h(u) = \lim_{\nu \to \infty} P_\nu(Z(u)) \tag{BT}$$

uniformly on K.

The idea of the proof is similar to many other approximation results: We convolve the data with a Gaussian kernel to obtain a sequence of holomorphic approximations whose power series we can truncate to get the desired polynomials.

Proof. First, we show that we can find suitable coordinates near p_0.

Lemma 5. *Take $(M, \mathcal{V}), p_0$ and $Z = (Z_1, \dots, Z_m)$ as in theorem BT. Then we can find local coordinates (x, y) near p_0, vanishing at p_0, with $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ and an invertible complex linear transformation L such that*

$$\tilde{Z}_j(x, y) = x_j + i\varphi_j(x, y), \tag{2.27}$$

for $j = 1, \dots, m$. Here $\tilde{Z}$ is the system of basic solutions defined by $\tilde{Z} = LZ$, the φ_j are smooth, real-valued functions defined near the origin in $\mathbb{R}^k$, $k = n + m$, with $\varphi(0) = 0$ and $\varphi_x(0) = 0$.

Proof of Lemma 5. $dZ_1, \dots, dZ_m$ are linearly independent near p_0, so we can find a system of coordinates $(u_1, \dots, u_k)$ vanishing at p_0 such that the matrix $A(u) = (\partial Z_j / \partial_l(u))_{1 \le j,l \le m}$ is invertible for u near 0 in $\mathbb{R}^k$. Then we take coordinates

$$x := \operatorname{Re} A(0)^{-1} Z(u), \;\; y_j := u_{j+m}, \quad j = 1, \dots, n.$$

From this, we see immediately that $\tilde{Z}(x, y) = A(0)^{-1} Z(u(x, y))$ satisfies (2.27). $\square$

Here is the basic idea of the rest of the proof, which is similar to one proof of the Weierstrass approximation theorem: We take a Gaussian-like kernel α_ν which is analytic in Z and consider H_ν, the convolutions of h with α_ν. These are again analytic and we can show that they converge uniformly to h as $\nu \to \infty$. Then we can extract our polynomials from this sequence by truncating the power series of H_ν.
Continuing with the proof of Theorem BT, we choose the coordinates (x, y) from Lemma 5 and take $\tilde{Z}$ as our new system of basic solutions but still denote it by Z. We give the domain K in terms of two parameters r and d to be determined later as

$$K := \{(x, y) : |x| \leq \frac{r}{4}, |y| \leq d\}.$$

Fix some smooth cutoff function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^m)$ with $\chi(x) = 1$ for $|x| \leq \frac{r}{2}$ and $\chi(x) = 0$ for $x \geq r$.
Temporarily, we will restrict to the case $h \in \mathcal{C}^1(M)$ and show how to adapt the proof to the continuous case later. Consider the family of $m-$forms $\alpha_\nu(x, y; z)$ for $z \in \mathbb{C}^m$ and $\nu \in \mathbb{Z}^+$ defined by

$$\alpha_\nu(x, y; z) := \Big(\frac{\nu}{\pi}\Big)^{m/2} \exp\big(-\nu(z - Z(x, y))^2\big)\chi(x)h(x, y)dZ(x, y),$$

where

$$dZ(x, y) = dZ_1(x, y) \wedge \ldots \wedge dZ_m(x, y),$$

and, for $v = (v_1, \ldots, v_m) \in \mathbb{C}^m$, we write $v^2 = v \cdot v = \sum_j v_j^2$. For $y \in \mathbb{R}^n$ with $0 < |y| < d$ we consider the cylinder D_y

$$D_y := \{(x', y') \in \mathbb{R}^k : \left|x'\right| < r, y' = ty, t \in (0, 1)\}.$$

By Stokes' Theorem we get

$$\int_{D_y} d\alpha_\nu(x'y'; z) = \int_{\partial D_y} \alpha_\nu(x', y'; z) \tag{2.28}$$

First we compute the right hand side of (2.28). Since $\chi(x) = 0$ for $|x| \geq r$ the only boundary remaining are

$$\begin{aligned}
&\int_{\partial D_y} \alpha_\nu(x', y'; z) = \\
&\Big(\frac{\nu}{\pi}\Big) \int_{\mathbb{R}^m} \exp\big(-\nu(z - Z(x', y))^2\big)\chi(x')h(x', y)d_{x'}Z(x', y) \\
&- \Big(\frac{\nu}{\pi}\Big) \int_{\mathbb{R}^m} \exp\big(-\nu(z - Z(x', 0))^2\big)\chi(x')h(x', 0)d_{x'}Z(x', 0)
\end{aligned} \tag{2.29}$$

where $d_{x'}$ denotes the differential with respect to the x' variables. The calculation of the left hand side of (2.28) will be aided by the following lemma.

Lemma 6. *Let Ω be an open subset of $\mathbb{R}^k$, $k = m+n$ and $(\Omega, \mathcal{V})$ an integrable structure with basic solutions $Z_1, \ldots, Z_m$ defined on Ω. A distribution f on Ω is a solution if and only if $d(fdZ) = 0$ on Ω, where $dZ(u) = dZ_1(u) \wedge \ldots \wedge dZ_m(u)$*

Proof. First assume $f \in \mathcal{C}^1(\Omega)$. Then f is a solution if and only if $df(p) \in \mathcal{V}_p^{\perp}$, which is further equivalent to $df(p)$ lying in the span of $\{dZ_1(p), \ldots, dZ_m(p)\}$. Hence,we can now calculate $d(fdZ) = df \wedge dZ = 0$ and the lemma is proved in the $\mathcal{C}^1$ case.
Without loss of generality, we may shrink Ω if necessary and assume that the Z_j are given by (2.27). Then $dZ_1, \ldots, dZ_m, dy_1, \ldots, dy_n$ form a basis for $\mathbb{C}T^*_{(x,y)}$ at every point $(x, y) \in \Omega$. Hence, we can express df for f $\mathcal{C}^1$ as

$$df = \sum_{j=1}^{m} (R_j f) dZ_j + \sum_{l=1}^{n} (S_l f) dy_l, \tag{2.30}$$

where R_j, $1 \leq j \leq m$ and S_l, $1 \leq l \leq n$ are vector fields. From the above arguments it follows that f is a solution if and only if $S_l f = 0$ for all $l = 1, \ldots, n$, i.e. if the vector fields S_l form a basis for the sections of $\mathcal{V}$ on Ω. Now, (2.30) also applies to a distribution f on Ω, and so f is a solution if and only if df is a linear combination of the dZ_j, this time with distribution coefficients. To finish the proof we can calculate

$$d(fdZ) = df \wedge dZ = \sum_{l=1}^{n} (S_l f) dy_l \wedge dZ$$

and we see that f is a solution if and only if $d(fdZ) = 0$. $\square$

We proceed with the calculation of the left hand side of (2.28). $\exp\{-\nu(z - Z(x', y'))^2\}$ is a holomorphic function of $Z(x', y')$ and the product of two solutions is again a solution. Thus, Lemma 6 eliminates most terms arising from the product rule and we are left with

$$d\alpha_\nu(x', y'; z) = \left(\frac{\nu}{2\pi}\right)^{m/2} \int_{\mathbb{R}^m} \exp\bigl(-\nu(z - Z(x', y'))^2\bigr) h(x', y') d\chi(x') \wedge dZ(x', y'). \tag{2.31}$$

Define a sequence $(H_\nu)_\nu$ of entire functions on $\mathbb{C}^m$ by

$$H_\nu(z) := \left(\frac{\nu}{\pi}\right)^{m/2} \int_{\mathbb{R}^m} \exp\bigl(-\nu(z - Z(x', 0))^2\bigr) \chi(x') h(x', 0) d_{x'} Z(x', 0). \tag{2.32}$$

We will show that H_ν converges uniformly to $h(x, y)$ on K as $\nu \to \infty$. To accomplish this, we prove that the left hand side of (2.28) converges to 0 when evaluated at $z = Z(x, y)$, uniformly in (x, y). On the other hand, the first integral on the right hand side of (2.29) converges uniformly to $h(x, y)$. This, we prove in the next two lemmas.

Lemma 7. *For r, d chosen sufficiently small and any continuous function f on Ω,*

$$\int_{D_y} \exp\bigl(-\nu(Z(x, y) - Z(x', y'))^2\bigr) f(x', y') d\chi(x') \wedge dZ(x', y') \to 0 \tag{2.33}$$

uniformly in $(x, y) \in K$ as $\nu \to \infty$.

Proof. $d\chi(x) = 0$ for $|x| \le \frac{r}{2}$ and $|x| > r$, so the integral in (2.33) is evaluated over the set $\{(x', y') : \frac{r}{2} \le |x'| \le r, y' \in [0, y]\}$. The exponential term is the only one containing ν so we need to estimate $\big|\exp\big(-\nu((Z(x,y) - Z(x',y'))^2)\big)\big|$. Using (2.27) we get

$$\operatorname{Re}(Z(x,y) - Z(x',y'))^2 = (x - x')^2 - (\varphi(x,y) - \varphi(x',y'))^2 \tag{2.34}$$

By the mean value theorem,

$$\big|\varphi(x,y) - \varphi(x',y')\big| \le \big|\varphi(x,y) - \varphi(x',y)\big| + \big|\varphi(x',y) - \varphi(x',y')\big| \le a\big|x - x'\big| + A\big|y - y'\big|,$$

with

$$a := \sup_{|x'| \le r, |y'| \le d} \big|\varphi_{x'}(x',y')\big|, \qquad \sup_{|x'| \le r, |y'| \le d} \big|\varphi_{y'}(x',y')\big|$$

By assumption $\varphi_{x'}(0) = 0$, so we can choose r, d sufficiently small such that $|a| \le \frac{1}{8}$. Then, we can shrink d further to also get $d \le \frac{r}{32A}$. Then, since $|x - x'| \ge \frac{r}{4}$,

$$\big|\exp\big(-\nu(Z(x,y) - Z(x',y'))^2\big)\big| \le \exp\left(-\nu \frac{53r^2}{1024}\right),$$

the lemma follows. $\square$

Lemma 8. *For r, d sufficiently small, f a continuous function on Ω,*

$$\left(\frac{\nu}{\pi}\right)^{m/2} \int_{x' \in \mathbb{R}^m} \exp\big(-\nu(Z(x,y) - Z(x',y))^2\big) f(x',y)\chi(x') d_{x'}Z(x',y) \to f(x,y)$$

uniformly in $(x,y) \in K$ as $\nu \to \infty$.

