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„Forcing combinatorics, compact partitions, cofinitary groups and Van Douwen families"

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Abstract

This doctoral thesis is a compilation of four submitted publications and in addition two further preprints containing various results in combinatorial set theory. All presented results aim to study applications of forcing to the existence of various maximal combinatorial families of reals of desired cardinalities, as well as the relations between the existence of combinatorial families of different types. The set of cardinalities of maximal combinatorial families of reals of a certain type is called its spectrum, whereas the minimal cardinal in the spectrum is called the corresponding cardinal characteristic. More explicitly, we study forcing notions to construct, extend and preserve various types of combinatorial families of reals in order to realize various spectra and separate different cardinal characteristics from one another. The types of families of central interest for this thesis are partitions of Baire space into compact sets, cofinitary groups and Van Douwen families, which correspond to the cardinal characteristics $\mathfrak{a}_{\mathrm{T}}$, $\mathfrak{a}_{\mathrm{g}}$ and $\mathfrak{a}_{\mathrm{v}}$, respectively.

The first paper constituting the thesis [2] studies a forcing notion to add a partition of Baire space into compact sets of desired size. The set of cardinalities of such partitions is denoted by $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ and its minimum is the cardinal characteristic $\mathfrak{a}_{\mathrm{T}}$. Under CH we construct a partition of Baire space into compact sets, which is indestructible by countably supported iterations or products of Sacks forcing of any length, thus answering a question of Newelski [9]. As an application, we provide an in-depth isomorphism-of-names argument for $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) = \{\aleph_1, \mathfrak{c}\}$ in product-Sacks models. Finally, we prove that Shelah's ultrapower model [13] for the consistency of $\mathfrak{d} < \mathfrak{a}$ satisfies $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$. Thus, consistently $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$ holds relative to a measurable.

The second paper [4] aims to generalize our construction of a partition of Baire space into compacts sets, which is indestructible by any product or iteration of Sacks forcing, to other combinatorial families. Say a combinatorial family of reals is universally Sacks-indestructible if it is indestructible by any countably supported iteration or product of Sacks forcing of any length. We introduce the notion of an arithmetical type of combinatorial family of reals, which serves to generalize different types of families such as mad families, maximal cofinitary groups, ultrafilter bases, splitting families and other similar types of families commonly studied in combinatorial set theory. We prove that every combinatorial family of reals of arithmetical type, which is indestructible by the countable product of Sacks forcing, is in fact universally Sacks-indestructible. Further, under CH we present a unified construction of universally Sacks-indestructible families for various arithmetical types of families. In particular, we prove the existence of a universally Sacks-indestructible maximal cofinitary group under CH.

The third paper [3] extends the state-of-the-art proof techniques for realizing various spectra of $\mathfrak{a}_{\mathrm{T}}$. The best result in this context by Brian may only realize a certain bounded spectrum of $\mathfrak{a}_{\mathrm{T}}$ once some minimum for the desired spectrum is fixed [1]. Under the additional assumption $\aleph_1 \in \mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ we remove this boundedness assumption in order to realize arbitrarily large spectra. Thus, we make significant progress in addressing the question posed by Brian in [1] if the boundedness assumption may be completely removed. As a by-product, we obtain many complete subforcings and an algebraic analysis of the automorphisms of the forcing which adds a witness for the spectrum of $\mathfrak{a}_{\mathrm{T}}$ of desired size.

The fourth paper [5] introduces the notion of tightness for maximal cofinitary groups, which captures forcing indestructibility of maximal cofinitary groups for a long list of partial orders, including Cohen, Sacks, Miller, Miller partition forcing and Shelah's poset for diagonalizing maximal ideals. We prove the existence of such a tight cofinitary group under $\mathsf{MA}(\sigma\text{-centered})$. Further, we establish the consistency of a co-analytic witness for $\mathfrak{a}_{\mathrm{g}}$ of size $\aleph_1$ together with $\mathfrak{d} = \aleph_1 < \aleph_2 = \mathfrak{c}$ and the existence of a Δ^1_3-definable well-order of the reals. Towards this end, we develop a new robust coding technique for cofinitary groups, where a real is coded into the lengths of orbits of every new word. Crucially, compared to other coding techniques for cofinitary groups (i.e. as in [6]) our new coding is parameter-less and hence may be applied to groups of uncountable size. Furthermore, as we code into orbits rather than actual function values, a more general generic hitting lemma required for tightness holds.

In the manuscript [12] we consider the isomorphism types of (maximal) cofinitary groups. In general, a full classification of the possible isomorphism types of (maximal) cofinitary groups is open, but there a various partial results. For example, Kastermans [8] proved that consistently $\bigoplus_{\aleph_1} \mathbb{Z}_2$ may have a cofinitary action. We improve this result by showing that ZFC already proves the existence of a cofinitary action of $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$.

Finally, in [11] we provide some new results regarding Van Douwen families. Van Douwen families are maximal eventually different families that remain maximal after restricting the domains of all functions to any infinite subset. First, we show that similar to the spectrum of $\mathfrak{a}$ and $\mathfrak{a}_{\mathrm{T}}$ (see [7] and [1], resp.), the spectrum of Van Douwen families is closed under singular limits. Further, for any maximal eventually different family in [10] Raghavan defined an associated ideal which measures how far the family is from being Van Douwen. Under CH we prove that every non-principal ideal is realized as the associated ideal of some maximal eventually different family, i.e. there are many different non Van Douwen families. Finally, we show that the standard forcing realizing a desired spectrum of $\mathfrak{a}_{\mathrm{e}}$ forces $\mathfrak{a}$ to have the same spectrum.

Zusammenfassung

Die Doktorarbeit ist eine Zusammenstellung von vier eingereichten Artikeln und zwei Preprints zu Resultaten im Gebiet der kombinatorischen Mengenlehre. Alle präsentierten Resultate zielen darauf ab, Anwendungen von Forcing auf die Existenz von verschiedenen maximalen kombinatorischen Familien gewünschter Kardinalität und die Relationen zwischen der Existenz solcher Familien zu studieren. Die Menge aller Kardinalitäten von maximalen kombinatorischen Familien reeller Zahlen eines bestimmten Types wird als Spektrum bezeichnet; dessen Minimum wird als Kardinalzahlcharakteristik bezeichnet. Expliziter werden Forcings studiert, welche kombinatorische Familien konstruieren, erweitern und erhalten können um verschiedene Spektra zu realisieren und Kardinalzahlcharakteristiken voneinander zu trennen. Die Typen von Familien zentraler Bedeutung für diese Arbeit sind Partitionen des Baire Raumes in kompakte Mengen, kofinitäre Gruppen und Van Douwen Familien, welche zu den Kardinalzahlcharakteristiken $\mathfrak{a}_{\mathrm{T}}$, $\mathfrak{a}_{\mathrm{g}}$ und $\mathfrak{a}_{\mathrm{v}}$ korrespondieren.

Der erste Artikel [2] studiert ein Forcing, welches neue Partitionen des Baire Raumes in kompakte Mengen gewünschter Größe hinzufügt. Die Menge der Kardinalitäten solcher Partitionen wird mit spec($\mathfrak{a}_{\mathrm{T}}$) bezeichnet; dessen Minimum ist die Kardinalzahlcharakteristik $\mathfrak{a}_{\mathrm{T}}$. Unter CH konstruieren wir eine Partition des Baire Raumes in kompakte Mengen, welche durch abzählbar gestützte Iterationen und Produkte von Sacks-Forcing jeder Länge erhalten bleibt. Dies beantwortet eine Frage von Newelski in [9]. Als Anwendung legen wir ein Isomorphie-von-Namen Argument für $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) = \{\aleph_1, \mathfrak{c}\}$ im Produkt-Sacks Modell vor. Abschließend zeigen wir, dass Shelah's Ultrapower Modell [13] für die Konsistenz von $\mathfrak{d} < \mathfrak{a}$ auch $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$ erfüllt. Dadurch ist $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$ konsistent relativ zu einer messbaren Kardinalzahl.

Der zweite Artikel [4] zielt darauf ab, unsere Konstruktion einer Partition des Baire Raumes in kompakte Mengen, welche durch abzählbar gestützte Iterationen und Produkte von Sacks-Forcing jeder Länge erhalten bleibt, für andere kombinatorische Familien zu generalisieren. Wir definieren, dass eine kombinatorische Familie universell Sacks-unzerstörbar ist, wenn diese von jeder abzählbar gestützten Iteration und Produkt von Sacks-Forcing jeder Länge erhalten bleibt. Wir führen den Begriff eines arithmetischen Typs einer kombinatorischen Familie ein, welche verschiedene Typen von Familien generalisiert, wie zum Beispiel maximale fast disjunkte Familien, maximale kofinitäre Gruppen, Ultrafilter Basen, Spaltungsfamilien und andere ähnliche Familien, die in kombinatorischer Mengenlehre studiert werden. Wir beweisen, dass jede arithmetische kombinatorische Familie, die von dem abzählbaren Produkt von Sacks-Forcing erhalten bleibt, universell Sacks-unzerstörbar ist. Des Weiteren präsentieren wir unter CH eine vereinheitlichte Konstruktion einer universell Sacks-unzerstörbaren Familie für verschieden arithmetischen Typen von Familien. Insbesondere beweisen wir die Existenz einer universell Sacks-unzerstörbaren maximalen kofinitären Gruppe unter CH.

Der dritte Artikel [3] erweitert die aktuellen Beweistechniken für die Realisierung verschiedener Spektra von $\mathfrak{a}_{\mathrm{T}}$. Das beste Resultat in diesem Kontext von Brian [1] kann, sobald ein Minimum für das Spektrum fixiert wurde, nur beschränkte Spektra von $\mathfrak{a}_{\mathrm{T}}$ realisieren. Unter der zusätzlichen Annahme von $\aleph_1 \in \mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ entfernen wir diese Beschränktheitsannahme um unbeschränkte Spektra realisieren zu können. Dadurch erhalten wir einen wesentlichen Fortschritt dabei, die Frage von Brian in [1] zu beantworten, ob diese Beschränktheitsannahme vollständig entfernt werden kann. Als ein Nebenprodukt erhalten wir dabei viele vollständige Unterforcings und eine algebraische Analyse der Automorphismen des Forcings welche einen Zeugen für das Spektrum von $\mathfrak{a}_{\mathrm{T}}$ gewünschter Größe hinzufügt.

Im vierten Artikel [5] führen wir den Begriff der Dichtheit für maximale kofinitäre Gruppen ein, welche Forcingunzerstörbarkeit von maximalen kofinitären Gruppen für eine Reihe von Forcings, wie Cohen, Sacks, Miller, Miller Partition Forcing und Shelah's Forcing um maximale Ideale zu diagonalisieren, umfasst. Wir zeigen die Existenz einer solchen dichten kofinitären Gruppe unter $\mathsf{MA}(\sigma\text{-centered})$. Des Weiteren begründen wir die Konsistenz eines koanalytischen Zeugen für $\mathfrak{a}_{\mathrm{g}}$ der Größe $\aleph_1$ zusammen mit $\mathfrak{d} = \aleph_1 < \aleph_2 = \mathfrak{c}$ und der Existenz einer Δ^1_3-definierbaren Wohlordnung der reellen Zahlen. Zu diesem Zweck entwickeln wir eine neue robuste Codierungstechnik für kofinitäre Gruppen, in dieser eine reelle Zahl in die Länge der Bahnen eines jeden neuen Wortes codiert wird. Entscheidend ist, dass verglichen zu anderen Codierungsmethoden für kofinitäre Gruppen (z.B. wie in [6]) unsere Codierungsmethode parameterlos ist und daher auf Gruppen überabzählbarer Größe angewendet werden kann. Des Weiteren, da wir in Bahnen kodieren, hält ein stärkeres generisches Treffargument, welches für Dichtheit benötigt wird.

Das erste Preprint [12] betrachtet die Isomorphietypen von (maximalen) kofinitären Gruppen. Im Allgemeinen ist die vollständige Klassifizierung der möglichen Isomorphietypen von (maximalen) kofinitären Gruppen offen, aber es gibt verschiedene Teilresultate. Zum Beispiel beweist Kastermans in [8], dass konsistent $\bigoplus_{\aleph_1} \mathbb{Z}_2$ eine kofinitäre Wirkung haben kann. Wir verbessern dieses Resultat indem wir zeigen, dass ZFC schon für den Beweis der Existenz einer kofinitären Wirkung von $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ ausreicht.

Abschließend stellen wir im zweiten Preprint [11] neue Resultate zu Van Douwen Familien vor. Van Douwen Familien sind maximale schließlich verschiedene Familien, welche maximal bleiben, wenn die Domänen aller Funktionen zu einer unendlichen Menge eingeschränkt werden. Wir zeigen, dass ähnlich wie das Spektrum von $\mathfrak{a}$ [7] und $\mathfrak{a}_{\mathrm{T}}$ [1] auch das Spektrum von Van Douwen Familien unter singulären Limiten abgeschlossen ist. Des Weiteren hat Raghavan in [10] für jede maximale schließlich verschiedene Familie ein assoziiertes Ideal eingeführt, welches misst, wie weit diese Familie davon entfernt ist Van Douwen zu sein. Unter CH zeigen wir, dass jedes nichtprinzipale Ideal als das assoziierte Ideal einer maximalen schließlich verschiedenen Familie realisiert werden kann, sodass viele verschiedene Familien existieren, die nicht Van Douwen sind. Abschließend beweisen wir, dass das Standardforcing, welches ein gewünschtes Spektrum von $\mathfrak{a}_{\mathrm{e}}$ realisiert, dasselbe Spektrum auch für $\mathfrak{a}$ erzwingt.

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Introduction

Undoubtedly, the origins of set theory can be traced back to the works of Georg Cantor. His definition of cardinality facilitated the precise comparison of infinities and thus the study of the hierarchy of infinite sets. In particular, he proved that the cardinality of the real numbers $2^{\aleph_0}$ is strictly bigger than the cardinality of the natural numbers $\aleph_0$. In other words, there is no surjection from the set of natural numbers onto the set of reals:

Theorem (Cantor, 1874, [6])**.** $2^{\aleph_0} > \aleph_0$.

Naturally, Cantor asked if the cardinality of $2^{\aleph_0}$ is the next smallest cardinality $\aleph_1$. This statement is now known as the continuum hypothesis CH:

Question (Cantor)**.** $2^{\aleph_0} = \aleph_1$?

Unfortunately, Cantor did not live long enough to witness an answer to his question, but surprisingly it turned out that the continuum hypothesis can neither be proven nor disproven from the standard axiom system of mathematics, i.e. the statement is independent from ZFC. Both consistency proofs have significantly shaped major subfields of set theory:

Theorem (Gödel, 1940, [16])**.** *Consistently,* $2^{\aleph_0} = \aleph_1$ *may hold.*

Gödel proved his theorem by defining the constructible universe L and showed that CH holds inside that model. Arguably, modern inner model theory builds upon this fundamental construction. On the other hand, Cohen proved that also the converse is consistent:

Theorem (Cohen, 1963, [7])**.** *Consistently,* $2^{\aleph_0} > \aleph_1$ *may hold.*

In order to obtain this result, Cohen introduced the method of forcing. Up to today, forcing has proven to be one of the most powerful methods in set theory. Moreover, the forcing method itself is an object of study of its own interest, beyond its numerous applications. Nowadays, set theorists have a diverse toolkit of forcing techniques at their disposal, for example:

- Finite support iterations
- Countable support iterations of proper forcing
- Template iterations
- Non-linear iterations
- Matrix iterations

Each of those techniques enjoys distinct advantages in their applications to problems in set theory, infinitary combinatorics and other areas of mathematics.

By Cohen's theorem the cardinality of the set of reals may be bigger than $\aleph_1$, in fact it may be any cardinal of uncountable cofinality. However, even if the continuum is large the situation may be very different for the cardinality of certain special set of reals. Usually, these special sets of interest satisfy some combinatorial property and are maximal with respect to this additional property. A well-studied example of such special sets of reals in set theory are maximal almost disjoint families (mad) families:

Definition. A family $\mathcal{A}$ of infinite subsets of the natural numbers is called almost disjoint (ad) iff for all $A \neq B$ in $\mathcal{A}$ we have that $A \cap B$ is finite. $\mathcal{A}$ is called maximal (mad) iff it is maximal with respect to inclusion.

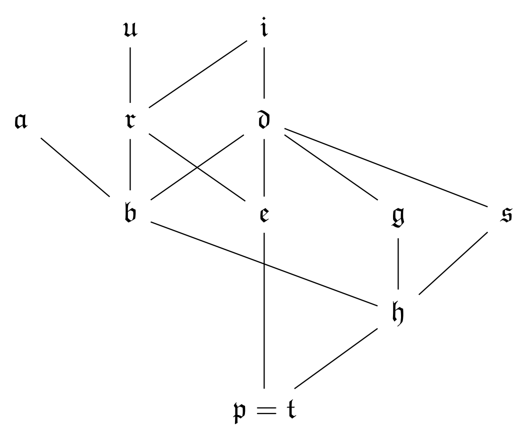
Given any such type of special set of reals, associated to it are its spectrum and cardinal characteristic. For the case of maximal almost disjoint families we obtain the following notions:

$$\begin{aligned} \operatorname{spec}(\mathfrak{a}) &:= \{|\mathcal{A}| \mid \mathcal{A} \text{ is an infinite mad family}\}, \\ \mathfrak{a} &:= \min(\operatorname{spec}(\mathfrak{a})). \end{aligned}$$

So, the spectrum of $\mathfrak{a}$ is the set of all possible sizes of infinite mad families. A standard Cantor-diagonal-style argument shows that its minimum, the almost-disjointness number $\mathfrak{a}$, is uncountable. Numerous other types of combinatorial families of reals naturally occur in set theory. However, also in other areas of mathematics there are special sets of reals with a similar combinatorial flavour, e.g.:

- Maximal eventually different families and maximal independent families in set theory,
- Partitions of Baire space into compact sets in set-theoretic topology,
- Maximal cofinitary groups in group theory/algebra,
- Maximal almost orthogonal families of projections in the Calkin algebra in functional analysis,
- Maximal sets of orthogonal measures in measure theory,
- ...

Hence, we obtain a whole zoo of spectra and cardinal characteristics. Remarkably, it turns out that for many of these families the existence of witnesses of particular sizes is not independent from one another. In fact, frequently the existence of a family of one type implies the existence of a family of another type. Thus, a rich theory emerges from the interplay between the different types of families and a key focus of classical combinatorial set theory has been the study of the relations between the associated cardinal characteristics. The following diagram gives a sneak peak of the intricacy of these relations:



Similar to the almost-disjointness number $\mathfrak{a}$ every letter represents a cardinal characteristic. A line between two cardinal characteristics indicates that there is a ZFC provable inequality between them. For example, the bounding number $\mathfrak{b}$ is never bigger than the almost-disjointness number $\mathfrak{a}$, i.e. $\mathfrak{b} \leq \mathfrak{a}$.

Note that the diagram is by no means complete in the following two senses: First, the chosen cardinal characteristics only represent a small subset of the many existing ones. Second, even for this small subset not all possible relations are known, for example we have the following famous open questions:

Question (Roitman's problem)**.** Is $\mathfrak{d} = \aleph_1 < \aleph_2 = \mathfrak{a}$ consistent?

Question (Vaughans's problem)**.** Is $\mathfrak{i} < \mathfrak{a}$ consistent?

Forcing appears as a natural counterpart to the ZFC-proofs depicted above. That is, given two cardinal characteristics $\mathfrak{x}$ and $\mathfrak{y}$ we can ask: Is there a model which separates them, i.e. either $\mathfrak{x} < \mathfrak{y}$ or $\mathfrak{y} < \mathfrak{x}$ holds? Approaching these types of questions has produced an abundance of new forcing techniques, e.g.:

- $\mathfrak{u} < \mathfrak{d}$ led to the development of matrix iterations [3],
- $\mathfrak{b} < \mathfrak{s}$ and $\mathfrak{b} < \mathfrak{a}$ led to the development of creature forcing [25],
- $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ led to the development of template iterations [26].

Hence, combinatorial set theory is not only a subject of study in its own interest, but also positively impacted the development of powerful new set theoretical tools. We may summarize one of the main guiding questions of the subject as follows:

Guiding Question. *How can set-theoretic techniques and forcing constructions be applied to the study of various types of combinatorial families of reals and their relations to each other?*

More specifically, in the context of forcing combinatorics, we may refine the above to:

Guiding Question. *How can we use forcing to preserve, construct, extend and destroy various types of combinatorial families of reals?*

We continue with a short exposition of the main contents and results of each of the six papers constituting the thesis.

1. Partitions of Baire space into compact sets

In the first paper [9] we study partitions of Baire space into compact sets. To this end, let ${}^{\omega}\omega$ denote the Baire space and define the following notions:

Definition. A family $\mathcal{C}$ of non-empty compact subsets of ${}^{\omega}\omega$ is called a partition iff

(1) $C \cap D = \emptyset$ for all $C \neq D$ in $\mathcal{C}$,
(2) $\bigcup \mathcal{C} = {}^{\omega}\omega$.

We then define the corresponding spectrum and cardinal characteristic to be:

$$\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) := \{|\mathcal{C}| \mid \mathcal{C} \text{ is a partition of Baire space into compact sets}\},$$
$$\mathfrak{a}_{\mathrm{T}} := \min(\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})).$$

We arbitrarily fixed ${}^{\omega}\omega$ as our Polish space of choice here. Note, that by a theorem of Brian [5] neither $\mathfrak{a}_{\mathrm{T}}$ nor its spectrum $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ depend on the underlying Polish space, nor do they depend on the choice of partitions into compact, closed or F_σ-sets. Miller first studied the cardinal characteristic $\mathfrak{a}_{\mathrm{T}}$ in 1980, where he constructed a model in which ${}^{\omega}2$ can be covered by $\aleph_1$-many meager sets, but not partitioned into $\aleph_1$-many disjoint non-empty closed sets. This model is now known as the Miller partition model. Further, we always have $\mathfrak{d} \leq \mathfrak{a}_{\mathrm{T}}$ [28], where $\mathfrak{d}$ is the dominating number. Towards the consistency of $\mathfrak{d} < \mathfrak{a}_{\mathrm{T}}$, in [28] Spinas proved that Miller partition forcing is ${}^{\omega}\omega$-bounding. Thus, $\mathfrak{d} = \aleph_1 < \aleph_2 = \mathfrak{a}_{\mathrm{T}} = 2^{\aleph_0}$ holds in the Miller partition model. However, as an iteration of proper forcings the technique cannot yield models with continuum larger than $\aleph_2$, so the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}_{\mathrm{T}}$ remained open. To this end, in [9] we prove that Shelah's ultrapower model for the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ [26] satisfies $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$, i.e. $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{a}_T$ is consistent relative to a measurable cardinal:

Theorem (Fischer, S., 2022, [9])**.** *Let κ be measurable and μ, λ be regular with $\kappa < \mu < \lambda$. Then consistently $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = \mathfrak{a}_T = \lambda = \mathfrak{c}$.*

Moreover, in [9] we study the forcing indestructibility of such partitions of Baire space into compact sets. In this context, Newelski [21] proved the consistency of a partition of Baire space into compact sets which is indestructible by countably supported product of Sacks-forcing of any size. Further, he showed that consistently there may be a random indestructible partition of Baire space into compact sets and asked the following follow-up question:

Question (Newelski, 1987, [21])**.** Consistently, is there a partition of Baire space into compact sets which is indestructible by countably supported iterations of Sacks-forcing?

In [9] we give a positive answer to Newelski's question. To this end, we show that the construction of a Sacks-indestructible maximal eventually different family, given in [13] can be adapted to partitions of Baire space into compact sets:

Theorem (Fischer, S., 2022, [9])**.** *Assume* CH*. Then there is a partition of Baire space into compact sets indestructible by any countably supported product or iteration of Sacks-forcing of any length.*

As a further application of such indestructible partitions, we give an in-depth isomorphism-of-names argument to compute the spectrum of $\mathfrak{a}_{\mathrm{T}}$ in product-Sacks models.

Theorem (Fischer, S., 2022, [9])**.** *Assume* CH *and let λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. Then*

$$\mathbb{S}^{\lambda} \Vdash \mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) = \{\aleph_1, \lambda\}.$$

2. Universally Sacks-indestructible combinatorial families of reals

Building on the similarities between Sacks-indestructibility of eventually different families and Sacks-indestructibility of partitions of Baire space into compact sets from [9], in the second paper constituting this thesis [11] we set to develop a general framework for the existence of Sacks-indestructible families of other types of combinatorial families. For this purpose we give the following definition:

Definition. A family is called universally Sacks-indestructible if it is indestructible by any countably supported product or iteration of Sacks-forcing of any length.

In order to prove theorems for many different combinatorial families of reals at the same time, we introduce the notion of an arithmetical type of combinatorial family of reals (see Definition 3.2 in [11]). It turns out, that most types of families usually considered in combinatorial set theory, e.g. mad families, ultrafilter bases, maximal cofinitary groups, splitting families, etc. fall into this framework. In this general context, in [11] we prove the following implication between forcing indestructibilities:

Theorem (Fischer, S., 2023, [11])**.** *Every $\mathbb{S}^{\aleph_0}$-indestructible family of arithmetical type is universally Sacks-indestructible.*

As a corollary, we obtain a number of results of the following form:

Corollary (Fischer, S., 2023, [11])**.** *Every $\mathbb{S}^{\aleph_0}$-indestructible mad family/ultrafilter basis/maximal cofinitary group/... is universally Sacks-indestructible.*

Therefore, our results imply that the indestructibility by $\mathbb{S}^{\aleph_0}$ can be seen as the strongest form of Sacks-indestructibility. Towards the proof of the previous theorem, we show that every arithmetical forcing statement about the generic sequence $\dot{s}_{\mathrm{gen}}$ added by $\mathbb{S}^{\aleph_0}$ is equivalent to a Π^1_3-formula in the following sense:

Lemma. *Let $\chi(v_1, \dots, v_k, w_1, \dots, w_l)$ be an arithmetical formula in $k+l$ real parameters. Further, let $p \in \mathbb{S}^{\aleph_0}$, $f_1, \dots, f_l \in {}^{\omega}\omega$ and $g_1, \dots, g_k$ be codes. Then the following are equivalent:*

(1) $p \Vdash \chi(g_1^*(\dot{s}_{gen}), \dots, g_k^*(\dot{s}_{gen}), f_1, \dots, f_l)$,
(2) $\forall q \leq p \; \exists r \leq q \; \forall x \in [r] \; \chi(g_1^*(x), \dots, g_k^*(x), f_1, \dots, f_l)$.

Here, every g is a code for a continuous function $g^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$. Furthermore, we provide a unified construction for the existence of such universally Sacks-indestructible families. In order to achieve this goal, we introduce the notion of elimination of intruders for any fixed type of combinatorial family (see Definition 4.1 in [11]) and prove the following theorem:

Theorem (Fischer, S., 2023, [11])**.** *Assume* CH *and elimination of intruders holds for an arithmetical type of combinatorial family. Then there is a universally Sacks-indestructible family of that type.*

Hence, the existence of universally Sacks-indestructible families under CH just reduces to verifying the elimination of intruders property. Consequently, in [11] we provide proofs that elimination of intruders holds for mad families, med families, partitions of Baire space into compact sets, maximal cofinitary groups and ultrafilter bases. In particular, our framework yields the following result:

Theorem (Fischer, S., 2023, [11])**.** *Assume* CH*. Then there is a universally Sacks-indestructible maximal cofinitary group.*

3. Realizing arbitrarily large spectra of $\mathfrak{a}_{\mathrm{T}}$

A more recent consideration in combinatorial set theory has been the study of the possible spectra of combinatorial families, expanding upon the classical theory of cardinal characteristics. In the third paper [10] we study the consistent spectra of $\mathfrak{a}_{\mathrm{T}}$.

In general, for any type of combinatorial family the classification of the possible spectra may be approached from two angles. One one hand, we may study the ZFC-provable properties of the spectrum. One the other hand, given a set of cardinals Θ with certain properties, we may consider forcing constructions to realize Θ as the spectrum. The ultimate goal is to reduce the required forcing assumptions on Θ until they agree with the provable properties in ZFC, thus yielding a complete classification of all possible spectra.

For example, Hechler [17] showed that $\mathrm{spec}(\mathfrak{a})$ is closed under singular limits and that for any set of uncountable cardinals Θ there is a model with $\Theta \subseteq \mathrm{spec}(\mathfrak{a})$. Blass [2] demonstrated that under certain assumptions on Θ in Hechler's model already $\mathrm{spec}(\mathfrak{a}) = \Theta$ holds. By employing a more sophisticated isomorphism-of-names argument Shelah and Spinas provided the currently best known result:

Theorem (Shelah, Spinas, 2015, [27])**.** *Assume* GCH *and let $\mathcal{C}$ be a set of uncountable cardinals such that*

(1) $\mathcal{C}$ is closed under singular limits,
(2) $\max(\mathcal{C})$ exists and $\mathrm{cof}(\max(\mathcal{C})) > \omega$,
(3) $\min(\mathcal{C})$ is regular.

Then there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}) = \mathcal{C}$ holds.

Note that the first two assumptions are necessary, whereas the third one is not as $\mathfrak{a}$ may be singular. Recently, similar theories have been developed for other types of combinatorial families. In particular, in [5] Brian proved that also $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ is closed under singular limits and provided a forcing construction realizing the following spectra of $\mathfrak{a}_{\mathrm{T}}$.

Theorem (Brian, 2021, [5])**.** *Assume* GCH *and let Θ be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) *If $\theta \in \Theta$ with $\mathrm{cof}(\theta) = \omega$, then $\theta^+ \in \Theta$,*
(IV) $\min(\Theta)$ *is regular,*
(V) $|\Theta| < \min(\Theta)$.

Then, there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) = \Theta$ holds.

As before, the first two assumptions are necessary. Surprisingly, in contrast to $\mathrm{spec}(\mathfrak{a})$ Brian also proved that a violation of (III) would imply the existence of an inner model with an inaccessible cardinal [4]. Hence, also (III) is necessary in ZFC. As $\mathfrak{a}_{\mathrm{T}}$ may be singular, (IV) is not necessary and since for any set of uncountable cardinals Θ there is a model with $\Theta \subseteq \mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$, also (V) is not necessary. Thus, Brian asked the following:

Question (Brian, 2021, [5])**.** Can assumption (V) be removed?