Proof. First, note that $d_{x'}Z(x',y) = \det(Z_{x'}(x',y))dx'$. Making the change of variables $\sqrt{\nu}(x' - x) = \xi$, the integral becomes

$$\begin{aligned} \frac{1}{\pi^{m/2}} \int_{\mathbb{R}^m} &\exp\left(-\left(\xi + i\nu^{1/2}\left(\varphi\left(x + \frac{\xi}{\sqrt{\nu}}, y\right) - \varphi(x,y)\right)\right)^2\right) \\ &\chi\left(x + \frac{\xi}{\sqrt{\nu}}\right) f\left(x + \frac{\xi}{\sqrt{\nu}}, y\right) \det\left(Z_x\left(x + \frac{\xi}{\sqrt{\nu}}, y\right)\right) d\xi. \end{aligned} \tag{2.35}$$

Again, choosing r and d sufficiently small such that

$$a = \sup_{|x| \le r, |y| \le d} |\varphi_x(x,y)| \le 1/2.$$

Then the integral (2.35) converges to

$$\frac{1}{\pi^{m/2}} \int_{\mathbb{R}^m} \exp\big(-(\xi + i\varphi_x(x,y)\xi)^2\big) f(x,y) \det(Z_x(x,y)) d\xi,$$

uniformly in $(x,y) \in K$ as $\nu \to \infty$. This, we can compute using the following lemma:

Lemma 9. *Let B be a real $m \times m$ matrix with norm less than* 1*. Set $A = I + iB$, then*

$$\frac{\det(A)}{\pi^{m/2}} \int_{\mathbb{R}^m} e^{-(A\xi)^2} d\xi = 1$$

Proof. $|B\xi| \leq \|B\||\xi|$, for all $\xi \in \mathbb{R}^m$, thus the integrand can be bounded by $e^{(1-\|B\|)\xi^2}$, so the integral is finite and we can make a change of variables $\xi' = A\xi$, since A is invertible by a Neumann series. We are then left with a Gaussian integral $\frac{1}{\pi^{m/2}} \int_{\mathbb{R}^m} e^{-(\xi')^2} d\xi'$, which can be evaluated to 1 in a multitude of ways (e.g. squaring the integral and factoring the integrand reduces to the two dimensional case which is easily handled using polar coordinates). □

This completes the proof of Lemma 8 □

We have now established that the entire functions H_ν defined by (2.32) satisfy $H_\nu(x, y) \to h(x, y)$ uniformly in $(x, y) \in K$. Now we can truncate the power series of the H_ν to get polynomials that converge uniformly to each of the H_ν on K, from these polynomials we can extract a diagonal sequence converging uniformly to h. Consequently, we have proved the theorem for continuously differentiable h. To extend to the continuous case, the only step that needs justification is the application of Stokes' theorem. (2.28) is still true in this case, even if coefficients of the forms are not necessarily $\mathcal{C}^1$, since neither side of (2.28) contains derivatives of h by Lemma 6. We have now finished the proof of Theorem BT. □

Theorem BT relates to hypersurfaces in $\mathbb{C}^{n+1}$ in the following way: Let $\mathbb{C}T\mathbb{C}^{n+1}$ be the complexified tangent bundle of $\mathbb{C}^{n+1}$. Taking $(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$ as coordinates of $\mathbb{C}^{n+1}$, we set

$$\frac{\partial}{\partial Z_j} := \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}, \quad \frac{\partial}{\partial \overline{Z}_j} := \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}$$

producing a basis $\{\frac{\partial}{\partial Z_j}|_p, \frac{\partial}{\partial \overline{Z}_j}|_p\}_{j=1,\ldots,n+1}$ for $\mathbb{C}T_p\mathbb{C}^{n+1}$. Setting $T^{1,0}_p\mathbb{C}^{n+1} = \text{span}_{\mathbb{C}}\{\frac{\partial}{\partial Z_j}|_p\}$ and $T^{0,1}_p\mathbb{C}^{n+1} = \text{span}_{\mathbb{C}}\{\frac{\partial}{\partial \overline{Z}_j}|_p\}$ we obtain a direct sum decomposition $\mathbb{C}T\mathbb{C}^{n+1} = T^{1,0}\mathbb{C}^{n+1} \oplus T^{0,1}\mathbb{C}^{n+1}$ of $\mathbb{C}T\mathbb{C}^n + 1$ into holomorphic and antiholomorphic vectors respectively. The complexified tangent bundle $\mathbb{C}TS$ of a hypersurface S inherits this direct sum decomposition from the ambient space and we can set $\mathcal{V}_p := \mathbb{C}T_pS \cap T^{0,1}\mathbb{C}^{n+1}$ for the space of antiholomorphic tangent vectors and define the CR bundle of S as the bundle with fiber $\mathcal{V}_p$ at $p \in S$.

Let $S, S' \subset \mathbb{C}^{n+1}$ be hypersurfaces with CR bundles $\mathcal{V}$ and $\mathcal{V}'$ respectively. A $\mathcal{C}^k$ map $H : S \to S'$ is CR if $T_pH(\mathcal{V}_p) \subset \mathcal{V}'_p$.

It is then possible to show that BT applies to CR functions between hypersurfaces in $\mathbb{C}^{n+1}$(see Proposition 2.1.5. of [1]).

We can now prove an extension result for CR functions on hypersurfaces, this uses the concept of analytic disks. For a hypersurface S in $\mathbb{C}^{n+1}$ we write $\mathcal{D}(S)$ for the analytic disks attached to S, i.e. $A \in \mathcal{D}(S)$ if $A : \overline{\Delta} \to \mathbb{C}^{n+1}$ is $\mathcal{C}^{k,\alpha}$, holomorphic on Δ and $A(b\Delta) \subset S$.

Proposition 1. *Let S be a hypersurface in $\mathbb{C}^{n+1}$ and $p_0 \in S$. Then there exists an open neighbourhood U of p_0 such that for any $\mathcal{C}^{k,\alpha}$ CR function f on U there exists a continuous function F defined on $W := \cup_{A \in \mathcal{D}(U)} A(\overline{\Delta})$ that fulfil for any $A \in \mathcal{D}(U)$*

- *$F \circ A$ is holomorphic on Δ*
- *$F(p) = f(p)$ for all $p \in U$*
- *F is holomorphic on the interior of W*

Proof. We choose U sufficiently small such that Theorem BT applies, i.e. such that for any CR function on U there exists a sequence of polynomials $p_\nu(Z)$ converging uniformly to f on U. The constant disk $A \equiv p$ for $p \in U$ lies in $\mathcal{D}(U)$, hence $U \subset W$. Now define F as follows: for $Z \in W$ choose $A \in \mathcal{D}(U)$ such that $Z \in A(\overline{\Delta})$. By construction, $p_\nu \circ A$ converges uniformly on $b\Delta$ and, by the Maximum Principle, also on $\overline{\Delta}$. Thus, we can set $F(Z) := \lim_{\nu \to \infty} p_\nu(Z)$ and we see that F is independent of the choice of A. F is also independent of the approximating sequence $(p_\nu)_\nu$ and the holomorphicity properties follow since the uniform limit of holomorphic functions is holomorphic. $\square$

3 A Jet Determination Result

In this chapter we apply the material to prove a finite jet determination result for CR-diffeomorphisms of hypersurfaces in $\mathbb{C}^{n+1}$. We follow the paper [4] and adopt the notation from there.

3.1 Stationary Disks

Let S be a finitely smooth hypersurface in $\mathbb{C}^{n+1}$, defined in a neighbourhood of 0 by its defining function r, we assume $0 \in S$. We will call functions $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ analytic disks and say that they are attached to S if $f(\zeta) \in S$ for all $\zeta \in b\Delta$.

Defiition 2. *A holomorphic disk $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ attached to $S = \{r = 0\}$ is called k_0-stationary if there exists a continuous map $c : b\Delta \to \mathbb{R} \setminus \{0\}$ such that $b\Delta \ni \zeta \mapsto \zeta^{k_0} c(\zeta)\partial r(f(\zeta))$ extends to a map in $(\mathcal{A}^{k,\alpha})^{n+1}$.*

Proposition 2. *Let $S \subset \mathbb{C}^{n+1}$ be a hypersurface as in Definition 2. Then there exists a neighbourhood U of 0 in $\mathbb{C}^{n+1}$ such that if H is a $\mathcal{C}^{k+1}$ CR diffeomorphism sending $S \cap U$ to a real hypersurface $S' \subset \mathbb{C}^{n+1}$ and $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ is a k_0-stationary disk attached to $S \cap U$ then $H \circ f \in (\mathcal{A}^{k,\alpha})^{n+1}$ gives a k_0-stationary disk attached to S'.*

Proof. Following the notation of Proposition 1, let $W = \bigcup \varphi(b\Delta)$ where φ ranges over all analytic disks attached to S. By the same result, H extends to a holomorphic map $\tilde{H}$ on W continuous up to $W \cap S$. Since $f(\overline{\Delta}) \subset W$, $H \circ f \in (\mathcal{C}^{k,\alpha}_{\mathbb{C}})^{n+1}$ extends analytically as $\tilde{H} \circ f \in (\mathcal{A}^{k,\alpha})^{n+1}$. To see that this disk is k_0-stationary, we note that $r \circ H^{-1}$ gives a defining function for S' and calculate

$$\zeta^{k_0} c(\zeta)\partial(r \circ H^{-1})(H \circ f(\zeta)) = \zeta^{k_0} c(\zeta)\partial r(f(\zeta))(\partial H(f(\zeta))^{-1}$$

where c is provided by the k_0-stationarity of f attached to S. $(\partial H(f(\zeta))^{-1} = (\partial \tilde{H}(f(\zeta))^{-1}$ so we have that the above expression extends analytically as desired. $\square$

There is also a geometric description of k_0-stationary disks

Defiition 3. *A holomorphic disk $f \in (\mathcal{A}^{k,\alpha})^{n+1}$ attached to $S = \{r = 0\}$ is k_0-stationary if there exists a holomorphic lift $\boldsymbol{f} = (f, \tilde{f})$ of f to the cotangent bundle $T^*\mathbb{C}^{n+1}$, which is continuous up to the boundary and such that for all $\zeta \in b\Delta$, $\boldsymbol{f}(\zeta) \in \mathcal{N}^{k_0}S(\zeta)$, where*

$$\mathcal{N}^{k_0}S(\zeta) := \{(z, w, \tilde{z}, \tilde{w}) \in T^*\mathbb{C}^{n+1} | (z, w) \in S, (\tilde{z}, \tilde{w}) \in \zeta^{k_0} N_z^* S \setminus \{0\}\}, \tag{3.1}$$

here $N_z^* S = \mathrm{span}_{\mathbb{R}}\{\partial r(z)\}$

3.2 The model surface

The jet determination result we will prove applies to a families of hypersurfaces which are perturbations of polynomial hypersurfaces. More specifically, we consider here generalizations of homogeneous polynomials: A (real) polynomial $P : \mathbb{C}^n \to \mathbb{C}$ is called weighted homogeneous with weight $M = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and weighted degree $d \in \mathbb{N}$ if for all $t \in \mathbb{R}$ and all $z \in \mathbb{C}^n$ we have

$$P(t^{m_1} z_1, \ldots, t^{m_n} z_n, t^{m_1} \overline{z}_1, \ldots, t^{m_n} \overline{z}_n) = t^d P(z, \overline{z}). \tag{3.2}$$

As an abbreviation we will write $t^M z = (t^{m_1} z_1, \ldots, t^{m_n} z_n)$, with this, the condition reads $P(t^M z, t^M \overline{z}) = t^d P(z, \overline{z})$. Evidently, a weighted homogeneous polynomial with weight $(1, \ldots, 1)$ is just a homogeneous polynomial.
For multi-indices $M = (m_1, \ldots, m_n), J = (j_1, \ldots, j_n)$ we write

$$M \cdot J = \sum_{l=1}^{n} m_l j_l.$$

Now we fix a weight vector $M = (m_1, \ldots, m_n)$ and a real-valued weighted homogeneous polynomial P with weight M and weighted degree d. With this notation we can write P in the form

$$P(z, \overline{z}) = \sum_{\substack{M\cdot J + M\cdot K = d \\ d-k_0 \le M\cdot J \le k_0}} a_{JK} z^J \overline{z}^K = \sum_{d-k_0}^{k_0} \underbrace{\left(\sum_{\substack{M\cdot J + M\cdot K = d \\ M\cdot K = l}} a_{JK} z^J \overline{z}^K \right)}_{=:P^{d-l,l}(z,\overline{z})} \tag{3.3}$$

where $\frac{d}{2} \le k_0 \le d-1$ is the largest number for which there exist multi-indices $\tilde{J}, \tilde{K}$ with $M \cdot \tilde{K} = k$ satisfying $\alpha_{\tilde{J}\tilde{K}} \neq 0$. The $P^{d-l,l}$ are the "bihomogeneous" components of P, satisfying $P^{d-l,l}(t^M z, s^M \overline{z} = t^{d-l} s^l P^{d-l,l}(z, \overline{z})$. Since P is assumed to be real-valued, we have that $\alpha_{JK} = \overline{\alpha}_{K,J}$ for all multi-indices J, K, and also, that $P^{d-l,l}(z, \overline{z}) = \overline{P}^{l,d-l}(\overline{z}, z)$.
We define the model hypersurface $S_P = \{\rho = 0\} \subset \mathbb{C}^{n+1}$ where the defining function ρ is given by

$$\rho(z, w) = -\operatorname{Re} w + P(z, \overline{z}) = -\operatorname{Re} z + \sum_{\substack{M\cdot J + M\cdot K = d \\ d-k_0 \le M\cdot J \le k_0}} a_{JK} z^J \overline{z}^K \tag{3.4}$$

For $v \in \mathbb{C}^n$ define the analytic disk $h^v : \Delta \to \mathbb{C}^n$

$$h^v(\zeta) := (1-\zeta)^M = ((1-\zeta)^{m_1} v_1, \ldots, (1-\zeta)^{m_n} v_n)$$

Then, for any disk (h^v, g^v) that is attached to S, the Levi Form takes the following form (cf. Prop. 2.2.10 in [1]):

$$P_{z\overline{z}}(h^v(\zeta), \overline{h^v(\zeta)}) = \begin{pmatrix} P_{z_1\overline{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \ldots & P_{z_1\overline{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \\ \vdots & \ddots & \vdots \\ P_{z_n\overline{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \ldots & P_{z_n\overline{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \end{pmatrix}.$$

Later on, expressions of the form $\zeta^{k_0} P_{z_i\overline{z}_j}(h^v(\zeta), \overline{h^v(\zeta)})$ will appear, we calculate

$$\begin{aligned}
\zeta^{k_0} P_{z_i\overline{z}_j}(h^v(\zeta), \overline{h^v(\zeta)}) &= \sum_{l=d-k_0}^{k_0} \zeta^{k_0} P^{d-l,l}_{z_i\overline{z}_j}((1-\zeta)^M v, (1-\overline{\zeta})^M \overline{v}) \\
&= \sum_{l=d-k_0}^{k_0} (1-\zeta)^{d-l-m_i}(1-\overline{\zeta})^{l-m_j} \zeta^{k_0} P^{d-l,l}_{z_i\overline{z}_j}(v, \overline{v}) \\
&= (1-\zeta)^{d-m_i-m_j} \sum_{l=d-k_0}^{k_0} (-1)^{l-m_j} \zeta^{k_0-l+m_j} P^{d-l,l}_{z_i\overline{z}_j}(v, \overline{v}) \\
&= (1-\zeta)^{d-m_i-m_j} \zeta^{k_0} P_{z_i\overline{z}_j}(v, (-\overline{\zeta})^M \overline{v})
\end{aligned}$$

and

$$\begin{aligned}
\zeta^{k_0} P_{z_i z_j}(h^v(\zeta), \overline{h^v(\zeta)}) &= \sum_{l=d-k_0}^{k_0} \zeta^{k_0} P^{d-l,l}_{z_i z_j}((1-\zeta)^M v, (1-\overline{\zeta})^M \overline{v}) \\
&= \sum_{l=d-k_0}^{k_0} (1-\zeta)^{d-l-m_i-m_j}(1-\overline{\zeta})^{l} \zeta^{k_0} P^{d-l,l}_{z_i z_j}(v, \overline{v}) \\
&= (1-\zeta)^{d-m_i-m_j} \sum_{l=d-k_0}^{k_0} (-1)^{l} \zeta^{k_0-l} P^{d-l,l}_{z_i z_j}(v, \overline{v}) \\
&= (1-\zeta)^{d-m_i-m_j} \zeta^{k_0} P_{z_i z_j}(v, (-\overline{\zeta})^M \overline{v}).
\end{aligned}$$

Thus, we can set

$$\begin{aligned}
\zeta^{k_0} P_{z_i\overline{z}_j}(h^v(\zeta), \overline{h^v(\zeta)}) &= (1-\zeta)^{d-m_i-m_j} Q^v_{i\overline{j}}(\zeta) \\
\zeta^{k_0} P_{z_i z_j}(h^v(\zeta), \overline{h^v(\zeta)}) &= (1-\zeta)^{d-m_i-m_j} S^v_{ij}(\zeta),
\end{aligned}$$

with $Q^v_{i\overline{j}}$ and S^v_{ij} holomorphic polynomials. $Q^v_{i\overline{j}}$ has degree at most $2k_0 - d + m_j$ and S^v_{ij} has degree at most $2k_0 - d$, furthermore, $Q^v_{i\overline{j}}$ is divisible by ζ^{m_j} : Comparing the condition $M \cdot J \leq k_0$ in the sum from (3.3) with the definition $Q^v_{i\overline{j}}(\zeta) = \zeta^{k_0} P_{z_i\overline{z}_j}(v, (-\overline{\zeta})^M \overline{v})$ we see that the corresponding terms here satisfy $M \cdot (M - e_j) \leq k_0 - m_j$. Hence, the $(-\overline{\zeta})^M$ can only cancel a $(k_0 - m_j)-$th power from the outside ζ^{k_0}. This is actually an essential ingredient in a later proof.

Defiition 4. *v is admissable for P if there exists a function $g^v : \overline{\Delta} \to \mathbb{C}$ such that for $f^v = (h^v, g^v)$ we have:*

- *$f^v(b\Delta) \subset S_P$, but $f^v(\Delta) \not\subseteq S_P$*
- $$Q^v(\zeta) = \det \begin{pmatrix} Q_{1\overline{1}}(\zeta) & \dots & Q_{1\overline{n}}(\zeta) \\ \vdots & \ddots & \vdots \\ Q_{n\overline{1}}(\zeta) & \dots & Q_{n\overline{n}}(\zeta) \end{pmatrix} \neq 0 \tag{3.5}$$

  for $\zeta \in b\Delta$.

In the above definition, $f^v(b\Delta) \subset S_P$, uniquely determines g^v and it can be explicitly computed, we will do this later. A more general method for attaching analytic disks is given in Chapter VI of [1].

Lemma 10. *Assume that S_P is generically Levi-nondegenerate and that the set of Levi-degenerate points $\Sigma_P := \{(z,w) \in S_P | \det P_{z\bar{z}}(z,\bar{z}) = 0\}$ does not have any branches of dimension $2n-1$ near 0. Then there exists an admissible vector v for P.*

Proof. First we show that for an open, dense subset of v's. Q^v only vanishes at 1.

$$\begin{aligned}(1-\zeta)^{nd-2|M|}Q^v(\zeta) &= \zeta^{nk_0} \det \begin{pmatrix} P_{z_1\bar{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \dots & P_{z_1\bar{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \\ \vdots & \ddots & \vdots \\ P_{z_n\bar{z}_1}(h^v(\zeta), \overline{h^v(\zeta)}) & \dots & P_{z_n\bar{z}_n}(h^v(\zeta), \overline{h^v(\zeta)}) \end{pmatrix} \\ &=: \zeta^{nk_0} D^v(\zeta, \bar{\zeta}),\end{aligned}$$

so $Q^v(\zeta) = 0$ for $\zeta \neq 1$ if and only if $(h^v(\zeta), \operatorname{Re} P(h^v(\zeta), \overline{h^v(\zeta)}) \in \Sigma_P$ is a Levi-degenerate point. Assuming towards a contradiction that there exists an open set of v's such that for each of them we have a $\zeta = \zeta_v$ with $D(\zeta_v) = 0$. Then we can pass to a smooth point of the real-algebraic variety Σ_P to see that its dimension would have to be a at least $2n-1$, contradicting our assumption on Σ_P.
Secondly, we claim that if v satisfies $D^v(i,-i) \neq 0$ then $Q^v(1) \neq 0$. For this we employ the coordinate change $\zeta = \frac{i-t}{i+t}$, where t is taken from the upper half plane. With this, we have that $(1-\zeta) = 2it + O(t^2)$, and that the boundary $b\Delta$ corresponds to $\mathbb{R}$, hence

$$\begin{aligned}D^v(\zeta, \bar{\zeta}) &= D^v(2it + O(t^2), -2it + O(t^2)) \\ &= (2t)^{nd-2|M|} D^v(i,-i) + O(t^{nd-2|M|}).\end{aligned}$$