The main goal in [10] is to provide a partial positive answer to Brian's question:

Theorem (Fischer, S., 2023, [10])**.** *Assume* GCH *and let* Θ *be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) *If* $\theta \in \Theta$ *with* $\mathrm{cof}(\theta) = \omega$*, then* $\theta^+ \in \Theta$*,*
(IV) $\aleph_1 \in \Theta$*.*

Then, there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}_\mathrm{T}) = \Theta$ *holds.*

Hence, (V) may be removed, however our current proof methods need the strengthening of Brian's (IV) to $\aleph_1 \in \Theta$. Nevertheless, we may realize arbitrarily large spectra of $\mathfrak{a}_\mathrm{T}$. The main idea is to employ a more sophisticated isomorphism-of-names argument as Shelah and Spinas did for $\mathrm{spec}(\mathfrak{a})$. However, the situation for $\mathfrak{a}_\mathrm{T}$ is far more complicated as there is no product-like forcing to add a witness for $\mathfrak{a}_\mathrm{T}$, but only an iteration. Furthermore, Shelah and Spinas' proof crucially depends on the following fact:

Fact. *Let* $\mathbb{H}_J$ *be Hechler's forcing for adding an almost disjoint family indexed by* J*. If* $I \subseteq J$ *then* $\mathbb{H}_I \lessdot \mathbb{H}_J$*, i.e.* $\mathbb{H}_I$ *is a complete subforcing of* $\mathbb{H}_J$*.*

The main part of our proof provides an analogue for the iterated forcing adding a witness for $\mathfrak{a}_\mathrm{T}$. However, with our current methods, we are only able to obtain a weaker result, where I is essentially required to be countable in the above fact.

Theorem (Fischer, S., 2023, [10])**.** *Let* $\Phi \subseteq \Psi$ *be a* Θ*-subindexing function and assume* Φ *is countable. Then,* $\mathbb{P}^\Phi_\alpha \lessdot \mathbb{P}^\Psi_\alpha$ *for all* $\alpha \leq \aleph_1$*.*

Here, $\mathbb{P}^\Psi_{\aleph_1}$ is the forcing realizing the desired spectrum of $\mathfrak{a}_\mathrm{T}$ and the indexing functions Φ and Ψ play similar roles as the index sets I and J above (see Definition 4.1 in [10]). Nevertheless, this weaker version suffices to prove our theorem, however at the cost of requiring the strengthening of (IV) discussed above. A central open question is if the latter theorem extends to uncountable index sets and longer iterations. As this is the only place where we require the strengthening of Brian's (IV), a positive answer would yield a complete answer to Brian's question.

4. Tight cofinitary groups

In the fourth paper constituting this thesis [12], we consider the combinatorics of maximal cofinitary groups. Recall the following notions:

Definition. Let S_ω be the group of permutations of ω. A cofinitary group is a subgroup $G \subseteq S_\omega$ such that every $g \in G \setminus \{\mathrm{id}\}$ only has finitely many fixpoints. It is maximal iff it is maximal with respect to inclusion. We define the associated spectrum and cardinal characteristic to be:

$$\mathrm{spec}(\mathfrak{a}_\mathrm{g}) := \{|G| \mid G \text{ is a maximal cofinitary group}\},$$
$$\mathfrak{a}_\mathrm{g} := \min(\mathrm{spec}(\mathfrak{a}_\mathrm{g})).$$

In [12] we consider a notion of strong maximality for cofinitary groups. In general, strong maximality refers to any kind of combinatorial strengthening of the usual notion of maximality. It turns out that these notions of strong maximality often capture the indestructibility by various forcing notions. For example, in [15] Fischer and Switzer introduced the notion of tightness for maximal eventually different families and proved their indestructibility by many different forcings, such as Cohen, Sacks, Miller, Miller partition forcing and Shelah's poset for diagonalizing maximal ideals. Further, they asked if there is a similar notion of tightness for maximal cofinitary groups. We positively answer this question in [12] and consider the following further applications of tight cofinitary groups. First, we show that Zhang's forcing [31] always adds a tight cofinitary group. Therefore, we obtain the following theorem:

Theorem (Fischer, S., Schrittesser, 2023, [12])**.** *Assume* $\mathsf{MA}(\sigma\text{-centered})$*. Then, every cofinitary group of size* $<\mathfrak{c}$ *is contained in a tight cofinitary group of size* $\mathfrak{c}$*.*

In [8] Fischer and Friedman introduced the method of coding with perfect trees to obtain models of $\mathfrak{c} = \aleph_2$, a light-face Δ^1_3-well-order of the reals and various cardinal characteristics constellations. We show that tight cofinitary groups are also indestructible by this Sacks-coding forcing, and thus obtain the following theorem:

Theorem (Fischer, S., Schrittesser, 2023, [12])**.** *Consistently, there is a co-analytic tight cofinitary group of size* $\aleph_1$ *(thus* $\mathfrak{a}_{\mathrm{g}} = \aleph_1$*), a* Δ^1_3 *well-order of the reals, and* $\mathfrak{c} = \aleph_2$*.*

In order to obtain a co-analytic witness in the previous theorem, we develop a new robust coding technique for cofinitary groups, where a real is coded into the lengths of orbits of every new word. Crucially, compared to other coding techniques for cofinitary groups (i.e. as in [14]) our coding is parameter-less and hence may be applied to groups of uncountable size. Furthermore, as we code into orbits rather than actual function values, a more general generic hitting lemma required for tightness holds. As we also have indestructibility of tight cofinitary groups by the same forcing notions as mentioned above for tight eventually different families, we obtain many different cardinal characteristics constellations for $\mathfrak{a}_{\mathrm{g}}$.

Corollary (Fischer, S., Schrittesser, 2023, [12])**.** *Each of the following cardinal characteristics constellations is consistent with the existence of a co-analytic tight witness to* $\mathfrak{a}_g$ *and a* Δ^1_3*-well-order of the reals:*

(1) $\mathfrak{a}_g = \mathfrak{u} = \mathfrak{i} = \aleph_1 < \mathfrak{c} = \aleph_2$,
(2) $\mathfrak{a}_g = \mathfrak{u} = \aleph_1 < \mathfrak{i} = \mathfrak{c} = \aleph_2$,
(3) $\mathfrak{a}_g = \mathfrak{i} = \aleph_1 < \mathfrak{u} = \mathfrak{c} = \aleph_2$,
(4) $\mathfrak{a}_g = \aleph_1 < \mathfrak{i} = \mathfrak{u} = \mathfrak{c} = \aleph_2$.

In addition, in each of the above constellations the characteristics $\mathfrak{a}$*,* $\mathfrak{a}_e$*,* $\mathfrak{a}_p$ *can have tight co-analytic witnesses of cardinality* $\aleph_1$*; in items* (1) *and* (2)*, the ultrafilter number* $\mathfrak{u}$ *can be witnessed by a co-analytic ultrafilter base for a p-point; in items* (1) *and* (3) *the independence number can be witnessed by a co-analytic selective independent family.*

5. $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ has a cofinitary representation

In [24] we consider the isomorphism types of (maximal) cofinitary groups. A complete classification of these isomorphism types is open, however for maximal cofinitary groups there are the following restrictions. Truss [29] and Adeleke [1] independently proved that maximal cofinitary groups have to be uncountable, i.e. $\mathfrak{a}_{\mathrm{g}} > \aleph_0$. Further, Kastermans [19] showed that a cofinitary group with infinitely many orbits is not maximal. As a consequence of this theorem, Blass noticed that in particular abelian cofinitary groups cannot be maximal [18].

Furthermore, one may use Zhang's forcing [31] in order to force the existence of certain cofinitary representations. In particular, Kastermans proved the following consistency result:

Theorem (Kastermans, [18])**.** *There exists a c.c.c. forcing which forces the existence of a cofinitary representation of* $\bigoplus_{\aleph_1} \mathbb{Z}_2$.

In [24] we improve this result by showing that ZFC already proves the existence of such a cofinitary representation, and even better for the group $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$.

Theorem (S., 2023, [24])**.** *There is a cofinitary representation of* $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$.

Note that by Blass' observation $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ cannot have a maximal cofinitary representation. Thus, we prove that there are always groups of size $\mathfrak{c}$ with cofinitary representations, but no maximal ones.

6. Van Douwen and many non Van Douwen families

Finally, in [23] we consider Van Douwen families, i.e. we consider the following types of families:

Definition. Let $\mathcal{F} \subseteq {}^{\omega}\omega$ and $A \in [\omega]^{\omega}$. Then we define $\mathcal{F} \restriction A := \{f \restriction A \mid f \in \mathcal{F}\}$. We call $\mathcal{F}$ Van Douwen iff $\mathcal{F} \restriction A$ is a maximal eventually different family for all $A \in [\omega]^{\omega}$. We define the associated spectrum and cardinal characteristic to be:

$$\begin{aligned} \mathrm{spec}(\mathfrak{a}_{\mathrm{v}}) &:= \{|\mathcal{F}| \mid \mathcal{F} \text{ is Van Douwen}\}, \\ \mathfrak{a}_{\mathrm{v}} &:= \min(\mathrm{spec}(\mathfrak{a}_{\mathrm{v}})). \end{aligned}$$

Clearly, we have that $\mathrm{spec}(\mathfrak{a}_{\mathrm{v}}) \subseteq \mathrm{spec}(\mathfrak{a}_{\mathrm{e}})$, so also $\mathfrak{a}_{\mathrm{e}} \leq \mathfrak{a}_{\mathrm{v}}$, where $\mathfrak{a}_{\mathrm{e}}$ is the minimal size of an eventually different family. These types of families are named after Van Douwen, as he asked in [20] if ZFC proves the existence of such families. Zhang demonstrated that the standard forcing for adding an eventually different family in fact adds a Van Douwen family [30]. Thus, consistently, and in particular under CH, there exist Van Douwen families. Answering Van Douwen's question Raghavan [22] proved that there always is a Van Douwen family of size $\mathfrak{c}$. The study of Van Douwen families is particularly interesting in the light of the following well-known open questions for the cardinal characteristic $\mathfrak{a}_{\mathrm{e}}$:

Question. Does ZFC prove $\mathfrak{a} \leq \mathfrak{a}_{\mathrm{e}}$?

Question. Is $\mathrm{spec}(\mathfrak{a}_{\mathrm{e}})$ closed under singular limits?

As discussed before, the second property holds for both $\mathfrak{a}$ and $\mathfrak{a}_{\mathrm{T}}$. It turns out that we may answer both of these questions for Van Douwen families. It is well known that $\mathfrak{a} \leq \mathfrak{a}_{\mathrm{v}}$ and in [23] we provide a proof for the second question for the spectrum of $\mathfrak{a}_{\mathrm{v}}$:

Theorem (S., 2023, [23])**.** $\mathrm{spec}(\mathfrak{a}_{\mathrm{v}})$ *is closed under singular limits.*

The key open questions regarding Van Douwen families are if $\mathfrak{a}_{\mathrm{v}} = \mathfrak{a}_{\mathrm{e}}$ and more specifically $\mathrm{spec}(\mathfrak{a}_{\mathrm{v}}) = \mathrm{spec}(\mathfrak{a}_{\mathrm{e}})$ hold. Note that a positive answer to the first question would yield a positive answer to the first open question for $\mathfrak{a}_{\mathrm{e}}$, and a positive answer to the second question together with our theorem would yield a positive answer for the second open question for $\mathfrak{a}_{\mathrm{e}}$. In order to answer these questions, it is particularly interesting to study non Van Douwen families. For any maximal eventually different family Raghavan [22] introduced the following associated ideal, which measures how far a family is from being Van Douwen.

Definition (Raghavan, 2010, [22])**.** Let $\mathcal{F}$ be m.e.d. family. Then we define

$$\mathcal{I}_0(\mathcal{F}) := \{A \in [\omega]^\omega \mid \mathcal{F} \restriction A \text{ is not a m.e.d. family}\} \cup \mathrm{Fin}\,.$$

We show that under CH any ideal may be realized as the associated ideal of some maximal eventually different family, i.e. many different non Van Douwen families exist.

Theorem (S., 2023, [23])**.** *Assume* CH *and let* $\mathcal{I}$ *be a non-principal ideal. Then there is a maximal eventually different family such that* $\mathcal{I} = \mathcal{I}_0(\mathcal{F})$*.*

Finally, we also use Van Douwen families to prove that the standard forcing for adding a maximal eventually different family of desired size $\mathbb{E}_{\mathcal{F}}(I)$ also adds a maximal almost disjoint family of the same size:

Theorem (S., 2023, [23])**.** *Let* $\mathcal{F}$ *be an e.d. family and* I *an uncountable index set. Then*

$$\mathbb{E}_{\mathcal{F}}(I) \Vdash \max(|\mathcal{F}|, |I|) \in \mathrm{spec}(\mathfrak{a}).$$

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Synopsis of the publications

All ideas and proofs in the manuscripts of this thesis were developed in close collaboration of the authors of the respective papers.

- **Partition of Baire space into compact sets**
  - Authors: Vera Fischer and Lukas Schembecker
  - Status: submitted, 18.08.2022
  - The author of this thesis wrote the paper.
- **Universally Sacks-indestructible combinatorial families of reals**
  - Authors: Vera Fischer and Lukas Schembecker
  - Status: submitted, 30.10.2023
  - The author of this thesis wrote the paper.
- **Realizing arbitrarily large spectra of $\mathfrak{a}_{\mathrm{T}}$**
  - Authors: Vera Fischer and Lukas Schembecker
  - Status: submitted, 19.12.2023
  - The author of this thesis wrote the paper.
- **Tight cofinitary groups**
  - Authors: Vera Fischer, Lukas Schembecker and David Schrittesser
  - Status: submitted, 12.01.2024
  - The author of this thesis wrote down the sections 'Tight cofinitary groups', 'Existence of tight cofinitary groups under Martin's Axiom' and 'Co-analyticity and Zhang's forcing with coding into orbits' except Theorem 43.
- **$\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ has a cofinitary representation**
  - Author: Lukas Schembecker
  - Status: preprint
- **Van Douwen and many non Van Douwen families**
  - Author: Lukas Schembecker
  - Status: preprint

PARTITIONS OF THE BAIRE SPACE INTO COMPACT SETS

V. FISCHER AND L. SCHEMBECKER

Abstract. We study a c.c.c. forcing which adds a maximal almost disjoint family of finitely splitting trees on ω (a.d.f.s. family) or equivalently a partition of the Baire space into compact sets of desired size. Further, under CH we construct a maximal a.d.f.s. family indestructible by any countably supported iteration or product of Sacks forcing of any length, which answers a question by Newelski [14]. As an application, we present an in-depth isomorphism-of-names argument to compute the spectrum of $\mathfrak{a}_{\mathrm{T}}$ in product Sacks-models as $\{\aleph_1, \mathfrak{c}\}$. Finally, we prove that Shelah's ultrapower model in [15] for the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ also satisfies $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$. Thus, consistently $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$ holds relative to a measurable.

1. Introduction

Given an uncountable Polish space X and a pointclass Γ of Borel sets we want to understand what the possible cardinalities of partitions of X into sets in Γ are. Here, with a partition we mean a collection of non-empty subsets of X, which are pairwise disjoint and union up to the whole space X.

We provide a brief summary of the existence of such partitions of size $\aleph_1$ for different choices for the pointclass Γ: First, for $\Gamma = F_{\sigma\delta}$, Hausdorff proved in [10] that every Polish space is the union of $\aleph_1$-many strictly increasing G_δ-sets. Thus, one immediately obtains that every uncountable Polish space may be partitioned into $\aleph_1$-many disjoint $F_{\sigma\delta}$-sets. For $\Gamma = G_\delta$, there is a close connection to the cardinal characteristic $\mathrm{cov}(\mathcal{M})$, the minimal size of a family of meager sets covering ${}^\omega 2$. In fact, in [8] Fremlin and Shelah showed that $\mathrm{cov}(\mathcal{M}) = \aleph_1$ if and only if ${}^\omega 2$ can be partitioned into $\aleph_1$-many G_δ-sets. Finally, the most interesting case for us will be partitions into closed/compact sets. In [13] Miller introduced a proper forcing notion - now known as the Miller partition forcing, which is ${}^\omega\omega$-bounding (see [16]) and destroys a given uncountable partition $\mathcal{C}$ of ${}^\omega 2$ into closed sets. This is achieved by adding a new real which is not in the closure of any element of $\mathcal{C}$ in the generic extension. Thus, by iterating the forcing $\aleph_2$-many times and using a suitable bookkeeping argument, Miller obtained in [13] a model in which ${}^\omega 2$ can be covered by $\aleph_1$-many meager sets, but there is no partition of ${}^\omega 2$ into $\aleph_1$-many closed sets.