This implies that $Q^v(1) = D^v(i,-i) \neq 0$. the set of all v's such that $D^v(i,-i) \neq 0$ is by assumption open and dense. Thirdly, we claim that the set of v's for which $h^v(\Delta) \nsubseteq S_P$ is a subset of the set of v's for which $P(v,\bar{v}) \neq 0$. Assume $f^v(\Delta) \subset S_P$, then can deduce that $g(\zeta) = 0$ for $\zeta \in \Delta$. Then we also get $P((1-\zeta)^M v, (1-\bar{\zeta})^M \bar{v}) = 0$ in Δ so $P(v,\bar{v}) = 0$.
To conclude, we have seen that admissable vectors have to lie in the three open dense sets that we have discussed and their intersection is obviously nonempty. □

From now on, we will fix an admissable vector v and set

$$f^0 := (h^v, g^0) = ((1-\zeta)^{m_1} v_1, \dots, (1-\zeta)^{m_n} v_n, g^0). \tag{3.6}$$

f^0 is a k_0-stationary disk attached to S_P satisfying $f^0(1) = 0$.

3.3 Deformations

Next we will parametrize a space of allowed higher order deformations of a model surface S_P.

3.3.1 Space of allowed deformations

Let $S_P = \{\rho = 0\}$ be a model surface of the form (3.4), $k > 0$ an integer. Choose $\delta > 0$ large enough such that the polydisc $\delta\Delta^{n+1} \subset \mathbb{C}^{n+1}$ contains $f^0(\overline{\Delta})$, where f^0 is the disc given by (3.6). We consider the affine Banach space X of functions $r \in \mathcal{C}^{k+3}\left(\overline{\delta\Delta^{n+1}}\right)$ which can be written as

$$r(z,w) = \rho(z,w) + \theta(z, \operatorname{Im} w)$$

with

$$\theta(z, \operatorname{Im} w) = \sum_{M\cdot J + M\cdot K = d+1} (z^J \overline{z}^K) r_{JK0}(z) + \sum_{l=1}^{d} \sum_{M\cdot J+M\cdot K=d-1} z^J \overline{z}^K (\operatorname{Im} w)^l \cdot r_{JKl}(z, \operatorname{Im} w) \quad (3.7)$$

where $r_{JK0} \in \mathcal{C}^{k+3}_{\mathbb{C}}\left(\overline{\delta\Delta^n}\right)$ and $r_{JKl} \in \mathcal{C}^{k+3}_{\mathbb{C}}\left(\overline{\delta\Delta^n} \times [-\delta, \delta]\right)$. We equip X with the following norm

$$\|r\|_X = \sup \|r_{JKl}\|_{\mathcal{C}^{k+3}}$$

so that X is isomorphic to a real, closed subspace of a suitable power of $\mathcal{C}^{k+3}_{\mathbb{C}}\left(\overline{\delta\Delta^n} \times [-\delta, \delta]\right)$ and, hence is a Banach space.

3.3.2 Defining equations for $\mathcal{N}^{k_0} S_P$

Let

$$S_P = \{\rho = 0\} = \{-\operatorname{Re} w + P(z, \overline{z}) = 0\} \subset \mathbb{C}^{n+1}$$

be the model hypersurface from (3.4). We will now give defining equations for the submanifold $\mathcal{N}^{k_0} S_P(\zeta) \subset \mathbb{C}^{2n+2}$ (see (3.1)) for each $\zeta \in b\Delta$. By definition, we have

$$(z, w, \tilde{z}, \tilde{w}) \in \mathcal{N}^{k_0} S_P(\zeta) \quad \Leftrightarrow \quad \begin{cases} \rho(z,w) = 0 \\ \\ \text{there exists } \; c : b\Delta \to \mathbb{R} \setminus \{0\} \text{ such that} \\ (\tilde{z}, \tilde{w}) = \zeta^{k_0} c(\zeta) \left(P_z(z, \overline{z}), -\frac{1}{2}\right). \end{cases}$$

Reading the $\tilde{w}$ component in the above, we have that $-2\tilde{w} = \zeta^{k_0} c(\zeta)$, so we can eliminate the function c and get the equivalent form

$$\Leftrightarrow \quad \begin{cases} \rho(z,w) = 0 \\ \\ \dfrac{\tilde{w}}{\zeta^{k_0}} \in \mathbb{R} \\ \\ \tilde{z}_i + 2\tilde{w} P_{z_i}(z, \overline{z}) = 0 \;\; \text{for } 1 \le i \le n. \end{cases}$$

By splitting these into real and imaginary part, we get $2n+2$ real defining equations for

$\mathcal{N}^{k_0} S_P(\zeta) \subset \mathbb{C}^{2n+2}$:

$$\begin{cases} \tilde{\rho}_1(\zeta)(z,w,\tilde{z},\tilde{w}) = -\operatorname{Re} w + P(z,\overline{z}) = 0 \\ \tilde{\rho}_2(\zeta)(z,w,\tilde{z},\tilde{w}) = (\tilde{z}_1 + 2\tilde{w}P_{z_1}(z,\overline{z})) + \left(\overline{\tilde{z}_1 + 2\tilde{w}P_{z_1}(z,\overline{z})}\right) = 0 \\ \tilde{\rho}_3(\zeta)(z,w,\tilde{z},\tilde{w}) = i\,(\tilde{z}_1 + 2\tilde{w}P_{z_1}(z,\overline{z})) - i\left(\overline{\tilde{z}_1 + 2\tilde{w}P_{z_1}(z,\overline{z})}\right) = 0 \\ \qquad \vdots \\ \tilde{\rho}_{2n}(\zeta)(z,w,\tilde{z},\tilde{w}) = (\tilde{z}_n + 2\tilde{w}P_{z_n}(z,\overline{z})) + \left(\overline{\tilde{z}_n + 2\tilde{w}P_{z_n}(z,\overline{z})}\right) = 0 \\ \tilde{\rho}_{2n+1}(\zeta)(z,w,\tilde{z},\tilde{w}) = i\,(\tilde{z}_n + 2\tilde{w}P_{z_n}(z,\overline{z})) - i\left(\overline{\tilde{z}_n + 2\tilde{w}P_{z_n}(z,\overline{z})}\right) = 0 \\ \tilde{\rho}_{2n+2}(\zeta)(z,w,\tilde{z},\tilde{w}) = i\dfrac{\tilde{w}}{\zeta^{k_0}} - i\zeta^{k_0}\overline{\tilde{w}} = 0. \end{cases}$$

We set

$$\tilde{\rho} := (\tilde{\rho}_1, \ldots, \tilde{\rho}_{2n+2}).$$

3.3.3 k_0-stationary disks attached to deformations

We introduce the following space to measure (lifts of) k_0-stationary disks in:

$$Y^{M,d} := \prod_{i=1}^{n} \left(\mathcal{A}^{k,\alpha}_{0^{m_i}}\right) \times \mathcal{A}^{k,\alpha}_0 \times \prod_{i=1}^{n} \left(\mathcal{A}^{k,\alpha}_{0^{d-m_i}}\right) \times \mathcal{A}^{k,\alpha}. \tag{3.8}$$

For a real hypersurface $S = \{r = 0\}$ with r in the space of allowed deformations X, we denote by $\mathcal{S}^{k_0,r}$ the set of lifts $\boldsymbol{f} \in Y^{M,d}$ of k_0-stationary disks attached to S. The disk (3.6) has the lift

$$\boldsymbol{f^0} = (h^0, g^0, \tilde{h}^0, \tilde{g}^0) = ((1-\zeta)^{m_1} v_1, \ldots, (1-\zeta)^{m_n} v_n, g^0, \tilde{h}^0, -\zeta^{k_0}/2) \in Y^{M,d}$$

where $\tilde{h}^0(\zeta) = \zeta^{k_0} P_z(h^0, \overline{h^0})$.

Theorem 8. *There exists an integer N, open neighborhoods $\rho \in V \subset X$ and $0 \in U \subset \mathbb{R}^N$, a real number $\eta > 0$ and a map*

$$\mathcal{F} : V \times U \to Y^{M,d}$$

of class $\mathcal{C}^1$ such that:

- $\mathcal{F}(\rho, 0) = \boldsymbol{f^0}$,
- *for all $r \in V$ the map*

$$\mathcal{F}(r,\cdot) : U \to \{\boldsymbol{f} \in \mathcal{S}^{k_0,r} | \left\|\boldsymbol{f} - \boldsymbol{f^0}\right\|_{Y^{M,d}} < \eta\}$$

  is one-to-one and onto.