Consequently, we denote with $\mathfrak{a}_{\mathrm{T}}$ the minimal size of an uncountable partition of ${}^\omega 2$ into closed sets. Thus, in [13] Miller established the relative consistency of $\mathrm{cov}(\mathcal{M}) = \aleph_1 < \aleph_2 = \mathfrak{a}_{\mathrm{T}}$. Notably, Miller partition forcing additionally preserves tight mad families (see [9]), and as recently shown

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in [4] also selective independent families and P-points. Thus, the same model also witnesses the consistency of $\mathfrak{d} = \mathfrak{a} = \mathfrak{i} = \mathfrak{u} = \aleph_1 < \aleph_2 = \mathfrak{a}_{\mathrm{T}}$ (see [4]).

Naturally, one might ask if the definition of $\mathfrak{a}_{\mathrm{T}}$ differs if we would have chosen any other uncountable Polish space than ${}^{\omega}2$. However, by a recent result of Brian [3], for any uncountable κ we have that some uncountable Polish space can be partitioned into κ-many closed sets if and only if every uncountable Polish space can be partitioned into κ-many closed sets. Hence, not only $\mathfrak{a}_{\mathrm{T}}$ is independent of the choice of the underlying Polish space, but so is its spectrum $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$, which is the set of all uncountable cardinalities of partitions of ${}^{\omega}2$ into closed sets. In fact, even more is true [3, Theorem 2.4]: The existence of partitions into κ-many closed sets is equivalent to the existence of partitions into κ-many compact or F_σ-sets. In order to explore what possible spectra of $\mathfrak{a}_{\mathrm{T}}$ may be realized, Brian [3] also defined a c.c.c. forcing, which adds partitions of ${}^{\omega}2$ into F_σ-sets of desired sizes and only those sizes. Therefore, by the aforementioned theorem his c.c.c. forcing also implicitly adjoins partitions of ${}^{\omega}2$ into closed or compact sets of these desired sizes, however not in some easily constructable way in terms of complexity.

In contrast, in this paper we define a c.c.c. forcing that explicitly adds a partition of ${}^{\omega}\omega$ into compact sets (see Definition 3.5). Our approach stems from a slightly different standpoint, as $\mathfrak{a}_{\mathrm{T}}$ can also be defined as the minimal size of a maximal almost disjoint family of finitely splitting (a.d.f.s.) trees on ω. The connection is given by König's Lemma, which implies that such families of trees may be identified with partitions of ${}^{\omega}\omega$ into compact sets.

In [14] Newelski proved the consistency of the existence of a partition of ${}^{\omega}2$ into F_σ-sets, which is indestructible by any product of Sacks-forcing as well as a random indestructible such partition. Newelski also asked if there may be a partition, which is indestructible by the iteration of Sacks forcing. Inspired by the the construction of a maximal eventually different family indestructible by any countably supported product or iteration of Sacks forcing by Fischer and Schrittesser in [6] we answer Newelski's question positively, by proving the following Theorem 4.17:

Theorem. *Assume* CH. *Then there is an a.d.f.s. family indestructible by any countably supported iteration or product of Sacks forcing of any length.*

As an application of such Sacks-indestructible partitions, we provide an in-depth isomorphism-of-names argument to compute the spectrum of $\mathfrak{a}_{\mathrm{T}}$ in product Sacks-models (see Theorem 4.18):

Theorem. *Assume* CH *and let λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. Then*

$$\mathbb{S}^{\lambda} \Vdash \mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) = \{\aleph_1, \lambda\}.$$

Further, by the previous discussion the Miller partition model witnesses the consistency of $\aleph_1 = \mathfrak{d} < \mathfrak{a}_{\mathrm{T}} = \aleph_2$ [13][16]. However, as an iteration of proper forcings this method cannot produce any models with larger continuum, i.e. the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}_{\mathrm{T}}$ remained open. To this end, we prove that in Shelah's model for the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$ relative to a measurable cardinal, we also have $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$. Hence, $\aleph_1 < \mathfrak{d} < \mathfrak{a}_{\mathrm{T}}$ is consistent (see Theorem 5.7).

Theorem. *Assume κ is measurable, and $\kappa < \nu < \lambda$, $\lambda = \lambda^{\omega}$, $\nu^{\kappa} < \lambda$ for all $\nu < \lambda$, are regular cardinals. Then there is a forcing extension satisfying $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}} = \mathfrak{c} = \lambda$.*

The paper is structured as follows: In the second section we review some common notions and definitions regarding trees. In the third section we define a c.c.c. forcing which extends a given almost disjoint family of finitely splitting trees by ω-many new finitely splitting trees (see Definition 3.5), so that the extended family is still almost disjoint (see Lemma 3.11). We then prove that the generic new trees satisfy a certain diagonalization property (see Definition 3.1 and 3.2 and Proposition 3.12), so that iterating the forcing yields a maximal almost disjoint family of finitely splitting trees of desired size κ for any κ of uncountable cofinality (see Theorem 3.13). Hence, the forcing may be also be used to add witnesses of $\mathfrak{a}_{\mathrm{T}}$ of desired sizes and realize large spectra of $\mathfrak{a}_{\mathrm{T}}$ (see Corollary 3.14 and Corollary 3.16). Finally, we conclude section 3 with a brief analysis if and when the forcing adds unbounded, Cohen and dominating reals (see Remark 3.18 and Propositions 3.17 and 3.19).

In the fourth section we first recall standard definitions for product-Sacks forcing and its fusion. We then prove the following key Lemma 4.8 towards the construction of a Sacks-indestructible a.d.f.s. family:

Lemma. *Let $\mathcal{T}$ be a countable a.d.f.s. family, λ be a cardinal, $p \in \mathbb{S}^\lambda$ and $\dot{f}$ be a $\mathbb{S}^\lambda$-name for a real such that for all $T \in \mathcal{T}$ we have*

$$p \Vdash \dot{f} \notin [T].$$

Then there is a finitely splitting tree S and $q \leq p$ such that $\mathcal{T} \cup \{S\}$ is an a.d.f.s. family and

$$q \Vdash \dot{f} \in [S].$$

We then review a nice version of continuous reading of names for iterations and products of Sacks forcing from [6] and under CH construct an a.d.f.s. family, which is indestructible by any countably supported product or iteration of Sacks-forcing of any length (Theorem 4.17). We then provide the isomorphism-of-names argument to show that $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) = \{\aleph_1, \mathfrak{c}\}$ holds in product-Sacks models (Theorem 4.18). Finally, we end the section with a discussion that we may also construct Sacks-indestructible partitions of ${}^{\omega}2$ into closed nowhere dense sets, which more precisely answers the question of Newelski in [14].

In the last section we recall the notion of nice forcing names and provide an average-of-names argument to show that ultrapowers of forcings may be used to destroy maximal a.d.f.s. families (see Lemma 5.6). As an application, we obtain that in Shelah's model for the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}_{\mathrm{T}}$ also $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$ holds (see Theorem 5.7).

2. Preliminaries

In the following every tree T will be a tree on ω, i.e. $T \subseteq {}^{<\omega}\omega$ is non-empty and closed under initial subsequences. For a tree T, we recall the following notions:

(1) If $s \in T$ and $n < \omega$, then $\mathrm{succ}_T(s) := \{t \in T \mid (\exists n < \omega)(t = s^\frown n)\}$ and $T \restriction n := T \cap {}^{\leq n}\omega$.
(2) T is pruned iff $\mathrm{succ}_T(s) \neq \emptyset$ for all $s \in T$.
(3) T is finitely splitting iff T is pruned and $\mathrm{succ}_T(s)$ is finite for all $s \in T$.
(4) $[T] := \{f \in {}^{\omega}\omega \mid (\forall n < \omega)(f \restriction n \in T)\}$.

For $n < \omega$ we call a non-empty and closed under initial subsequences $T \subseteq {}^{\leq n}\omega$ an n-tree and use the same definitions as above, where we replace (1) and (4) with (1*) and (4*), respectively:

(1*) T is pruned iff $\mathrm{succ}_T(s) \neq \emptyset$ for all $s \in T \cap {}^{<n}\omega$.
(4*) $[T] = T \cap {}^{n}\omega$.

For trees $T \subseteq {}^{<\omega}\omega$ we call $[T]$ the branches of T and for n-trees $T \subseteq {}^{\leq n}\omega$ we call $[T]$ the leaves of T. Recall, that the non-empty closed sets of ${}^{\omega}\omega$ are in bijection with pruned trees on ω in the following way. Let T be a pruned tree and $C \subseteq {}^{\omega}\omega$ be closed and non-empty, then the maps

$$T \mapsto [T] \quad \text{and} \quad C \mapsto T_C := \{s \in {}^{<\omega}\omega \mid (\exists f \in C)(s \subseteq f)\}$$

are inverse to each other. Furthermore, the bijection restricts to a bijection between finitely splitting trees on ω and non-empty compact subsets of ${}^{\omega}\omega$. The following definition will be of central interest:

Definition 2.1. The spectrum of $\mathfrak{a}_{\mathrm{T}}$ is the set

$$\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) := \{|\mathcal{C}| \mid \mathcal{C} \text{ is a partition of } {}^{\omega}\omega \text{ into compact sets}\}$$

and we define the cardinal characteristic $\mathfrak{a}_{\mathrm{T}} := \min(\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}))$.

As ${}^{\omega}\omega$ is not σ-compact we have $\aleph_1 \leq \mathfrak{a}_{\mathrm{T}}$ and the partition of ${}^{\omega}\omega$ into singletons witnesses that $\mathfrak{a}_{\mathrm{T}} \leq \mathfrak{c}$. Furthermore, by the aforementioned result of Brian in [3], $\mathfrak{a}_{\mathrm{T}}$ and $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ do not depend on the choice of the underlying Polish space and also do not depend whether we consider partitions into compact, closed or F_σ-sets.

Definition 2.2. A family $\mathcal{T}$ of finitely splitting trees is called an almost disjoint family of finitely splitting trees (or an a.d.f.s. family) iff S and T are almost disjoint, i.e. $S \cap T$ is finite, for all $S, T \in \mathcal{T}$. $\mathcal{T}$ is called maximal iff it is maximal with respect to inclusion.

Notice, that by König's lemma for finitely splitting trees S and T we have that S and T are almost disjoint iff $[T] \cap [S] = \emptyset$. Thus, using the above identification of finitely splitting trees and non-empty compact subsets of ${}^{\omega}\omega$, we can also identify maximal a.d.f.s. families with partitions of ${}^{\omega}\omega$ into compact sets. Moreover, an a.d.f.s. family $\mathcal{T}$ is maximal iff for all reals $f \in {}^{\omega}\omega$ there is a $T \in \mathcal{T}$ such that $f \in [T]$. Finally, we note that $\mathfrak{d} \leq \mathfrak{a}_{\mathrm{T}}$ [16] and $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ is closed under singular limits [3].

3. Forcing maximal a.d.f.s. families

In this chapter we will define and analyse a c.c.c. forcing that allows us for any κ of uncountable cofinality to explicitly add a maximal a.d.f.s. family of size κ or equivalently a partition of ${}^{\omega}\omega$ into κ-many compact sets.

Definition 3.1. Let $\mathcal{T}$ be an a.d.f.s. family. We define

$$\mathcal{W}(\mathcal{T}) := \{f \in {}^{\omega}\omega \mid (\forall T \in \mathcal{T})(f \notin [T])\}.$$

Furthermore, we define

$$\mathcal{I}^+(\mathcal{T}) := \{T \mid T \text{ is a finitely splitting tree with } [T] \subseteq \mathcal{W}(\mathcal{T})\}.$$

Note that $\mathcal{W}(\mathcal{T})$ is the set of all reals that $\mathcal{T}$ is missing to be maximal and $S \in \mathcal{I}^+(\mathcal{T})$ iff S is almost disjoint from every $T \in \mathcal{T}$. The notion $\mathcal{I}^+(\mathcal{T})$ should merely emphasize the similarity to the positive sets $\mathcal{I}^+(\mathcal{A})$ associated to an a.d. family $\mathcal{A}$ in the subsequent diagonalization property. However, contrary to $\mathcal{I}^+(\mathcal{A})$ the set $\mathcal{I}^+(\mathcal{T})$ is generally not induced by an ideal.

Definition 3.2. Let $\mathcal{T}$ be an a.d.f.s. family, $\mathbb{P}$ a forcing notion and G a $\mathbb{P}$-generic filter. We say that a set of finitely splitting trees $\mathcal{S}$ in $V[G]$ diagonalizes $\mathcal{T}$ iff $\mathcal{T} \cup \mathcal{S}$ is an a.d.f.s. family and for all $T \in \mathcal{I}^+(\mathcal{T})^V$ we have that $\{T\} \cup \mathcal{S}$ is not almost disjoint.

Proposition 3.3. *The above definition is equivalent to: $\mathcal{T} \cup \mathcal{S}$ is an a.d.f.s. family and for all $f \in \mathcal{W}(\mathcal{T})^V$ there is an $S \in \mathcal{S}$ with $f \in [S]$.*

Proof. We argue in $V[G]$. Let $f \in \mathcal{W}(\mathcal{T})^V$. Then $T_f := \{s \in {}^{<\omega}\omega \mid s \subseteq f\} \in \mathcal{I}^+(\mathcal{T})^V$. By assumption choose $S \in \mathcal{S}$ such that $[T_f] \cap [S] \neq \emptyset$. But $[T_f] = \{f\}$, so $f \in [S]$.

Conversely, let $T \in \mathcal{I}^+(\mathcal{T})^V$. Choose $f \in \mathcal{W}(\mathcal{T})^V$ such that $f \in [T]$. By assumption choose $S \in \mathcal{S}$ such that $f \in [S]$. But then $f \in [S] \cap [T]$, so S and T are not almost disjoint. □

We approximate families of diagonalizing trees of size ω with finite conditions as follows:

Definition 3.4. $\mathbb{T}$ is the forcing consisting of finite partial functions $p : \omega \times {}^{<\omega}\omega \to 2$, such that

(1) $\mathrm{dom}(p) = F_p \times {}^{\leq n_p}\omega$, where $n_p \in \omega$ and $F_p \in [\omega]^{<\omega}$;
(2) for all $i \in F_p$, $T_p^i := \{s \in {}^{\leq n_p}\omega \mid p(i)(s) = 1\}$ is a finitely splitting n_p-tree.

We order $\mathbb{T}$ by $q \leq p$ iff $q \supseteq p$.

Definition 3.5. Let $\mathcal{T}$ be an a.d.f.s. family. $\mathbb{T}(\mathcal{T})$ is the forcing notion of all pairs (p, w_p) such that $p \in \mathbb{T}$ and $w_p : \mathcal{W}(\mathcal{T}) \to \omega$ is a partial function, so that $\mathrm{dom}(w_p) = H_p$ for some $H_p \in [\mathcal{W}(\mathcal{T})]^{<\omega}$ and for all $f \in H_p$ we have $w_p(f) \notin F_p$ or $f \restriction n_p \in T_p^{w_p(f)}$. We order $\mathbb{T}(\mathcal{T})$ by

$$(q, w_q) \leq (p, w_p) \quad \text{iff} \quad q \leq_{\mathbb{T}} p \text{ and } w_q \supseteq w_p.$$

Notice that for every maximal a.d.f.s. family $\mathcal{T}$ we have $\mathcal{W}(\mathcal{T}) = \emptyset$, so $\mathbb{T}(\mathcal{T}) \cong \mathbb{T} \cong \mathbb{C}$ as $\mathbb{T}$ is countable. Intuitively, $\mathbb{T}(\mathcal{T})$ adds ω-many new finitely splitting trees, where the side conditions w_p ensure that every element of $\mathcal{W}(\mathcal{T})$ is contained in the branches of exactly one of those new trees. The main difference to the forcing sketched in [3, Theorem 3.1] is that only one of the new trees is allowed to contain an element of $\mathcal{W}(\mathcal{T})$ as its branch, which ensures that the new trees are almost disjoint. Thus, this forcing may be used to explicitly add partitions of ${}^{\omega}\omega$ into compact sets rather than only partitions into F_σ-sets. A further subtle difference is that our forcing is only Knaster instead of σ-centered.

Lemma 3.6. *Let $\mathcal{T}$ be an a.d.f.s. family. Then $\mathbb{T}(\mathcal{T})$ is Knaster.*

Proof. Let $A \subseteq \mathbb{T}(\mathcal{T})$ be of size ω_1. Since $\mathbb{T}$ is countable, we may assume that $p = q$ for all $(p, w_p), (q, w_q) \in A$. Moreover, by the Δ-system lemma applied to $\{H_p \mid (p, w_p) \in A\}$, we may assume that there is a root $R \in [\mathcal{W}(\mathcal{T})]^{<\omega}$, i.e. that $H_p \cap H_q = R$ for all $(p, w_p) \neq (q, w_q) \in A$. However, there are only countably many functions from $R \to \omega$, so we may assume $w_p \restriction R = w_q \restriction R$ holds for all $(p, w_p), (q, w_q) \in A$.

It remains to observe, that this implies that all elements of A are pairwise compatible. Indeed, let $(p, w_p), (q, w_q) \in A$. By choice of R, $w_{p\cup q} = w_p \cup w_q$ is a function with $\mathrm{dom}(w_{p\cup q}) = H_p \cup H_q$. We claim that $(p, w_{p\cup q}) \in \mathbb{T}(\mathcal{T})$. If this is the case, then the result follows as $p = q = p \cup q$ implies that $(p \cup q, w_{p\cup q}) \leq (p, w_p), (q, w_q)$.

Let $f \in H_p \cup H_q$. If $f \in H_p$, then either $w_p(f) \in F_p$ and thus $f \restriction n \in T_p^{w_p(f)} = T_{p\cup q}^{w_{p\cup q}(f)}$. Otherwise, we have $w_p(f) \notin F_p$ and thus $w_p(f) \notin F_q$ and $w_p(f) \notin F_{p\cup q}$. On the other hand, the case $f \in H_q$ works analogously. □

Definition 3.7. Let $\mathcal{T}$ be an a.d.f.s. family and let G be a $\mathbb{T}(\mathcal{T})$-generic filter. In $V[G]$ we let $\mathcal{S}_G := \{S_{G,i} \mid i \in \omega\}$, where $S_{G,i} := \{s \in {}^{<\omega}\omega \mid \exists (p, w_p) \in G \text{ with } p(i)(s) = 1\}$ for $i \in \omega$.

Next, we show that in $V[G]$ the family $\mathcal{S}$ diagonalizes $\mathcal{T}$. First, we prove that $\mathcal{T} \cup \mathcal{S}$ is an a.d.f.s. family. We will make use of the following density arguments:

Proposition 3.8. *Let $\mathcal{T}$ be an a.d.f.s. family. Let $(p, w_p) \in \mathbb{T}(\mathcal{T})$ and $i_0 \in \omega \setminus F_p$. Then there is $(q, w_p) \in \mathbb{T}(\mathcal{T})$ with $(q, w_p) \leq (p, w_p)$ and $\mathrm{dom}(q) = (F_p \cup \{i_0\}) \times {}^{\leq n_p}\omega$.*

Proof. Choose any finitely splitting n_p-tree T which contains the leaf $f \restriction n_p$ for every $f \in H_p$ with $w_p(f) = i_0$ and define $q : (F_p \cup \{i_0\}) \times {}^{\leq n_p}\omega \to 2$ by

$$T_q^i := \begin{cases} T & \text{if } i = i_0, \\ T_p^i & \text{otherwise.} \end{cases}$$

Then $q \in \mathbb{T}$ and $q \leq p$. Furthermore, we have that $(q, w_p) \in \mathbb{T}(\mathcal{T})$ by choice of T and also $(q, w_p) \leq (p, w_p)$ holds by definition of q. □

Proposition 3.9. *Let $\mathcal{T}$ be an a.d.f.s. family. Let $(p, w_p) \in \mathbb{T}(\mathcal{T})$ and $m > n_p$. Then there is $(q, w_p) \in \mathbb{T}(\mathcal{T})$ with $(q, w_p) \leq (p, w_p)$ with $\mathrm{dom}(q) = F_p \times {}^{\leq m}\omega$.*

Proof. For every $i \in F_p$ and for every $f \in H_p$ with $w_p(f) = i$ we have $f \restriction n_p \in T_p^i$. Hence, we may choose a finitely splitting m-tree T_i which extends T_p^i and contains $f \restriction m$ for every $f \in H_p$ with $w_p(f) = i$. Define $q : F_p \times {}^{\leq m}\omega \to 2$ by $T_q^i = T_i$ for every $i \in F_p$. Then $q \in \mathbb{T}$ and $q \leq p$. Furthermore, we have that $(q, w_p) \in \mathbb{T}(\mathcal{T})$ by choice of the T_i's and also $(q, w_p) \leq (p, w_p)$ holds by definition of q. □

Proposition 3.10. *Let $\mathcal{T}$ be an a.d.f.s. family and $T \in \mathcal{T}$. Let $(p, w_p) \in \mathbb{T}(\mathcal{T})$. Then there is $(q, w_p) \in \mathbb{T}(\mathcal{T})$ with $(q, w_p) \leq (p, w_p)$ and $\mathrm{dom}(q) = F_p \times {}^{\leq n_q}\omega$ as well as $[T_q^i] \cap [T_q^j] = \emptyset$ and $[T_q^i] \cap [T \restriction n_q] = \emptyset$ for all $i \neq j \in F_p$.*

Proof. Choose $m > n_p$ so that $f \restriction m \neq g \restriction m$ for all $f \neq g \in H_p$ and $f \restriction m \notin T$ for all $f \in H_p$. By Proposition 3.9 choose $(q_0, w_p) \leq (p, w_p)$ with $\mathrm{dom}(q_0) = F_p \times {}^{\leq m-1}\omega$.

For every $i \in F_p$ and $s \in [T_{q_0}^i]$ the set

$$K_s^i := \{k < \omega \mid s^\frown k \in T \text{ or } s^\frown k = f \restriction m \text{ for some } f \in H_p \text{ with } w_p(f) \neq i\}$$

is finite, so there are pairwise different $\{k_s^i < \omega \mid i \in F_p, s \in [T_{q_0}^i]\}$ such that $k_s^i \notin K_s^i$.

We define $q : F_p \times {}^{\leq m}\omega \to 2$ for $i \in F_p$ and $t \in {}^{\leq m}\omega$ by

$$q(i)(t) := \begin{cases} q_0(i)(t) & \text{if } t \in {}^{\leq m-1}\omega, \\ 1 & \text{if } t = s^\frown k \text{ with } s \in [T^i_{q_0}] \text{ and } k = k^i_s, \\ 1 & \text{if } t = f \restriction m \text{ with } f \in H_p \text{ with } w_p(f) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Then T^i_q is a finitely splitting m-tree for every $i \in F_p$, $\mathrm{dom}(q) = F_p \times {}^{\leq m}\omega$, $(q, w_p) \in \mathbb{T}(\mathcal{T})$ and $(q, w_p) \leq (p, w_p)$. It remains to show that (q, w_p) has the desired properties, so let $i \neq j \in F_p$.

Let $s^\frown k \in [T^i_q] \cap [T^j_q]$. Assume $k = k^j_s$. Then $k^i_s \neq k^j_s = k$, so choose $f \in H_p$ with $w_p(f) = i$ and $f \restriction m = s^\frown k$. Then $k \in K^j_s$, which contradicts $k = k^j_s \notin K^j_s$. The case $k = k^i_s$ follows analogously. So assume $k^i_s \neq k \neq k^j_s$. Choose $f, g \in H_p$ with $w_p(f) = i, w_p(g) = j$ and $f \restriction m = s^\frown k = g \restriction m$. This contradicts the choice of m.

Finally, let $s^\frown k \in [T^i_q] \cap [T \restriction m]$. Then $k \in K^i_s$, so $k \neq k^i_s$. Now, choose $f \in H_p$ with $w_p(f) = i$ and $f \restriction m = s^\frown k$. Again, this contradicts the choice of m. □

Lemma 3.11. *Let $\mathcal{T}$ be an a.d.f.s. family and let G be $\mathbb{T}(\mathcal{T})$-generic. Then, in $V[G]$ we have that $\mathcal{T} \cup \mathcal{S}_G$ is an a.d.f.s. family.*

Proof. Let $i \in \omega$. First, we show that $S_{G,i}$ is a finitely splitting tree in $V[G]$. Let $t \in S_{G,i}$ and $s \trianglelefteq t$. Then we can choose $(p, w_p) \in G$ with $p(i)(t) = 1$. But T^i_p is an n_p-tree, so also $p(i)(s) = 1$, i.e. $s \in S_{G_i}$ witnesses that S_{G_i} is a tree.

Next, we show that $S_{G,i}$ is finitely splitting, so let $s \in S_{G_i}$. Again, choose $(p, w_p) \in G$ with $p(i)(s) = 1$. By Proposition 3.9 there is $(q, w_p) \in G$ with $(q, w_p) \leq (p, w_p)$ and $\mathrm{dom}(q) = F_p \times {}^{\leq n_q}\omega$ for $n_q > |s|$. But then T^i_q is an n_q-tree and $s \in T^i_q \setminus [T^i_q]$, which implies that $\mathrm{succ}_{S_{G,i}}(s) = \mathrm{succ}_{T^i_q}(s)$, so $S_{G,i}$ is finitely splitting in $V[G]$.

Let $i_0 \neq j_0 \in \omega$. We show that S_{G,i_0} and S_{G,j_0} are almost disjoint in $V[G]$. Consider $(p, w_p) \in \mathbb{T}(\mathcal{T})$. By Proposition 3.8 there is $(q, w_p) \leq (p, w_p)$ with $\mathrm{dom}(q) = (F_p \cup \{i_0, j_0\}) \times {}^{\leq n_p}\omega$. By Proposition 3.10 there is $(r, w_p) \leq (p, w_p)$ with $\mathrm{dom}(r) = (F_p \cup \{i_0, j_0\}) \times {}^{\leq n_r}\omega$ and $[T^i_r] \cap [T^j_r] = \emptyset$ for all $i \neq j \in F_p \cup \{i_0, j_0\}$. But then we have that

$$(r, w_p) \Vdash [S_{\dot{G},i_0} \restriction n_r] \cap [S_{\dot{G},j_0} \restriction n_r] = [T^{i_0}_r] \cap [T^{j_0}_r] = \emptyset,$$

so the set of all conditions which force that $S_{\dot{G},i_0}$ and $S_{\dot{G},j_0}$ are almost disjoint is dense.

Finally, let $i_0 \in \omega$ and $T \in \mathcal{T}$. We show that S_{G,i_0} and T are almost disjoint in $V[G]$. Consider $(p, w_p) \in \mathbb{T}(\mathcal{T})$. By Proposition 3.8 there is $(q, w_p) \leq (p, w_p)$ with $\mathrm{dom}(q) = (F_p \cup \{i_0\}) \times {}^{\leq n_p}\omega$. By Proposition 3.10 there is $(r, w_p) \leq (r, w_p)$ with $\mathrm{dom}(r) = (F_p \cup \{i_0\}) \times {}^{\leq n_r}\omega$ and $[T^i_r] \cap [T \restriction n_r] = \emptyset$ for all $i \in F_p \cup \{i_0\}$. But then

$$(r, w_p) \Vdash [S_{\dot{G},i_0} \restriction n_r] \cap [T \restriction n_r] = [T^{i_0}_r] \cap [T \restriction n_r] = \emptyset,$$

so the set of all conditions which force that $S_{\dot{G},i_0}$ and T_j are almost disjoint is dense. □

Proposition 3.12. *Let $\mathcal{T}$ be an a.d.f.s. family. Then $\mathbb{T}(\mathcal{T}) \Vdash$ '$\mathcal{S}_G$ diagonalizes $\mathcal{T}$'.*

Proof. Let $(p, w_p) \in \mathbb{T}(\mathcal{T})$ and $f \in \mathcal{W}(\mathcal{T})$. If $f \notin H_p$ we can choose an $i \in \omega \setminus F_p$ and consider $(q, w_q) = (p, w_p \cup (f, i)) \leq (p, w_p)$. Then, $f \in H_q$ and $(q, w_q) \Vdash$ '$f \in [S_{\dot{G}, w_q(f)}]$'. □

Using this diagonalization property we obtain a maximal a.d.f.s. family as follows:

Theorem 3.13. *Let κ be an cardinal of uncountable cofinality. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\gamma \mid \alpha \leq \kappa, \gamma < \kappa \rangle$ be the finite support iteration, where $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\mathbb{T}(\dot{\mathcal{T}}_\alpha)$ and $\dot{\mathcal{T}}_\alpha$ is a $\mathbb{P}_\alpha$-name for $\bigcup_{\beta<\alpha} \mathcal{S}^{\beta}_{\dot{G}_{\beta+1}/\dot{G}_\beta}$. Let G be $\mathbb{P}_\kappa$-generic, then $\mathcal{T}_\kappa := \bigcup_{\alpha<\kappa} \mathcal{S}^{\alpha}_{G}$ is a maximal a.d.f.s. family.*

Proof. By Lemma 3.11, $\mathcal{T}_\kappa$ is an a.d.f.s. family in $V[G]$. Assume there was a finitely splitting tree T almost disjoint from $\mathcal{T}_\kappa$. As $\mathbb{P}_\kappa$ is c.c.c. we can choose $\alpha < \kappa$ such that $T \in V[G_\alpha]$, where $G_\alpha = G \cap \mathbb{P}_\alpha$ is a $\mathbb{P}_\alpha$-generic filter. In $V[G_\alpha]$ by assumption $T \in \mathcal{I}^+(\mathcal{T}_\alpha)$, so by Proposition 3.12 in $V[G_{\alpha+1}]$ there is an $i < \omega$ such that T has non-empty intersection with $S^{\alpha+1}_{G_{\alpha+1}/G_\alpha, i}$, which is a contradiction. □

Corollary 3.14. *Assume CH and let κ be regular and $\lambda \geq \kappa$ be of uncountable cofinality. Then there is c.c.c. extension in which $\mathfrak{d} = \mathfrak{a}_{\mathrm{T}} = \kappa \leq \lambda = \mathfrak{c}$ holds.*

Proof. Use $\mathbb{C}_\lambda * \dot{\mathbb{P}}_\kappa$, where $\mathbb{C}_\lambda$ is λ-Cohen forcing and $\dot{\mathbb{P}}_\kappa$ is the forcing from the previous theorem. In the generic extension clearly $\lambda = \mathfrak{c}$ holds and we have $\mathfrak{a}_{\mathrm{T}} \leq \kappa$ by the previous theorem. Further, $\mathfrak{d} \geq \kappa$, because $\dot{\mathbb{P}}_\kappa$ adds Cohen reals cofinally often, which follows either from the fact the we use finite support and add Cohen reals at limit steps of countable cofinality or from the fact that already $\mathbb{T}(\mathcal{T})$ adds Cohen reals, which we will prove at the end of this section. □

On the other hand, if we want to realize singular values κ of $\mathfrak{a}_{\mathrm{T}}$, we have the following two cases. First, if κ of uncountable cofinality any model of $\mathfrak{d} = \kappa = \mathfrak{c}$ will witness $\mathfrak{a}_{\mathrm{T}} = \kappa$ since $\mathfrak{d} \leq \mathfrak{a}_{\mathrm{T}}$. However, it is open if additionally the continuum may be large:

Question. Let $\kappa < \lambda$ be any cardinals of uncountable cofinality and assume κ is singular. Then, is $\mathfrak{a}_{\mathrm{T}} = \kappa < \lambda = \mathfrak{c}$ consistent?