Proof. Define the following map between Banach spaces

$$\mathcal{H} : X \times Y^{M,d} \to \mathcal{C}_0^{k,\alpha} \times \prod_{i=1}^{n} \left(\left(\mathcal{C}_{0^{d-m_i}}^{k,\alpha} \right)^2 \right) \times \mathcal{C}^{k,\alpha}$$

by

$$\mathcal{H}(r, \boldsymbol{f}) := \tilde{r}(\boldsymbol{f}).$$

Since $\boldsymbol{f} \in Y^{M,d}$ is a k_0-stationary disk attached to $S = \{r = 0\}$ if and only if $\tilde{r}(\boldsymbol{f}) = 0$, we have that for each $r \in X$, the zero set of $\mathcal{H}(\tilde{r}, \cdot)$ is exactly $\mathcal{S}^{k_0,r}$. Thus, the theorem follows by applying a version of the implicit function theorem to $\mathcal{H}$.
The partial derivative of $\mathcal{H}$ with respect to $Y^{M,d}$ at $(\rho, \boldsymbol{f^0})$ is

$$\boldsymbol{f'} \mapsto 2\,\mathrm{Re}(\overline{G(\zeta)}\boldsymbol{f'}) \tag{3.9}$$

where the matrix

$$G(\zeta) := \left(\tilde{\rho}_{\overline{z}}(\boldsymbol{f^0}), \tilde{\rho}_{\overline{w}}(\boldsymbol{f^0}), \tilde{\rho}_{\overline{\overline{z}}}(\boldsymbol{f^0}), \tilde{\rho}_{\overline{\overline{w}}}(\boldsymbol{f^0})\right) \in M_{2n+2}(\mathbb{C})$$

is given by

$$G(\zeta) = \begin{pmatrix} P_{\overline{z}_1}(h^0, \overline{h^0}) & \dots & P_{\overline{z}_n}(h^0, \overline{h^0}) & -1/2 & 0 & \cdots & 0 & 0 \\ & & & 0 & 1 & \ddots & 0 & \overline{2P_{z_1}(h^0, \overline{h^0})} \\ & & & 0 & -i & \ddots & 0 & \overline{-2iP_{z_1}(h^0, \overline{h^0})} \\ & B(\zeta) & & 0 & 0 & \ddots & 0 & \overline{2P_{z_2}(h^0, \overline{h^0})} \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots \\ & & & 0 & 0 & \ddots & -i & \overline{-2iP_{z_n}(h^0, \overline{h^0})} \\ 0 & \cdots & 0 & 0 & 0 & \ddots & 0 & -i\zeta^{k_0} \end{pmatrix}.$$

Using the notation $d_{\ell j} := d - m_\ell - m_j$, the entries of the $2n \times n$ matrix $B(\zeta)$ are given by

$$B_{2\ell-1,j}(\zeta) = -(1-\zeta)^{d_{\ell j}} \left(Q_{\ell\bar{j}}(\zeta) + \frac{\overline{S}_{\ell j}(\zeta)}{\zeta^{d_{\ell j}}} \right)$$

for odd $1 \leq 2l - 1 \leq 2n - 1$ and

$$B_{2\ell,j}(\zeta) = -i(1-\zeta)^{d_{\ell j}} \left(Q_{\ell\bar{j}}(\zeta) - \frac{\overline{S}_{\ell j}(\zeta)}{\zeta^{d_{\ell j}}} \right)$$

for even $2 \leq 2\ell \leq 2n$.
For our application of the implicit function theorem, we care about the surjectivity of $\boldsymbol{f'} \mapsto 2\,\mathrm{Re}(\overline{G(\zeta)}\boldsymbol{f'})$ and the dimension of its kernel. We can permute the columns of $G(\zeta)$ to obtain the operator

$$L_1 : \mathcal{A}_0^{k,\alpha} \times \prod_{i=1}^{n} \left(\mathcal{A}_{0^{d-m_i}}^{k,\alpha} \times \mathcal{A}_{0^{m_i}}^{k,\alpha} \right) \times \mathcal{A}^{k,\alpha} \to \mathcal{C}_0^{k,\alpha} \times \prod_{i=1}^{n} \left(\left(\mathcal{C}_{0^{d-m_i}}^{k,\alpha} \right)^2 \right) \times \mathcal{C}^{k,\alpha}$$

defined by

$$L_1(g', \tilde{h}'_1, h'_1, \cdots, \tilde{h}'_n, h'_n, \tilde{g}') := 2\,\mathrm{Re}\left[\overline{G_1(\zeta)}(g', \tilde{h}'_1, h'_1, \cdots, \tilde{h}'_n, h'_n, \tilde{g}')\right],$$

where

$$G_1(\zeta) = \begin{pmatrix} -1/2 & & (*) \\ & A(\zeta) & \\ (0) & & -i\zeta^{k_0} \end{pmatrix}$$

and where $A(\zeta)$ is

$$\begin{pmatrix} 1 & B_{1,1}(\zeta) & \dots & 0 & B_{1,n}(\zeta) \\ -i & B_{2,1}(\zeta) & \dots & 0 & B_{2,1}(\zeta) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & B_{2n-1,1}(\zeta) & \dots & 1 & B_{2n-1,n}(\zeta) \\ 0 & B_{2n,1}(\zeta) & \dots & -i & B_{2n,n}(\zeta) \end{pmatrix}.$$

At this point, we have gone from a nonlinear problem to a linear one but we still have the problem that $G(1)$ is singular so we can not apply the machinery we have developed. Now our results concerning different function spaces come in handy. Specifically, for $\varphi \in \mathcal{C}_0^{k,\alpha} \times \prod_{j=1}^n \left(\left(\mathcal{C}^{k,\alpha}_{0^{d-m_j}}\right)^2\right)$ transform the linear system

$$2\,\mathrm{Re}\left(\overline{G_1(\zeta)}(g', \tilde{h}'_1, h'_1, \dots, \tilde{h}'_n, h'_n, \tilde{g}')\right) = \varphi$$

in the following way: We divide the first line by $(1-\zeta)$ and the $(2l-1)-$th and $(2l)-$th lines by $(1-\zeta)^{d-m_l}$ for $l = 1, \dots, n$. Then we multiply the $(2l-1)-$th and $(2l)-$th lines by ζ^{s_l}, where $s_l = (d-m_l)/2$ for $l = 1, \dots, n$. By Lemma 1 these transformations correspond to isomorphisms transforming L_1 to an operator

$$L_2 : \left(\mathcal{A}^{k,\alpha}\right)^{2n+2} \to \mathcal{R}_1 \times (\mathcal{R}_0)^{2n} \times \mathcal{C}^{k,\alpha}$$

which is equivalent to L_1 in the sense that L_1 is surjective with finite dimensional kernel if and only if L_2 is surjective with finite dimensional kernel of the same dimension. We have thus reduced the problem to studying the linear operator

$$L_3 : (\mathcal{A}^{k,\alpha})^{2n} \to (\mathcal{R}_0)^{2n}$$

defined by

$$L_3(\tilde{h}'_1, h'_1, \cdots, \tilde{h}'_n, h'_n) := 2\,\mathrm{Re}\left[\overline{A(\zeta)}(\tilde{h}'_1, -h'_1, \cdots, \tilde{h}'_n, -h'_n)\right]$$

where the corresponding matrix, still denoted by $A(\zeta)$, is

$$\begin{pmatrix} \overline{\zeta}^{s_1} & Q_{1\overline{1}}\zeta^{s_1-m_1} + \overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & Q_{1\overline{n}}\zeta^{s_1-m_n} + \overline{S}_{1n}\overline{\zeta}^{s_1} \\ -i\overline{\zeta}^{s_1} & iQ_{1\overline{1}}\zeta^{s_1-m_1} - i\overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & iQ_{1\overline{n}}\zeta^{s_1-m_n} - i\overline{S}_{1n}\overline{\zeta}^{s_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & Q_{n\overline{1}}\zeta^{s_n-m_1} + \overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & \overline{\zeta}^{s_n} & Q_{n\overline{n}}\zeta^{s_n-m_n} + \overline{S}_{nn}\overline{\zeta}^{s_n} \\ 0 & iQ_{n\overline{1}}\zeta^{s_n-m_1} - i\overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & -i\overline{\zeta}^{s_n} & iQ_{n\overline{n}}\zeta^{s_n-m_n} - i\overline{S}_{nn}\overline{\zeta}^{s_n} \end{pmatrix}.$$

Out of convenience, we set $Q'_{\ell\bar{j}} = Q_{\ell\bar{j}}\zeta^{-m_j}$ and therefore

$$A(\zeta) = \begin{pmatrix} \overline{\zeta}^{s_1} & Q'_{1\bar{1}}\zeta^{s_1} + \overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & Q'_{1\bar{n}}\zeta^{s_1} + \overline{S}_{1n}\overline{\zeta}^{s_1} \\ -i\overline{\zeta}^{s_1} & iQ'_{1\bar{1}}\zeta^{s_1} - i\overline{S}_{11}\overline{\zeta}^{s_1} & \dots & 0 & iQ'_{1\bar{n}}\zeta^{s_1} - i\overline{S}_{1n}\overline{\zeta}^{s_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & Q'_{n\bar{1}}\zeta^{s_n} + \overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & \overline{\zeta}^{s_n} & Q'_{n\bar{n}}\zeta^{s_n} + \overline{S}_{nn}\overline{\zeta}^{s_n} \\ 0 & iQ'_{n\bar{1}}\zeta^{s_n} - i\overline{S}_{n1}\overline{\zeta}^{s_n} & \dots & -i\overline{\zeta}^{s_n} & iQ'_{n\bar{n}}\zeta^{s_n} - i\overline{S}_{nn}\overline{\zeta}^{s_n} \end{pmatrix}.$$

Multiplying the $2l-1-$th row with i and adding it to the $2l-$th row for $l = 1, \dots, n$, we can eliminate all the $-i\overline{\zeta}^{s_l}$'s. Then, expanding along the $2l-1-$th columns for $l = 1, \dots, n$ we get

$$\det A(\zeta) = (2i)^n Q'(\zeta) \tag{3.10}$$

where

$$Q'(\zeta) = \zeta^{-(m_1+\ldots+m_n)} Q(\zeta).$$

We have now transformed the problem into a form where the machinery we have developed in section 2.3 can be applied.