Secondly, it is known that $\mathfrak{a}$ may have countable cofinality [2]. However, the analogous question for $\mathfrak{a}_{\mathrm{T}}$ is still open:

Question 3.15. Is $\mathrm{cof}(\mathfrak{a}_{\mathrm{T}}) = \aleph_0$ consistent? In particular is $\mathfrak{a}_{\mathrm{T}} = \aleph_{\aleph_0}$ consistent?

We can also add many maximal a.d.f.s. families at the same time with our forcing. We use an analogous construction as Fischer and Shelah in [5], where they add many maximal independent families at the same time.

Corollary 3.16. *Assume GCH. Let λ be a regular cardinal of uncountable cofinality, $\theta \leq \lambda$ a regular cardinal and $\langle \kappa_\beta \mid \beta < \theta \rangle$ a sequence of regular uncountable cardinals with $\mathrm{cof}(\lambda) \leq \kappa_\beta \leq \lambda$ for all $\beta < \theta$. Then there is a c.c.c. extension in which $\mathfrak{c} = \lambda$ and $\kappa_\beta \in \mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ for all $\beta < \theta$.*

Proof. As $\mathrm{cof}(\lambda) \leq \kappa_\beta$ for all $\beta < \theta$ we may choose a partition of (possibly a subset of) λ into θ-many disjoint sets $\langle I_\beta \mid \beta < \theta\rangle$ such that $|I_\beta| = \kappa_\beta$ and I_β is cofinal in λ for all $\beta < \theta$. We define a finite support iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \lambda\rangle$ of c.c.c. forcings as follows: We will iteratively add θ-many a.d.f.s. families $\langle \mathcal{T}_\beta \mid \beta < \theta\rangle$. Denote with $\dot{\mathcal{T}}^\alpha_\beta$ the name for the β-th family after iteration step α. Initially we let $\dot{\mathcal{T}}^0_\beta$ be a name for the empty set for all $\beta < \theta$. Now, assume that $\mathbb{P}_\alpha$ and $\dot{\mathcal{T}}^\alpha_\beta$ have been defined for all $\beta < \theta$. If there is no $\beta < \theta$ such that $\alpha \in I_\beta$ let $\dot{\mathbb{Q}}_\alpha$ be a name for Cohen forcing. Otherwise, let $\beta_0 < \theta$ be the unique index such that $\alpha \in I_{\beta_0}$ and let $\dot{\mathbb{Q}}_\alpha$ be a name for $\mathbb{T}(\dot{\mathcal{T}}^\alpha_{\beta_0})$. Furthermore, for $\beta \neq \beta_0$ let $\dot{\mathcal{T}}^{\alpha+1}_\beta$ be a name for the same family as $\dot{\mathcal{T}}^\alpha_\beta$ and for β_0 let $\dot{\mathcal{T}}^{\alpha+1}_\beta$ be a name for $\dot{\mathcal{T}}^\alpha_\beta \cup \mathcal{S}_{\dot{G}_{\alpha+1}/\dot{G}_\alpha}$.

Let G be $\mathbb{P}_\lambda$-generic. As I_β is cofinal in λ for all $\beta < \theta$ and by the results of the previous section we obtain in $V[G]$ that $\mathcal{T}_\beta$ is a maximal a.d.f.s. family for all $\beta < \theta$. Furthermore, as $|I_\beta| = \kappa_\beta$ and every iteration step extends at most one family by ω-many new trees we obtain that $\mathcal{T}_\beta$ has size κ_β for all $\beta < \theta$. But this shows that $\kappa_\beta \in \mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$. Finally, GCH, the c.c.c.-ness of all $\mathbb{P}_\alpha$ and $\mathrm{cof}(\lambda) > \aleph_0$ imply that $\mathfrak{c} = \lambda$ by counting nice names. $\square$

Finally, we briefly consider if the forcing $\mathbb{T}(\mathcal{T})$ adds unbounded, Cohen and dominating reals.

Proposition 3.17. *$\mathbb{T}(\mathcal{T})$ adds Cohen reals.*

Proof. Let G be $\mathbb{T}(\mathcal{T})$-generic. Define a real $c : \omega \to 2$ by $c(i) = 1$ iff $\langle 0\rangle \in S_{G,i}$. We show that c defines a Cohen real over V. In V let $D \subseteq \mathbb{C}$ be dense and $(p, w_p) \in \mathbb{T}(\mathcal{T})$. By Propositions 3.8 and 3.9 we may assume that $n_p > 0$, $F_p = [0, N]$ and $\mathrm{ran}(H_p) \subseteq F_p$ for some $N < \omega$.

Define $s(i) = 1$ iff $\langle 0\rangle \in T^i_p$ for $i \leq N$. By definition of c we get $(p, w_p) \Vdash$ '$s \subseteq \dot{c}$'. By density of D choose $t \in D$ such that $s \subseteq t$. Let S, T be any n-trees, such that $\langle 0\rangle \in S$ and $\langle 0\rangle \notin T$. Then we extend (p, w_p) to (q, w_p) with $\mathrm{dom}(q) = |t| \times {}^{\leq n_p}\omega$ by

$$T^i_q = \begin{cases} T^i_p & \text{if } i \leq N, \\ S & \text{if } i > N \text{ and } t(i) = 1, \\ T & \text{otherwise.} \end{cases}$$

But then we get that $(q, w_p) \Vdash$ '$t \subseteq \dot{c}$', where $t \in D$. $\square$

Remark 3.18. Let $\mathcal{T}$ be an a.d.f.s. family. Apart from the Cohen reals above, there is also a second kind of unbounded real added by $\mathbb{T}(\mathcal{T})$. Let G be a $V[G]$-generic filter. For $i_0 < \omega$ in $V[G]$ define

$$f_{i_0}(n) := \max\{k < \omega \mid \exists s \in S_{G,i_0} \text{ with } s(n) = k\}.$$

To show that f_{i_0} is unbounded over V, consider any $g \in {}^\omega\omega \cap V$, $n_0 < \omega$ and $(p, w_p) \in \mathbb{T}(\mathcal{T})$. By Propositions 3.8 and 3.9 we may assume that $i_0 \in F_p$ and $n_0 \leq n_p$. As before, given $i \in F_p$ we can choose an $(n_p + 1)$-tree T_i which extends T^i_p and contains $f \restriction (n_p + 1)$ for all $f \in H_p$ with $w_p(f) = i$, but for i_0 additionally also contains an $s \in [T_{i_0}]$ such that $s(n_p) = g(n_p) + 1$. Define $q : F_p \times {}^{\leq n_p+1}\omega \to 2$ by $T^i_q := T_i$. Then $(q, w_p) \leq (p, w_p)$ and $(q, w_p) \Vdash g(n_p) < s(n_p) \leq f_{i_0}(n)$.

Finally, we show that $\mathbb{T}(\mathcal{T})$ can both add and not add dominating reals, depending on the properties of $\mathcal{T}$. Note that the following characterization is sufficient but not necessary.

Proposition 3.19. *Let $\mathcal{T}$ be an a.d.f.s. family. If $\mathcal{I}^+(\mathcal{T})$ is a dominating family then $\mathbb{T}(\mathcal{T})$ adds a dominating real. In particular $\mathbb{T}(\emptyset)$ adds dominating reals.*

Proof. Assume $\mathcal{I}^+(\mathcal{T})$ is dominating. Let G be $\mathbb{T}(\mathcal{T})$-generic. In $V[G]$ choose f such that $f_i \leq^* f$, where f_i is defined as in the previous remark. We claim that f is dominating, so in V let $g \in {}^\omega\omega$ and $(p, w_p) \in \mathbb{T}(\mathcal{T})$. Choose $h \in \mathcal{I}^+(\mathcal{T})$ and $N < \omega$ such that for all $n \geq N$ we have $g(n) \leq h(n)$. By possibly extending (p, w_p) we may assume that $h \in H_p$. Let $i := w_p(h)$ and choose $(q, w_q) \leq (p, w_p)$ and $M < \omega$ with $N \leq M < \omega$ such that

$$(q, w_q) \Vdash \text{For all } n \geq M \text{ we have } f_i(n) \leq f(n).$$

But then $i = w_p(h)$ implies that

$$(q, w_q) \Vdash \text{For all } n \geq M \text{ we have } g(n) \leq h(n) \leq f_i(n) \leq f(n).$$

Hence, $(q, w_q) \Vdash g \leq^* f$. □

Remark 3.20. On the other hand, if $\mathcal{T}$ is a maximal a.d.f.s. family, we have $\mathbb{T}(\mathcal{T}) \cong \mathbb{C}$, so that $\mathbb{T}(\mathcal{T})$ does not add dominating reals.

Question 3.21. Is there a nice combinatorial characterization of those families $\mathcal{T}$ for which $\mathbb{T}(\mathcal{T})$ adds a dominating real?

4. A Sacks-indestructible maximal a.d.f.s. family

Similar to the construction of a Sacks-indestructible maximal eventually different family in [6] in this section in this section we will prove that CH implies the existence of a maximal a.d.f.s. family indestructible by any countably supported iteration or product of Sacks forcing of any length. Together with an isomorphism-of-names argument we will then compute the spectrum of $\mathfrak{a}_{\mathrm{T}}$ in product-Sacks models as $\{\aleph_1, \mathfrak{c}\}$. First, we recall standard definitions of Sacks forcing and countably supported Sacks forcing and their fusion sequences.

Definition 4.1. Let $T \subseteq {}^{<\omega}2$ be a tree.

1. Let $s \in T$ then $T_s := \{t \in T \mid s \subseteq t \text{ or } t \subseteq s\}$.
2. $\mathrm{spl}(T) := \{s \in T \mid s^\frown 0 \in T \text{ and } s^\frown 1 \in T\}$ is the set of all splitting nodes of T.
3. T is perfect iff for all $s \in T$ there is a $t \in \mathrm{spl}(T)$ with $s \trianglelefteq t$.
4. $\mathbb{S} := \{T \subseteq {}^{<\omega}2 \mid T \text{ is a perfect tree}\}$ ordered by inclusion is Sacks forcing.

Definition 4.2. Let $T \in \mathbb{S}$. We define the fusion ordering for Sacks forcing as follows:

1. Let $s \in T$ then $\mathrm{succspl}_T(s)$ is the unique minimal splitting node in T extending s.
2. $\mathrm{stem}(T) := \mathrm{succspl}_T(\emptyset)$.
3. $\mathrm{spl}_0 := \{\mathrm{stem}(T)\}$ and for $n < \omega$ we set

   $$\mathrm{spl}_{n+1} := \{\mathrm{succspl}_T(s^\frown i) \mid s \in \mathrm{spl}_n(T), i \in 2\}.$$

   $\mathrm{spl}_n(T)$ is called the n-th splitting level of T.
4. Let $n < \omega$ and $S, T \in \mathbb{S}$. We write $S \leq_n T$ iff $S \leq T$ and $\mathrm{spl}_n(S) = \mathrm{spl}_n(T)$.

The following lemmas are well-known (e.g. see [11]), but as their proofs are short we provide them for completeness.

Lemma 4.3. *Let $\langle T_n \in \mathbb{S} \mid n < \omega\rangle$ be a sequence of trees such that $T_{n+1} \leq_n T_n$ for all $n < \omega$. Then $T := \bigcap_{n<\omega} T_n \in \mathbb{S}$ and $T \leq_n T_n$ for all $n < \omega$.*

Proof. The only non-trivial property to verify is that $T \in \mathbb{S}$, so let $s \in T$. Then $s \in T_0$. Let $t := \mathrm{succspl}_{T_0}(s)$ and choose $n < \omega$ such that $t \in \mathrm{spl}_n(T_0)$. As $T_n \leq T_0$ choose $u \in \mathrm{spl}_n(T_n)$ such that $u \leq t$. But then $u \leq s$ and by definition of $\leq_n$ we get that $u \in \mathrm{spl}(T)$. □

Definition 4.4. Let λ be a cardinal. $\mathbb{S}^\lambda$ is the countably supported product of Sacks forcing of size λ. Moreover,

(1) for $A \subseteq \mathbb{S}^\lambda$ let $\bigcap A$ be the function with $\mathrm{dom}(\bigcap A) := \bigcup_{p\in A} \mathrm{dom}(p)$ and for all $\alpha < \lambda$ we have $(\bigcap A)(\alpha) := \bigcap_{p\in A} p(\alpha)$. Notice that we do not necessarily have $\bigcap A \in \mathbb{S}^\lambda$.
(2) Let $n < \omega$, $p, q \in \mathbb{S}^\lambda$ and $F \in [\mathrm{dom}(q)]^{<\omega}$. Write $p \leq_{F,n} q$ iff $p \leq q$ and $p(\alpha) \leq_n q(\alpha)$ for all $\alpha \in F$.

Lemma 4.5. *Let $\langle p_n \in \mathbb{S}^\lambda \mid n < \omega\rangle$ and $\langle F_n \mid n < \omega\rangle$ be sequences such that*

(1) $p_{n+1} \leq_{F_n,n} p_n$ for all $n < \omega$,
(2) $F_n \subseteq F_{n+1}$ for all $n < \omega$ and $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} \mathrm{dom}(p_n)$.

Then $p := \bigcap_{n<\omega} p_n \in \mathbb{S}^\lambda$ and $p \leq_{F_n,n} p_n$ for all $n < \omega$.

Proof. Again, we only need to verify that $p \in \mathbb{S}^\lambda$. Clearly, $\mathrm{dom}(p) = \bigcup_{n<\omega} \mathrm{dom}(p_n)$ is countable. Let $\alpha \in \mathrm{dom}(p)$. Choose $N < \omega$ such that $\alpha \in F_N$. By assumption we get that $\langle p_n(\alpha) \in \mathbb{S} \mid n \geq N\rangle$ is a fusion sequence, so $p(\alpha) = \bigcap_{n<\omega} p_n(\alpha) \in \mathbb{S}$ by the previous lemma. □

Definition 4.6. Let $p \in \mathbb{S}^\lambda$, $F \in [\mathrm{dom}(p)]^{<\omega}$, $n < \omega$ and $\sigma : F \to V$ be a suitable function for p, F and n, i.e. $\sigma(\alpha) \in \mathrm{spl}_n(p(\alpha)) ^\frown 2$ for all $\alpha \in F$. Then we define $p \restriction \sigma \in \mathbb{S}^\lambda$ by

$$(p \restriction \sigma)(\alpha) := \begin{cases} p(\alpha)_{\sigma(\alpha)} & \text{if } \alpha \in F, \\ p(\alpha) & \text{otherwise.} \end{cases}$$

Notice that for fixed $p \in \mathbb{S}^\lambda$, $n < \omega$ and $F \in [\mathrm{dom}(p)]^{<\omega}$ there are only finitely many σ which are suitable for p, F and n. Also, if $q \leq_{F,n} p$, then q and p have the same suitable functions for F and n. Furthermore, the set

$$\{p \restriction \sigma \mid \sigma : F \to V \text{ is a suitable function for } p, F \text{ and } n\}$$

is a maximal antichain below p.