Lemma 11. *The linear operator $L_3 : (\mathcal{A}^{k,\alpha})^{2n} \to (\mathcal{R}_0)^{2n}$ is onto.*

Proof. This follows from the first part of Theorem 5 if we can show that the partial indices of

$$\overline{A^{-1}(\zeta)}A(\zeta) = \frac{1}{\det A(\zeta)} A'(\zeta) = \frac{1}{(2i)^n Q'(\zeta)} A'(\zeta)$$

are all greater than or equal to -1. For $1 \le j, \ell \le 2n$ we denote by $A'_{j\ell}$ the (j,ℓ)-entry of A'. A direct computation gives for $\ell, p = 1, \cdots, n$

$$A'_{2\ell-1,2p} = (-2i)^n \zeta^{s_1+\cdots+s_n-s_\ell} \det \begin{pmatrix} Q'_{l\overline{p}}\zeta^{s_\ell} & S_{l1}\zeta^{s_\ell} & S_{l2}\zeta^{s_\ell} & \cdots & S_{ln}\zeta^{s_\ell} \\ \overline{S}_{1p}\overline{\zeta}^{s_1} & \overline{Q}'_{1\bar{1}}\overline{\zeta}^{s_1} & \overline{Q}'_{1\bar{2}}\overline{\zeta}^{s_1} & \cdots & \overline{Q}'_{1\bar{n}}\overline{\zeta}^{s_1} \\ \overline{S}_{2p}\overline{\zeta}^{s_2} & \overline{Q}'_{2\bar{1}}\overline{\zeta}^{s_2} & \overline{Q}'_{2\bar{2}}\overline{\zeta}^{s_2} & \cdots & \overline{Q}'_{2\bar{n}}\overline{\zeta}^{s_2} \\ \vdots & \vdots & \vdots & \vdots \cdots & \vdots \\ \overline{S}_{np}\overline{\zeta}^{s_n} & \overline{Q}'_{n\bar{1}}\overline{\zeta}^{s_n} & \overline{Q}'_{n\bar{2}}\overline{\zeta}^{s_n} & \cdots & \overline{Q}'_{n\bar{n}}\overline{\zeta}^{s_n} \end{pmatrix}$$

$$= (-2i)^n \det \underbrace{\begin{pmatrix} Q'_{\ell\overline{p}} & S_{\ell 1} & S_{\ell 2} & \cdots & S_{\ell n} \\ \overline{S}_{1p} & \overline{Q}'_{1\bar{1}} & \overline{Q}'_{1\bar{2}} & \cdots & \overline{Q}'_{1\bar{n}} \\ \overline{S}_{2p} & \overline{Q}'_{2\bar{1}} & \overline{Q}'_{2\bar{2}} & \cdots & \overline{Q}'_{2\bar{n}} \\ \vdots & \vdots & \vdots & \vdots \cdots & \vdots \\ \overline{S}_{np} & \overline{Q}'_{n\bar{1}} & \overline{Q}'_{n\bar{2}} & \cdots & \overline{Q}'_{n\bar{n}} \end{pmatrix}}_{:=B_{2\ell-1,2p}}$$

$$= (-2i)^n a'_{2\ell-1,2p}$$

where $a'_{2\ell-1,2p} = \det B_{2\ell-1,2p}$. We use $C_{j,l}(B)$ to denote the $(j,l)-$cofactor of a square matrix B. All entries of the first row of $B_{2l-1,p}$ except the first only depend on l, so we have for all $j = 1, \ldots, n$ and any $p, p' = 1, \ldots, n$:

$$C_{j,1}(B_{2l-1,p}) = C_{j,1}(B_{2l-1,p'}).$$

We write $C_{j,1;l} = C_{j,1}(B_{2l-1,p})$, which is well-defined by the above. Similarly, we get

$$C_{1,j}(B_{2l-1,p}) = C_{1,j}(B_{2l'-1,p})$$

for any $p = 1, \ldots, n$ and any $l, l' = 1, \ldots, n$ which we denote by $C^p_{1,j}$. A computation yields

$$A'_{2l-1,2p-1} = (-2i)^n C_{p+1,1;l}$$

and

$$A'_{2l,2p} = (-2i)^n C^p_{1,l+1}$$

for $l, p = 1, \ldots, n$. Denote by D_{lp} the $n \times n$ matrix obtained by removing the first row and the $(l+1)-$th column of $B_{2l-1,2p}$, namely,

$$D_{\ell p} = \begin{pmatrix} \overline{S}_{1p} & \overline{Q}'_{1\overline{1}} & \overline{Q}'_{1\overline{2}} & \cdots & \overline{Q}'_{1\overline{\ell-1}} & \overline{Q}'_{1\overline{\ell+1}} & \cdots & \overline{Q}'_{1\overline{n}} \\ \overline{S}_{2p} & \overline{Q}'_{2\overline{1}} & \overline{Q}'_{2\overline{2}} & \cdots & \overline{Q}'_{2\overline{\ell-1}} & \overline{Q}'_{2\overline{\ell+1}} & \cdots & \overline{Q}'_{2\overline{n}} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \overline{S}_{np} & \overline{Q}'_{n\overline{1}} & \overline{Q}'_{n\overline{2}} & \cdots & \overline{Q}'_{n\overline{\ell-1}} & \overline{Q}'_{n\overline{\ell+1}} & \cdots & \overline{Q}'_{n\overline{n}} \end{pmatrix}$$

for $l, p = 1, \ldots, n$. Note that

$$\det(D_{lp}) = (-1)^l C^p_{1,l+1}$$

and

$$C_{j,1}(D_{lp}) = C_{j,1}(D_{lp'})$$

which we denote by $c_{j,1;l}$. By a direct computation, we see

$$A'_{2l,2p-1} = (-1)^{l+1}(-2i)^n c_{p,1;l}.$$

Now we have notation for the entries of $A'(\zeta)$:

$$\frac{A'(\zeta)}{(-2i)^n} = \begin{pmatrix} C_{2,1;1} & a'_{1,2} & C_{3,1;1} & a'_{1,4} & \cdots & C_{n+1,1;1} & a'_{1,2n} \\ c_{1,1;1} & C^1_{1,2} & c_{2,1;1} & C^2_{1,2} & \cdots & c_{n,1;1} & C^n_{1,2} \\ C_{2,1;2} & a'_{3,2} & C_{3,1;2} & a'_{3,4} & \cdots & C_{n+1,1,2} & a'_{3,2n} \\ -c_{1,1;2} & C^1_{1,3} & -c_{2,1;2} & C^2_{1,3} & \cdots & -c_{n,1;2} & C^n_{1,3} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ C_{2,1;n} & a'_{2n-1,2} & C_{3,1;n} & a'_{2n-1,4} & \cdots & C_{n+1,1;n} & a'_{2n-1,2n} \\ \frac{c_{11,n}}{(-1)^{n+1}} & C^1_{1,n+1} & \frac{c_{2,1;n}}{(-1)^{n+1}} & C^2_{1,n+1} & \cdots & \frac{c_{n,1;n}}{(-1)^{n+1}} & C^n_{1,n+1} \end{pmatrix}. \quad (3.11)$$

We set C_p for the $p-$th column of $A'(\zeta)$. By performing the column operation

$$C_{2p} \to C_{2p} - \sum_{j=1}^{n} \overline{S}_{jp} C_{2j-1} \tag{3.12}$$

for each $p = 1, \dots, n$ $A'(\zeta)$ transforms into

$$A'(\zeta) \to (-2i)^n \begin{pmatrix} C_{2,1;1} & Q'_{1\bar{1}}\overline{Q'} & C_{3,1;1} & Q'_{1\bar{2}}\overline{Q'} & \cdots & C_{n+1,1;1} & Q'_{l\bar{n}}\overline{Q'} \\ c_{1,1;2} & 0 & c_{2,1;2} & 0 & \cdots & c_{n,1;2} & 0 \\ C_{2,1;2} & Q'_{2\bar{1}}\overline{Q'} & C_{3,1;2} & Q'_{2\bar{2}}\overline{Q'} & \cdots & C_{n+1,1;2} & Q'_{2\bar{n}}\overline{Q'} \\ -c_{1,1;3} & 0 & -c_{2,1;3} & 0 & \cdots & -c_{n,1;3} & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ C_{2,1;n} & Q'_{n\bar{1}}\overline{Q'} & C_{3,1;n} & Q'_{n\bar{2}}\overline{Q'} & \cdots & C_{n+1,1;n} & Q'_{n\bar{n}}\overline{Q'} \\ c\frac{{}_{1,1;n}}{(-1)^{n+1}} & 0 & \frac{c_{2,1;n}}{(-1)^{n+1}} & 0 & \cdots & \frac{c_{n,1;n}}{(-1)^{n+1}} & 0 \end{pmatrix}. \tag{3.13}$$

Now consider the Birkhoff factorization of $\overline{A^{-1}}A$ (cf. section 2.2): Let $\kappa_1 \geq \ldots \geq \kappa_{2n}$ be the partial indices of $\overline{A^{-1}}A$, $\Lambda = \operatorname{diag}(\zeta^{\kappa_1}, \dots, \zeta^{\kappa_{2n}})$, and $\Theta : \overline{\Delta} \to GL_{2n}(\mathbb{C})$ a smooth map , holomorphic on Δ such that

$$\Theta \overline{A^{-1}} A = \Lambda \overline{\Theta}. \tag{3.14}$$

Let $\lambda = (\lambda_1, \mu_1, \dots, \lambda_n, \mu_n)$ be the last row of the matrix Θ. Substituting (3.11) into (3.14) results in the system:

$$\left.\begin{aligned} \sum_{k=1}^{n} C_{j+1,1;k}\lambda_k + \sum_{k=1}^{n} (-1)^{k+1} c_{j,1;k+1}\mu_k &= \overline{Q'}\zeta^{\kappa_{2n}}\overline{\lambda}_j \\ \sum_{k=1}^{n} a'_{2k-1,2j}\lambda_k + \sum_{k=1}^{n} C^{j}_{1,k+1}\mu_k &= \overline{Q}'\zeta^{\kappa_{2n}}\overline{\mu}_j \end{aligned}\right\} j = 1, \dots, n.$$

After performing the column operations (3.12) the second line above reads (see (3.13)):

$$\overline{Q'} \sum_{k=1}^{n} Q'_{k\bar{j}}\lambda_k = \overline{Q'}\zeta^{\kappa_{2n}}\overline{\mu}_j - \sum_{k=1}^{n} \overline{S}_{kj}\overline{Q'}\zeta^{\kappa_{2n}}\overline{\lambda}_k, \quad j = 1, \dots, n$$

Recall that we are working with an admissable vector (3.5), which means that $\overline{Q'}$ does not vanish on $b\Delta$, so we can divide to get

$$\sum_{k=1}^{n} Q'_{k\bar{j}}\lambda_k = \zeta^{\kappa_{2n}}\overline{\mu}_j + \sum_{k=1}^{n} \overline{S}_{kj}\zeta^{\kappa_{2n}}\overline{\lambda}_k, \quad j = 1, \dots, n.$$

In section 3.2, we saw that $Q_{i\bar{j}}$ is divisible by ζ^{m_j}, thus $Q'_{i\bar{j}} = Q_{i\bar{j}}\zeta^{-m_j}$ is holomorphic. Assuming now that $\kappa_{2n} \leq -1$, the right hand side of each of the equations above is antiholomorphic and divisible $\overline{\zeta}$, while the left hand side is holomorphic. This is only possible if both sides vanish, yielding the system

$$\sum_{k=1}^{n} Q'_{k\bar{j}} \lambda_k = 0, \quad 1, \ldots, n.$$

Since the determinant of the system is $Q' \neq 0$, we must then have that λ_j vanishes identically. Inserting this above implies that the μ_j also vanish identically. But the matrix Θ is invertible so its last row $\lambda = (\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n)$ cannot be 0. Hence we must have $\kappa_{2n} \geq 0$, proving Lemma 11. $\square$

Lemma 12. *The kernel of $L_3 : (\mathcal{A}^{k,\alpha})^{2n} \to (\mathcal{R}_0)^{2n}$ has finite real dimension less than or equal to $2n(2k_0 - d) + 2n$.*

Proof. Here we apply the second part of Theorem 5 (with $m = 0$) which states that the kernel of L_3 has real dimension $\kappa + 2n$, where κ is the Maslov index of $\overline{A^{-1}}A$ which is given by (Ind):

$$\operatorname{ind}\det\left(-\overline{A^{-1}}A\right) = \frac{1}{2i\pi}\int_{b\Delta} \frac{\left(\det\left(-\overline{A(\zeta)}^{-1}A(\zeta)\right)\right)'}{\det\left(-\overline{A(\zeta)}^{-1}A(\zeta)\right)} d\zeta.$$

By (3.10), we have

$$\det \overline{A^{-1}}A = (-1)^n \frac{2'(\zeta)}{\overline{Q'(\zeta)}} = (-1)^n \zeta^{-2(m_1+\ldots+m_n)} \frac{Q(\zeta)}{\overline{Q(\zeta)}}$$

Therefore

$$\begin{aligned}
\operatorname{ind}\det\left(-\overline{A^{-1}}A\right) &= -2\sum_{i=1}^{n} m_i + 2\operatorname{ind}Q \\
&\leq -2\sum_{i=1}^{n} m_i + 2\left(n(2k_0 - d) + \sum_{i=1}^{n} m_i\right) = 2n(2k_0 - d).
\end{aligned}$$

$\square$

$\square$

Lemma 13. *Let $S = \{r = 0\} \subset \mathbb{C}^{n+1}$ be an admissable real sufficiently smooth hypersurface. Consider the scaling $\Lambda_t(z, w) = (t^{m_1}z_1, \ldots, t^{m_n}z_n, t^d w)$. For $t > 0$ small enough, the defining function $r_t = \frac{1}{t^d} r \circ \Lambda_t$ lies in the neighbourhood V of Theorem 8*

Proof. Recall that we can write r as $\rho + \theta$ with θ as in 3.7, accordingly we can decompose r_t as

$$r_t = \frac{1}{t^d}\rho \circ \Lambda_t + \frac{1}{t^d}\theta \circ \Lambda_t.$$

First, we note that the “model” part of the above is invariant under the scaling, since P is weighted homogeneous:

$$\frac{1}{t^d}\rho\circ\Lambda_t(z,w)=\frac{1}{t^d}(-\operatorname{Re}t^dw+P(t^Mz,\overline{t^Mz}))=-\operatorname{Re}w+P(z,\overline{z})=\rho(z,w).$$

V is a neighborhood of ρ, so we need to show that the θ part vanishes as $t\to 0$.

$$\begin{aligned}\frac{1}{t^d}\theta\circ\Lambda_t(z,\operatorname{Im}w)&=\frac{1}{t^d}\sum_{M\cdot(J+K)=d+1}t^{d+1}(z^J\overline{z}^K)r_{JK0}(t^Mz)\\&\quad+\frac{1}{t^d}\sum_{l=1}^{d}\sum_{M\cdot(J+K)=d-1}t^{d-l}z^J\overline{z}^Kt^{d\cdot l}(\operatorname{Im}w)^l\cdot r_{JKl}(t^Mz,t^d\operatorname{Im}w)\end{aligned}$$

Now we can estimate $\|tr_{JK0}\|_{\mathcal{C}^{k+3}}\leq t\|r\|_X$ and, since $d-l+d\cdot l\geq d+1$, $\left\|t^{d\cdot l-l}r_{JKl}\right\|_{\mathcal{C}^{k+3}}\leq t\|r\|_X$. Combined, these estimates show that $\|r_t\|_X\leq t\|r\|$, proving the lemma. □

Putting this lemma together with Theorem 8, we get

Theorem 9. *Let $S\subset\mathbb{C}^{n+1}$ an admissable $\mathcal{C}^{d+k+4}$ real hypersurface. There exists a finite-dimensional biholomorphically invariant manifold of small $\mathcal{C}^{k,\alpha}$ k_0-stationary disks attached to S.*

3.4 Finite jet determination

With our theorem concerning the existence of k_0-stationary disks in hand we are now ready to prove the following jet determination result for CR diffeomorphisms:

Theorem 10. *Let $P(z,\overline{z})$ be a weighted homogeneous polynomial of weighted degree d. Then there exists an integer $\ell_0\leq 6nd$ such that the following holds: Let $S\subset\mathbb{C}^{n+1}$ be an admissable $\mathcal{C}^{d+\ell_0+4}$ real hypersurface through $0\in\mathbb{C}^{n+1}$ with model S_P. If H is a germ of a $\mathcal{C}^{\ell_0+1}$ CR diffeomorphism of S at 0 satisfying $j_0^{\ell_0+1}H=I$, then $H=\operatorname{id}$.*

Proof. For the proof, we will first show that we can reduce the case of an arbitrary allowed deformation to one whose defining equation lies in the neighborhood of ρ from Theorem 8. Indeed, we take r_t and Λ_t as in Lemma 13 and set

$$S_t:=\{r_t=0\}\quad H_t=\Lambda_t^{-1}\circ H\circ\Lambda_t.$$

Then H_t and S_t satisfy all the assumptions of Theorem 10. We will now use the following facts whose proof we give later:

- The jet map j_{ℓ_0} for disks at 1 is injective on $T_{\boldsymbol{f}^0}\mathcal{S}^{k_0,\rho}$.
- Pullbacks $H_t^*(\boldsymbol{f})(\zeta):=(H_t(f(\zeta)),\tilde{f}(\zeta)(\partial H_t(f(\zeta))^{-1})$ of disks along H_t satisfy the estimate:

$$\|H_t^*(\boldsymbol{f})-\boldsymbol{f}\|_{Y^{m,d}}\lesssim t\|\boldsymbol{f}\|_{Y^{M,d}}\quad\forall t\in(0,1),\|\boldsymbol{f}\|_{Y^{M,d}}\leq 1$$

- The family of disks f^v associated to admissable vectors cover an open subset of $\mathbb{C}^{n+1}$ with their images.

Putting these facts together there exists and open neighborhood O_1 of $\boldsymbol{f}^0$ in $Y^{M,d}$, such that the jet map j_{ℓ_0} is injective on the intersection of this neighborhood with $\mathcal{S}^{k_0,r_t}$ for all t small enough. From the estimate we can find an even smaller neighborhood $O_2 \subset O_1$ of $\boldsymbol{f}^0$ such that pullbacks along H_t lie in O_1, i.e. $H_t^*(O_2) \subset O_1$. Since we have the jet identity $j_{\ell_0} H_t^*(\boldsymbol{f}) = j_{\ell_0}\boldsymbol{f}$ for $\boldsymbol{f} \in O_2$, we already have that $H_t^*(\boldsymbol{f}) = \boldsymbol{f}$ by injectivity of j_{ℓ_0} on O_1 (it is here that we need $\ell_0 + 1$ jet determination since the conormal part of the pullback contains derivatives of H). Since the disks f^v cover an open subset of $\mathbb{C}^{n+1}$ we can also cover an open subset with the nearby disks from O_2. We can sample this identity on there to obtain $\tilde{H}_t(q) = q$ for q in the open subset covered by the disks, here $\tilde{H}_t$ is the holomorphic extension of H_t from Proposition 1. This implies $\tilde{H}_t = \mathrm{id}$ by the identity theorem, so $H_t = \mathrm{id}$ and in turn $H = \mathrm{id}$.

□

First we sketch a proof of the estimate

$$\|H_t^*(\boldsymbol{f}) - \boldsymbol{f}\|_{Y^{m,d}} \lesssim t\|\boldsymbol{f}\|_{Y^{M,d}} \quad \forall t \in (0,1), \|\boldsymbol{f}\|_{Y^{M,d}} \leq 1.$$

The Leibniz rule implies the following estimate for products in the Hölder space:

$$\|uv\|_{\mathcal{C}^{k,\alpha}} \lesssim \|u\|_{\mathcal{C}^{k,\alpha}} \|v\|_{\mathcal{C}^{k,\alpha}} \quad \forall u, v \in \mathcal{C}^{k,\alpha}.$$

This can be used in our setting since the space $Y^{M,d}$ is a product space with $\mathcal{C}^{k,\alpha}$ norms in each factor. To get the difference $H_t^*(\boldsymbol{f}) - \boldsymbol{f}$ into a form where we can apply this inequality and extract the factor of t from the higher order terms, write

$$H = \mathrm{id} + \sum_j P_j \psi_j,$$

where the P_j are polynomials with no constant term and ψ_j are $\mathcal{C}^{\ell_0}$ functions. The same method works for the cotangent part of the pullback.

3.4.1 Injectivity of the jet map

Let $\ell_0, m, N \in \mathbb{N}$. Consider the the linear map $j_{\ell_0} : Y^{M,d} \to \mathbb{C}^{(2n+2)(\ell_0+1)}$ sending $\boldsymbol{f}$ to its ℓ_0−jet at $\zeta = 1$

$$j_{\ell_0}(\boldsymbol{f}) = (\boldsymbol{f}(1), \partial\boldsymbol{f}(1), \ldots, \partial_{\ell_0}\boldsymbol{f}(1))) \in \mathbb{C}^{N(\ell_0+1)}$$

where $\partial_l \boldsymbol{f}(1) \in \mathbb{C}^N$ denotes the vector $\frac{\partial^l \boldsymbol{f}}{\partial \zeta^l}(1)$ for all $l = 1, \ldots, \ell_0$.