Lemma 4.7. *Let $p \in \mathbb{S}^\lambda$, $D \subseteq \mathbb{S}^\lambda$ be dense open below p, $n < \omega$ and $F \in [\mathrm{dom}(p)]^{<\omega}$. Then there is $q \leq_{F,n} p$ such that for all σ suitable for p (or equivalently q), F and n we have $q \restriction \sigma \in D$.*

Proof. Let $\langle \sigma_i \mid i < N\rangle$ enumerate all suitable functions for p, F and n. Set $q_0 := p$. We will define a $\leq_{F,n}$-decreasing sequence $\langle q_i \mid i \leq N\rangle$ so that all of the q_i have the same suitable functions as p for F and n. Assume $i < N$ and q_i is defined. Choose $r_i \leq q_i \restriction \sigma_i$ in D and define

$$q_{i+1}(\alpha) := \begin{cases} r_i(\alpha) \cup \bigcup \{q_i(\alpha)_s \mid s \in \operatorname{spl}_n(q_i(\alpha)) ^\frown 2 \text{ and } s \neq \sigma(\alpha)\} & \text{if } \alpha \in F, \\ r_i(\alpha) & \text{otherwise.} \end{cases}$$

Clearly, $q_{i+1} \leq_{F,n} q_i$ and $q_{i+1} \restriction \sigma = r_i$. Now, set $q = q_N$ and let σ be suitable for p, F and n. Choose $i < N$ such that $\sigma = \sigma_i$. Then we have $q \restriction \sigma \leq q_{i+1} \restriction \sigma = r_i \in D$, so $q \restriction \sigma \in D$ as D is open. □

By routine fusion arguments both $\mathbb{S}$ and $\mathbb{S}^\lambda$ are proper and ${}^\omega\omega$-bounding. Towards the construction of the Sacks-indestructible a.d.f.s. family, we will need the following key lemma:

Lemma 4.8. *Let $\mathcal{T}$ be a countable a.d.f.s. family, λ be a cardinal, $p \in \mathbb{S}^\lambda$ and $\dot{f}$ be a $\mathbb{S}^\lambda$-name for a real such that for all $T \in \mathcal{T}$ we have*

$$p \Vdash \dot{f} \notin [T].$$

Then there is a finitely splitting tree S and $q \leq p$ such that $\mathcal{T} \cup \{S\}$ is an a.d.f.s. family and

$$q \Vdash \dot{f} \in [S].$$

Proof. Enumerate $\mathcal{T} = \{T_n \mid n < \omega\}$. By assumption for every $n < \omega$ the set

$$D_n := \{r \in \mathbb{S}^\lambda \mid \text{There is a } k < \omega \text{ such that } r \Vdash \dot{f} \restriction k \notin T_n\}$$

is open dense below p. Set $q_0 = p$. We define a fusion sequence $\langle q_n \mid n < \omega \rangle$ as follows. Fix with a suitable bookkeeping argument a sequence $\langle F_n \in [\operatorname{dom}(q_n)]^{<\omega} \mid n < \omega \rangle$ such that that $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} \operatorname{dom}(q_n)$. Now, let $n < \omega$, assume q_n has been defined and F_n is given. Apply Lemma 4.7 to D_n, q_n, n and F_n to obtain $q_{n+1} \leq_{F_n,n} q_n$ such that for all suitable functions σ for q_n, F_n and n there is a $k_\sigma < \omega$ such that

$$q_{n+1} \restriction \sigma \Vdash \dot{f} \restriction k_\sigma \notin T_n.$$

Set $k_n := \max\{k_\sigma \mid \sigma \text{ is a suitable function for } q_n, F_n \text{ and } n\}$. Then we have

$$q_{n+1} \Vdash \dot{f} \restriction k_n \notin T_n.$$

Let $q_\omega := \bigcap_{n<\omega} q_n$. Finally, $\mathbb{S}^\lambda$ is ${}^\omega\omega$-bounding, so we may choose $q \leq q_\omega$ such that

$$S := \{s \in {}^{<\omega}\omega \mid \exists r \leq q \text{ with } r \Vdash s \subseteq \dot{f}\}$$

is a finitely splitting tree. Clearly, $q \Vdash \dot{f} \in [S]$ by definition of S. Furthermore, for $n < \omega$ we have that S and T_n are almost disjoint, since $q \leq q_{n+1}$, $q_{n+1} \Vdash \dot{f} \restriction k_n \notin T_n$ and the definition of S imply

$$q \Vdash [T \restriction k_n] \cap [S \restriction k_n] = \emptyset,$$

so that T and S are almost disjoint. □

Finally, we will need a nice version of continuous reading of names for $\mathbb{S}^\lambda$ developed in [6]. To state this version of continuous reading of names we summarize the most important definitions. First, we slightly modify the presentation of [12, Lemmas 2.5 and 2.6] to code continuous functions $f^* : {}^\omega({}^\omega 2) \to {}^\omega\omega$:

Definition 4.9.

(1) For $s, t \in {}^{<\omega}({}^{<\omega}2)$ write $s \trianglelefteq t$ iff $\mathrm{dom}(s) \leq \mathrm{dom}(t)$ and for all $n \in \mathrm{dom}(s)$, $s(n) \trianglelefteq t(n)$.
(2) A function $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ is monotone if for all $s \trianglelefteq t \in {}^{<\omega}({}^{<\omega}2)$, $f(s) \trianglelefteq f(t)$.
(3) A function $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ is proper iff for all $x \in {}^{\omega}({}^{\omega}2)$:
$$|\,\mathrm{dom}(f(x \restriction n \times n))| \overset{n\to\infty}{\longrightarrow} \infty.$$
(4) For a monotone, proper function $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ define a continuous function
$$f^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega \text{ via } f^*(x) := \bigcup_{n<\omega} f(x \restriction n \times n).$$
In this case f is called a code for f^*.

Remark 4.10. Conversely, for every continuous function $f^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$ there is a code for it. In fact, in general this is true for continuous functions between any two effective Polish spaces.

Remark 4.11. For all $p, q \in \mathbb{S}$ there is a natural bijection $\pi : \mathrm{spl}(p) \to \mathrm{spl}(q)$ which for every $n < \omega$ restricts to bijections $\pi \restriction \mathrm{spl}_n(p) : \mathrm{spl}_n(p) \to \mathrm{spl}_n(q)$ and which preserves the lexicographical ordering. We can extend it to a monotone and proper function $\pi : p \to q$ in a similar sense as above. π then codes a homeomorphism $\pi : [p] \to [q]$, which we call the induced homeomorphism.

Definition 4.12. Let $\mathbb{P}$ be the countable support iteration of Sacks forcing of length $\lambda \geq \omega$ and let $p \in \mathbb{P}$. By density we may always assume that $|\,\mathrm{dom}(p)| = \omega$ and $0 \in \mathrm{dom}(p)$.

(1) A standard enumeration of $\mathrm{dom}(p)$ is a sequence
$$\Sigma = \langle \sigma_k \mid k < \omega \rangle$$
such that $\sigma_0 = 0$ and $\mathrm{ran}(\Sigma) = \mathrm{dom}(p)$.
(2) Let $[p]$ be a $\mathbb{P}$-name such that
$$p \Vdash [p] = \langle x \in {}^{\mathrm{dom}(p)}({}^{\omega}2) \mid \text{For all } \alpha \in \mathrm{dom}(p) \text{ we have } x(\alpha) \in [p(\alpha)] \rangle.$$
(3) Let Σ be a standard enumeration of $\mathrm{dom}(p)$. For $k < \omega$ let $\dot{e}_k^{p,\Sigma}$ be a $\mathbb{P} \restriction \sigma_k$-name such that
$$p \restriction \sigma_k \Vdash \dot{e}_k^{p,\Sigma} \text{ is the induced homeomorphism between } [p(\sigma_k)] \text{ and } {}^{\omega}2.$$
Finally, let $\dot{e}^{p,\Sigma}$ be a $\mathbb{P}$-name such that
$$p \Vdash \dot{e}^{p,\Sigma} : [p] \to {}^{\omega}({}^{\omega}2) \text{ such that } \dot{e}^{p,\Sigma}(x) = \langle \dot{e}_k^{p,\Sigma}(x(\sigma_k)) \mid k < \omega \rangle \text{ for all } x \in [p].$$

Remark 4.13. For the countable support product of Sacks forcing we define the analogous notions. In fact in this simpler case we do not have to define $[p]$, $\dot{e}_k^{p,\Sigma}$ and $\dot{e}^{p,\Sigma}$ as names.

Definition 4.14. Let $\mathbb{P}$ be the countable support iteration or product of Sacks forcing of any length. Let $q \in \mathbb{P}$ and $\dot{f}$ be a $\mathbb{P}$-name such that $q \Vdash$ '$\dot{f} \in {}^{\omega}\omega$'. Let $\Sigma = \langle \sigma_k \mid k < \omega \rangle$ be a standard enumeration of $\mathrm{dom}(q)$ and $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ be a code for a continuous function $f^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$ such that
$$q \Vdash \dot{f} = (f^* \circ \dot{e}^{q,\Sigma})(s_{\dot{G}} \restriction \mathrm{dom}(q)),$$
where $s_{\dot{G}}$ is the sequence of Sacks reals. We say $\dot{f}$ is read continuously below q (by f and Σ).

Lemma 4.15 (Lemma 4 of [6]). *Let $\mathbb{P}$ be the countable support iteration or product of Sacks forcing of length λ. Suppose $p \in \mathbb{P}$ and $\dot{f}$ is a $\mathbb{P}$-name such that $p \Vdash$ '$\dot{f} \in {}^{\omega}\omega$'. Then there is $q \leq p$ such that $\dot{f}$ is read continuously below q.*

Remark 4.16. For any $p \in \mathbb{P}$ and $\mathbb{P}$-name $\dot{f}$ such that $p \Vdash$ '$\dot{f} \in {}^{\omega}\omega$' it is easy to see that if $\dot{f}$ is read continuously below p then for all $q \leq p$ also $\dot{f}$ is read continuously below q. Thus, the previous lemma shows that the set

$$\{q \in \mathbb{P} \mid \dot{f} \text{ is read continuously below } q\}$$

is dense open below p.

Theorem 4.17. *Assume* CH. *Then there is an a.d.f.s. family indestructible by any countably supported iteration or product of Sacks forcing of any length.*

Proof. By CH let $\langle f_\alpha \mid \alpha < \aleph_1 \rangle$ enumerate all codes $f_\alpha : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ for continuous functions $f^*_\alpha : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$. We define an increasing and continuous sequence $\langle \mathcal{T}_\alpha \mid \alpha < \aleph_1 \rangle$ of a.d.f.s. families as follows. Set $\mathcal{T}_0 := \emptyset$. Now, assume $\mathcal{T}_\alpha$ is defined. If for all $T \in \mathcal{T}_\alpha$ we have that

$$\mathbb{S}^{\aleph_0} \Vdash_{\mathbb{S}^{\aleph_0}} f^*_\alpha(s_{\dot{G}}) \notin [T],$$

then by Lemma 4.8 there is a finitely splitting tree S and $p \in \mathbb{S}^{\aleph_0}$ such that $\mathcal{T}_\alpha \cup \{S\}$ is an a.d.f.s. family and

$$p \Vdash_{\mathbb{S}^{\aleph_0}} f^*_\alpha(s_{\dot{G}}) \in [S].$$

Set $\mathcal{T}_{\alpha+1} := \mathcal{T}_\alpha \cup \{S\}$. In the other case we set $\mathcal{T}_{\alpha+1} := \mathcal{T}_\alpha$.

This finishes the construction and we set $\mathcal{T} := \bigcup_{\alpha<\aleph_1} \mathcal{T}_\alpha$. By construction we have that for any code $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ for a continuous function $f^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$ there is a $p \in \mathbb{S}^{\aleph_0}$ and $T \in \mathcal{T}$ such that

$$p \Vdash_{\mathbb{S}^{\aleph_0}} f^*(s_{\dot{G}}) \in [T].$$

We claim that this implies for all $x \in [p]$ that $f^*(x) \in [T]$, for if $x \in [p]$ and $n < \omega$ we define $p_{x \restriction n \times n} \leq p$ as follows: For $m < \omega$ let

$$p_{x \restriction n \times n}(m) := \begin{cases} (p(m))_{x_m \restriction n} & \text{if } m < n, \\ p(m) & \text{otherwise.} \end{cases}$$

This is well-defined and $p_{x \restriction n \times n} \in \mathbb{S}^{\aleph_0}$ since $x_m \restriction n \in p(m)$. But then we have

$$p_{x \restriction n \times n} \Vdash_{\mathbb{S}^{\aleph_0}} f(s_{\dot{G}} \restriction n \times n) \in T \text{ and } s_{\dot{G}} \restriction n \times n = x \restriction n \times n,$$

which yields $f(x \restriction n \times n) \in T$. Thus, we have shown $f^*(x) \in [T]$.

Now, assume that $\mathcal{T}$ is not maximal in some iterated Sacks-extension with countable support. The argument for countably supported product of Sacks forcing follows similarly. So let λ be an ordinal and $\mathbb{P}$ the countably supported iteration of Sacks forcing of length λ; we may assume that $\lambda \geq \omega$. Further, let $p \in \mathbb{P}$ and $\dot{f}$ be a $\mathbb{P}$-name for a real such that for all $T \in \mathcal{T}$ we have

$$p \Vdash_{\mathbb{P}} \dot{f} \notin [T].$$

By Lemma 4.15 we may choose $q \leq p$ and a standard enumeration $\Sigma = \langle \sigma_k \mid k \in \omega \rangle$ of $\mathrm{dom}(q)$ and $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ a code for a continuous function $f^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$ such that

$$q \Vdash_{\mathbb{P}} \dot{f} = f^*(\dot{e}^{q,\Sigma}(s_{\dot{G}})).$$

By construction of $\mathcal{T}$ we may choose $T \in \mathcal{T}$ and $p \in \mathbb{S}^{\aleph_0}$ such that for all $x \in [p]$ we have $f^*(x) \in [T]$. Let r be the 'pull-back' of p under $\dot{e}^{q,\Sigma}$, i.e. for all $k < \omega$ we have

$$q \Vdash_{\mathbb{P}} [r(\sigma_k)] = (\dot{e}^{q,\Sigma}_k)^{-1}[[p(k)]] \quad \text{so that} \quad r \Vdash_{\mathbb{P}} \dot{e}^{q,\Sigma}(s_{\dot{G}}) \in [p].$$

By definition of $\dot{e}^{q,\Sigma}$ we have that $r \leq q$. By Π^1_1-absoluteness we have

$$\Vdash_{\mathbb{P}} \text{For all } x \in [p] \text{ we have } f^*(x) \in [T]$$

which yields that

$$r \Vdash \dot{f} = f^*(\dot{e}^{q,\Sigma}(s_{\dot{G}})) \in [T],$$

a contraction. □

Finally, we use the previous theorem together with an isomorphism-of-names argument to compute the spectrum of $\mathfrak{a}_{\mathrm{T}}$ in product-Sacks models. For similar isomorphism-of-names arguments in other contexts, also see [3] and [7].