Lemma 14. *There exists an integer $\ell_0 \leq 6nd$ such that the restriction of j_{ℓ_0} to the kernel of the operator $\boldsymbol{f}' \mapsto 2\,\mathrm{Re}\left(\overline{G(\zeta)}\boldsymbol{f}'\right)$ (see (3.9)) is injective.*

Proof. Following the notation of the proof of Theorem 8, we prove there exists an integer ℓ_0 such that the restriction of j_{ℓ_0} to the kernel of L_2 (from the proof of Theorem 8) is injective. Using a Birkhoff factorization, we write

$$-\overline{G_2^{-1}}G_2 = \Theta_2^{-1}\Lambda\overline{\Theta}_2$$

where $\Theta_2 : \overline{\Delta} \to GL_{2n+2}(\mathbb{C})$ is a smooth map holomorphic on Δ, and $\Lambda = \operatorname{diag}(\zeta^{\kappa_1}, \dots, \zeta^{\kappa_{2n+2}})$ with κ_j being the Maslov indices of $-\overline{G_2^{-1}}G_2$. For $\boldsymbol{f} \in \ker L_1$, we then have

$$\boldsymbol{f} = \overline{-G_2^{-1}}G_2\overline{\boldsymbol{f}} = \Theta_2^{-1}\Lambda\overline{\Theta}_2\overline{\boldsymbol{f}}$$

which we can rearrange to

$$\Theta_2\boldsymbol{f} = \Lambda\overline{\Theta_2\boldsymbol{f}}.$$

$\Theta_2\boldsymbol{f}$ is holomorphic, which implies that the $j-$th component of $\Theta_2\boldsymbol{f}$ is a polynomial of degree at most κ_j. Hence $\Theta_2\boldsymbol{f}$ is determined by its $\ell_0 = \max\{\kappa_1, \dots, \kappa_{2m+2}\}-$jet at 1. We now have to show that j_{ℓ_0} restricted to $\ker L_1$ is injective. To this end, we observe that, by the product rule, for any $l \geq 0$,

$$\partial_l(\Theta_2\boldsymbol{f})(1) = \Theta_2(1)\partial_l\boldsymbol{f}(1) + R_{l-1}$$

where R is a linear function of the $(l-1)-$jet of $\boldsymbol{f}$ at 1. Hence, we can get a well-defined linear map $\Theta_{l_2} : \mathbb{C}^{(2n+2)(\ell_0+1)} \to \mathbb{C}^{(2n+2)(\ell_0+1)}$ by $j_{\ell_0}(\boldsymbol{f}) \mapsto j_{\ell_0}(\Theta_2\boldsymbol{f})$. Additionally, we have a matrix representation of this map, block-triangular matrix whose $(2n+2)\times(2n+2)$ diagonal blocks are the invertible matrix $\Theta_2(1)$. Thus, Θ_{l_2} is invertible and we can combine the identity $j_{\ell_0} \circ \Theta_2 = \Theta_{l_2} \circ j_{\ell_0}$ with the injectivity of j_{ℓ_0} on $\Theta_2(\ker L_1)$ to get injectivity of j_{ℓ_0} on $\ker L_1$. Finally, to get the bound on ℓ_0, we estimate ℓ_0 by the Maslov index of $\overline{G_2^{-1}}G_2$:

$$\begin{aligned}\operatorname{ind}\det\left(-\overline{G_2^{-1}}G_2\right) &= -2\sum_{i=1}^{n} m_i + 2\mathrm{ind}Q + 2k_0 \\ &\leq 4n(2k_0 - d)) + \sum_{i=1}^{n} m_i + 2k_0 \leq 6nd.\end{aligned}$$

3.4.2 Disks covering an open subset

Let v be admissable as in Definition 4 with associated disk $f^v = (h^v, g^v)$. f^v is k_0-stationary, since

$$\partial\rho \circ f^v = (P_{z_1}(h^v, \overline{h}^v), \dots, P_{z_n}(h^v, \overline{h}^v), -\frac{1}{2})$$

and the degree in $\overline{\zeta}$ for each of the components is at most k_0 so $\zeta^{k_0}\partial\rho \circ f^v$ extends holomorphically to Δ. We can find an explicit form for g^v : It is the unique holomorphic function satisfying

$$g^v(0) = 0 \quad \text{and} \quad \operatorname{Re} g^v(\zeta) = P(h^v(\zeta), \overline{h^v(\zeta)}) \quad (\zeta \in b\Delta).$$

Thus we can compute

$$\begin{aligned}\operatorname{Re} g^v(\zeta) &= \sum_{j=d-k_0}^{k_0} (1-\zeta)^j (1-\overline{\zeta})^{d-j} P^{j,d-j}(v,\overline{v}) \\ &= \operatorname{Re} \sum_{j=d-k_0}^{k_0} \left(\sum_{\ell} \binom{j}{\ell} \binom{d-j}{\ell} \right) P^{j,d-j}(v,\overline{v}) \\ &\quad + 2 \operatorname{Re} \sum_{j=d-k_0}^{k_0} \sum_{e=1}^{|d-2j|} (-1)^e \zeta^e \left(\sum_{\ell} \binom{j}{e+\ell} \binom{d-j}{\ell} \right) P^{j,d-j}(v,\overline{v})\end{aligned}$$

and we can read off g^v from the sum inside the second real part. We can get $(g^v)'(0)$ as the ζ coefficient:

$$(g^v)'(0) = -2 \sum_{j=d-k_0}^{k_0} \left(\sum_{\ell} \binom{j}{1+\ell} \binom{d-j}{\ell} \right) P^{j,d-j}(v,\overline{v})$$

so $(g^v)'(0) \neq 0$ for an open, dense subset of v's. We will now see how to use this fact to cover an open subset of $\mathbb{C}^n$ with disks. Take the evaluation map $C : (v,\zeta) \mapsto f^v(\zeta)$, we will show that it has full rank at points $(v,0)$ where $(g^v(0))' \neq 0$. To this end, consider the Jacobi matrix

$$DC = \begin{pmatrix} \frac{\partial C}{\partial(v,\zeta)} & \frac{\partial C}{\partial\overline{(v,\zeta)}} \\ \frac{\partial \overline{C}}{\partial(v,\zeta)} & \frac{\partial \overline{C}}{\partial\overline{(v,\zeta)}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial v} h^v & \frac{\partial}{\partial \zeta} h^v & \frac{\partial}{\partial \overline{v}} h^v & \frac{\partial}{\partial \overline{\zeta}} h^v \\ \frac{\partial}{\partial v} g^v & \frac{\partial}{\partial \zeta} g^v & \frac{\partial}{\partial \overline{v}} g^v & \frac{\partial}{\partial \overline{\zeta}} g^v \\ \frac{\partial}{\partial v} \overline{h}^v & \frac{\partial}{\partial \zeta} \overline{h}^v & \frac{\partial}{\partial \overline{v}} \overline{h}^v & \frac{\partial}{\partial \overline{\zeta}} \overline{h}^v \\ \frac{\partial}{\partial v} \overline{g}^v & \frac{\partial}{\partial \zeta} \overline{g}^v & \frac{\partial}{\partial \overline{v}} \overline{g}^v & \frac{\partial}{\partial \overline{\zeta}} \overline{g}^v \end{pmatrix}$$

The first column is given by

$$\begin{aligned}&\frac{\partial}{\partial v} h^v(v,\zeta) = \operatorname{diag}(1-\zeta)^M \\ &\frac{\partial}{\partial v} g^v(v,\zeta) = \sum_{j=d-k_0}^{k_0} \sum_{e=1}^{|d-2j|} (-1)^e \zeta^e \left(\sum_{\ell} \binom{j}{e+\ell} \binom{d-j}{\ell} \right) P_z^{j,d-j}(v,\overline{v}) \Rightarrow \frac{\partial}{\partial v} g^v(v,0) = 0 \\ &\frac{\partial}{\partial v} \overline{h}^v(v,\zeta) = 0 \\ &\frac{\partial}{\partial v} g^v(v,\zeta) = \sum_{j=d-k_0}^{k_0} \sum_{e=1}^{|d-2j|} (-1)^e \zeta^e \left(\sum_{\ell} \binom{j}{e+\ell} \binom{d-j}{\ell} \right) P_z^{d-j,j}(v,\overline{v}) \Rightarrow \frac{\partial}{\partial v} \overline{g}^v(v,0) = 0,\end{aligned}$$

the second column by

$$\frac{\partial}{\partial\zeta}h^v(v,\zeta) = (-m_1(1-\zeta)^{m_1}v_1, \dots, -m_n(1-\zeta)^{m_n}v_n$$

$$\frac{\partial}{\partial\zeta}g^v(v,\zeta) = \sum_{j=d-k_0}^{k_0}\sum_{e=1}^{|d-2j|}(-1)^e\zeta^{e-1}\left(\sum_{\ell}\binom{j}{e+\ell}\binom{d-j}{\ell}\right)P^{j,d-j}(v,\overline{v}) \Rightarrow \frac{\partial}{\partial\zeta}g^v(v,0) = (g^v)'(0)$$

$$\frac{\partial}{\partial\zeta}\overline{h}^v(v,\zeta) = 0, \frac{\partial}{\partial\zeta}\overline{g}^v(v,\zeta) = 0.$$

The other columns are then determined by conjugation and we arrive at

$$DC(v,0) = \begin{pmatrix} I_n & \frac{\partial}{\partial\zeta}h^v(v,0) & & \\ & & & 0 \\ 0 & (g^v)'(0) & & \\ & & I_n & \frac{\partial}{\partial\overline{\zeta}}\overline{h}^v(v,0) \\ & 0 & & \\ & & 0 & \overline{(g^v)'(0)} \end{pmatrix}.$$

Hence $\det DC(v,0) = |(g^v)'(0)|^2 \neq 0$ for an open dense subset of $v's$, proving

Lemma 15. $\cup_v f^v(\Delta)$ *contains an open subset of* $\mathbb{C}^{n+1}$.

□

Bibliography

[1] M.S. Baouendi, P. Ebenfelt, and L.P. Rothschild. *Real Submanifolds in Complex Space and Their Mappings.* Princeton mathematical series. Princeton University Press, 1999.

[2] S.R. Bell. *The Cauchy Transform, Potential Theory and Conformal Mapping.* Studies in Advanced Mathematics. Taylor & Francis, 1992.

[3] Florian Bertrand and Giuseppe Della Sala. Riemann-hilbert problems with constraints, 2014.

[4] Florian Bertrand, Giuseppe Della Sala, and Bernhard Lamel. Jet determination of smooth cr automorphisms and generalized stationary discs, 2018.

[5] Josip Globevnik. Perturbing analytic discs attached to maximal real submanifolds of cn. *Indagationes Mathematicae*, 7(1):37–46, 1996.

[6] N P Vekua, Alan G. Gibbs, G. M. Simmons, and Joel H. Ferziger. Systems of singular integral equations. 1967.