Theorem 4.18. *Assume* CH *and let λ be a cardinal such that $\lambda^{\aleph_0} = \lambda$. Then*

$$\mathbb{S}^\lambda \Vdash \mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) = \{\aleph_1, \lambda\}.$$

Remark 4.19. We may code finitely splitting trees with reals, so by the previous discussion on continuous reading of names, every name for a finitely splitting tree can be continuously read. Furthermore, CH implies that $\mathbb{S}^\lambda$ has the $\aleph_2$-c.c. Hence, if we have a nice name for a finitely splitting tree, then its evaluation only depends on $\aleph_1$-many conditions in $\mathbb{S}^\lambda$.

Proof. First, we argue that $\mathbb{S}^\lambda \Vdash \mathfrak{c} = \lambda$. As $\lambda^{\aleph_0} = \lambda$ we have $|\mathbb{S}^\lambda| = \lambda$. Further, for any $p \in \mathbb{S}^\lambda$ by CH there are at most $\aleph_1$-many standard enumerations of $\mathrm{dom}(p)$ and at most $\aleph_1$-many codes $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ for continuous functions $f^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$. Hence, at most $\aleph_1$-many different functions can be read continuously below p. By Lemma 4.15 we obtain

$$\mathbb{S}^\lambda \Vdash \mathfrak{c} \leq |\mathbb{S}^\lambda| \cdot \aleph_1 = \lambda \cdot \aleph_1 = \lambda.$$

Conversely, $\mathbb{S}^\lambda$ adds λ-many different Sacks reals so we get $\mathbb{S}^\lambda \Vdash$ '$\mathfrak{c} \geq \lambda$'. Furthermore, by CH Theorem 4.17 implies the existence of a Sacks-indestructible a.d.f.s. family, so $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}}) \supseteq \{\aleph_1, \lambda\}$. Now, consider any $\aleph_1 < \kappa < \lambda$ and for a contradiction assume $\langle \dot{T}_\alpha \mid \alpha < \kappa \rangle$ are nice $\mathbb{S}^\lambda$-names for finitely splitting trees and $p \in \mathbb{S}^\lambda$ is such that

$$p \Vdash \langle \dot{T}_\alpha \mid \alpha < \kappa \rangle \text{ is a maximal a.d.f.s. family.}$$

For any $\alpha < \aleph_2$ choose $p_\alpha \leq p$ with $\mathrm{dom}(p_\alpha) = U_\alpha \in [\lambda]^{\aleph_0}$ such that $\dot{T}_\alpha$ is read continuously below p_α. By CH we may use the Δ-system lemma to choose $I_0 \in [\aleph_2]^{\aleph_2}$ and a root U of $\langle U_\alpha \mid \alpha \in I_0 \rangle$. By a counting argument there is $I_1 \in [I_0]^{\aleph_2}$ such that $|U_\alpha \setminus U| = |U_\beta \setminus U|$ for all $\alpha, \beta \in I_1$.

Then, for every $\alpha \in I_1$, we may choose a bijection $\phi_\alpha : U_\alpha \to \omega$ such that $\phi_\alpha \restriction U = \phi_\beta \restriction U$ for all $\alpha, \beta \in I_1$. Now, for $\alpha, \beta \in I_1$ define an involution $\pi_{\alpha,\beta} : \lambda \to \lambda$ by

$$\pi_{\alpha,\beta}(i) := \begin{cases} (\phi_\beta)^{-1}(\phi_\alpha(i)) & \text{for } i \in U_\alpha, \\ (\phi_\alpha)^{-1}(\phi_\beta(i)) & \text{for } i \in U_\beta, \\ i & \text{otherwise.} \end{cases}$$

Clearly, this is well-defined as $U_\alpha \cap U_\beta = U$ and $\phi_\alpha \restriction U = \phi_\beta \restriction U$. Moreover, for all $\alpha, \beta, \gamma \in I_1$ we have

(1) $\pi_{\alpha,\beta}$ maps U_α onto U_β and U_β onto U_α, but is the identity on the rest of λ,
(2) $\pi_{\alpha,\beta}$ is the identity on U, $\pi_{\alpha,\alpha} = \mathrm{id}_\lambda$ and $\pi_{\alpha,\beta} = \pi_{\beta,\alpha}$,
(3) if α, β, γ are pairwise distinct, then $\pi_{\alpha,\gamma} = \pi_{\alpha,\beta} \circ \pi_{\beta,\gamma} \circ \pi_{\alpha,\beta}$.

$\pi_{\alpha,\beta}$ naturally induces to an automorphism of $\mathbb{S}^\lambda$, which we also denote with $\pi_{\alpha,\beta}$. Note that the three properties above also hold for the induced maps. Fix $\gamma_1 \in I_1$. For any $\alpha \in I_1$ we have that $\mathrm{dom}(\pi_{\alpha,\gamma_1}(p_\alpha)) \subseteq U_{\gamma_1}$. But by CH we have $|\mathbb{S}| = \aleph_1$ and $\aleph_1^{\aleph_0} = \aleph_1$, so there are only $\aleph_1$-many conditions $p \in \mathbb{S}^\lambda$ with $\mathrm{dom}(p) \subseteq U_{\gamma_1}$. By a counting argument, we can find $I_2 \in [I_1 \setminus \{\gamma_1\}]^{\aleph_2}$ such that $\pi_{\alpha,\gamma_1}(p_\alpha) = \pi_{\beta,\gamma_1}(p_\beta)$ for all $\alpha, \beta \in I_2$. But by (3) this implies that

$$\pi_{\alpha,\beta}(p_\alpha) = \pi_{\alpha,\gamma_1} \circ \pi_{\gamma_1,\beta} \circ \pi_{\alpha,\gamma_1}(p_\alpha) = \pi_{\alpha,\gamma_1} \circ \pi_{\gamma_1,\beta} \circ \pi_{\beta,\gamma_1}(p_\beta) = \pi_{\alpha,\gamma_1}(p_\beta) = p_\beta$$

for all $\alpha \neq \beta \in I_2$. Notice that the last equality follows from $\mathrm{dom}(p_\beta) \cap (U_\alpha \cup U_{\gamma_1}) = U$.

Again, fix some $\gamma_2 \in I_2$. The automorphism $\pi_{\alpha,\beta}$ on $\mathbb{S}^\lambda$ extends to $\mathbb{S}^\lambda$-names and properties (1) to (3) still hold for this extension. By the automorphism theorem we get that $\pi_{\alpha,\gamma_2}(\dot{T}_\alpha)$ can be read continuously below $\pi_{\alpha,\gamma_2}(p_\alpha) = p_{\gamma_2}$ for all $\alpha \in I_2$. But by CH there are only $\aleph_1$ many different names for finitely splitting trees that can be read continuously below p_{γ_2}. Again, by a counting argument we can find $I_3 \in [I_2 \setminus \{\gamma_2\}]^{\aleph_2}$ such that for all $\alpha, \beta \in I_3$ we have

$$p_{\gamma_2} \Vdash \pi_{\alpha,\gamma_2}(\dot{T}_\alpha) = \pi_{\beta,\gamma_2}(\dot{T}_\beta).$$

We may assume that for all $\alpha < \kappa$, all maximal antichains in the nice name of $\dot{T}_\alpha$ restrict to maximal antichains below p_α. Then, we write $\dot{T}_\alpha \restriction p_\alpha$ for the restriction of the name $\dot{T}_\alpha$ below p_α. Then $\dot{T}_\alpha \restriction p_\alpha$ is again a nice name and $p_\alpha \Vdash$ '$\dot{T}_\alpha = \dot{T}_\alpha \restriction p_\alpha$'. Since $\dot{T}_\alpha$ can be continuously read below p_α, we may additionally assume that $\mathrm{dom}(q) \subseteq U_\alpha$ for every $q \in \mathbb{S}^\lambda$ on which the nice name $\dot{T}_\alpha \restriction p_\alpha$ depends.

We now define a new $\mathbb{S}^\lambda$-name $\dot{T}_\kappa$ for a finitely splitting tree. By assumption, all names $\dot{T}_\alpha$ are nice, so we may choose $\{p_{\alpha,i} \mid i < \aleph_1, \alpha < \kappa\}$ such that the evaluation of $\dot{T}_\alpha$ only depends on $\{p_{\alpha,i} \mid i < \aleph_1\}$. Let $W_\alpha := \bigcup_{i<\aleph_1} \mathrm{dom}(p_{\alpha,i})$ and $W := \bigcup_{\alpha<\kappa} W_\alpha$. Then $|W| \leq \kappa < \lambda$, so choose $U_\kappa \in [\lambda]^{\aleph_0}$ with $U_\kappa \cap W = U$ and $|U_\alpha \setminus U| = |U_\kappa \setminus U|$ for all $\alpha \in I_1$. Choose a bijection $\phi_\kappa : U_\kappa \to \omega$ such that $\phi_\kappa \restriction U = \phi_\alpha \restriction U$ for all $\alpha \in I_1$. Thus, we can extend the system of involutions to $I_1 \cup \{\kappa\}$ by defining $\pi_{\alpha,\kappa}$ by the same equation as above, thus preserving properties (1) to (3). Fix $\gamma_3 \in I_3$ and set $p_\kappa := \pi_{\gamma_3,\kappa}(p_{\gamma_3})$ and $\dot{T}_\kappa := \pi_{\gamma_3,\kappa}(\dot{T}_{\gamma_3})$. We claim that p_κ and $\dot{T}_\kappa$ are independent of the choice of γ_3 in the following sense: For all $\alpha \in I_2$ (in particular for γ_2) we have $\pi_{\alpha,\kappa}(p_\alpha) = p_\kappa$ and for all $\alpha \in I_3$ we have

$$p_\kappa \Vdash \pi_{\alpha,\kappa}(\dot{T}_\alpha) = \dot{T}_\kappa = \pi_{\gamma_3,\kappa}(\dot{T}_{\gamma_3}).$$

For γ_3 the first claim holds trivially and for $\alpha \in I_2 \setminus \{\gamma_3\}$ we use (3) to compute

$$\pi_{\alpha,\kappa}(p_\alpha) = \pi_{\alpha,\gamma_3} \circ \pi_{\gamma_3,\kappa} \circ \pi_{\alpha,\gamma_3}(p_\alpha) = \pi_{\alpha,\gamma_3} \circ \pi_{\gamma_3,\kappa}(p_{\gamma_3}) = \pi_{\alpha,\gamma_3}(p_\kappa) = p_\kappa,$$

where the last equality follows from $\mathrm{dom}(p_\kappa) \cap (U_\alpha \cup U_{\gamma_3}) = U$. Again, the second holds trivially for γ_3, so let $\alpha \in I_3 \setminus \{\gamma_3\}$. Since $\gamma_3 \in I_3$ we have

$$p_{\gamma_2} \Vdash \pi_{\alpha,\gamma_2}(\dot{T}_\alpha) = \pi_{\gamma_3,\gamma_2}(\dot{T}_{\gamma_3}).$$

Now, apply the automorphism theorem with $\pi_{\gamma_2,\kappa}$ to obtain

$$p_\kappa \Vdash \pi_{\gamma_2,\kappa}(\pi_{\alpha,\gamma_2}(\dot{T}_\alpha)) = \pi_{\gamma_2,\kappa}(\pi_{\gamma_3,\gamma_2}(\dot{T}_{\gamma_3})).$$

Then, property (3) and simplifying yields

$$p_\kappa \Vdash \pi_{\alpha,\gamma_2}(\pi_{\alpha,\kappa}(\dot{T}_\alpha)) = \pi_{\gamma_3,\gamma_2}(\pi_{\gamma_3,\kappa}(\dot{T}_{\gamma_3})).$$

To prove the claim it remains to show that

$$p_\kappa \Vdash \pi_{\alpha,\kappa}(\dot{T}_\alpha) = \pi_{\alpha,\gamma_2}(\pi_{\alpha,\kappa}(\dot{T}_\alpha)) \text{ and } \pi_{\gamma_3,\kappa}(\dot{T}_{\gamma_3}) = \pi_{\gamma_3,\gamma_2}(\pi_{\gamma_3,\kappa}(\dot{T}_{\gamma_3})).$$

For the first equation, apply the automorphism theorem with $\pi_{\alpha,\kappa}$ to

$$p_\alpha \Vdash \dot{T}_\alpha = \dot{T}_\alpha \restriction p_\alpha$$

in order to obtain

$$p_\kappa \Vdash \pi_{\alpha,\kappa}(\dot{T}_\alpha) = \pi_{\alpha,\kappa}(\dot{T}_\alpha \restriction p_\alpha).$$

Further, $\dot{T}_\alpha \restriction p_\alpha$ only depends on U_α. Hence, $\pi_{\alpha,\kappa}(\dot{T}_\alpha)$ only depends on U_κ. Thus,

$$\pi_{\alpha,\gamma_2}(\pi_{\alpha,\kappa}(\dot{T}_\alpha \restriction p_\kappa)) = \pi_{\alpha,\kappa}(\dot{T}_\alpha \restriction p_\kappa).$$

But this implies

$$p_\kappa \Vdash \pi_{\alpha,\gamma_2}(\pi_{\alpha,\kappa}(\dot{T}_\alpha)) = \pi_{\alpha,\gamma_2}(\pi_{\alpha,\kappa}(\dot{T}_\alpha \restriction p_\alpha)) = \pi_{\alpha,\kappa}(\dot{T}_\alpha \restriction p_\alpha) = \pi_{\alpha,\kappa}(\dot{T}_\alpha).$$

With an analogous argument we can also verify that

$$p_\kappa \Vdash \pi_{\gamma_3,\kappa}(\dot{T}_{\gamma_3}) = \pi_{\gamma_3,\gamma_2}(\pi_{\gamma_3,\kappa}(\dot{T}_{\gamma_3}))$$

holds, thus proving the claim.

Finally, let $\beta < \kappa$. Choose $\alpha \in I_3$ such that $U_\alpha \cap W_\beta \subseteq U$. This is possible, as we have $|W_\beta| \leq \aleph_1 < |I_3| = \aleph_2$ and for every $i \in W_\beta \setminus U$ there is at most one $\alpha \in I_3$ such that $i \in U_\alpha$ as $\langle U_\alpha \mid \alpha \in I_3 \rangle$ is a Δ-system. But then $\pi_{\alpha,\kappa}(\dot{T}_\beta) = \dot{T}_\beta$ and $p_\kappa \Vdash$ '$\dot{T}_\kappa = \pi_{\alpha,\kappa}(\dot{T}_\alpha)$' by the previous computation. Now, applying the automorphism theorem with $\pi_{\alpha,\kappa}$ to

$$p_\alpha \Vdash \dot{T}_\alpha \text{ and } \dot{T}_\beta \text{ are almost disjoint}$$

yields that

$$p_\kappa \Vdash \dot{T}_\kappa \text{ and } \dot{T}_\beta \text{ are almost disjoint.}$$

But $p_\alpha \leq p$ and $\mathrm{dom}(p) \subseteq U$ imply $p_\kappa = \pi_{\alpha,\kappa}(p_\alpha) \leq \pi_{\alpha,\kappa}(p) = p$, contradicting

$$p \Vdash \langle \dot{T}_\alpha \mid \alpha < \kappa \rangle \text{ is a maximal a.d.f.s. family,}$$

This finishes the proof. □

Finally, we discuss how our results answer Question 2 of Newelski in [14]. In his work he constructed a forcing which adds a partition of ${}^{\omega}2$ into F_σ-sets, which is indestructible by any countably supported product of Sacks-forcing and asked if there can be such a partition, which is indestructible by any countable support iteration of Sacks-forcing. However, if we replace the use of Lemma 4.8 in the proof of Theorem 4.17 by the following lemma:

Lemma 4.20. *Let $\mathcal{T}$ be a countable family of almost disjoint nowhere dense trees on ${}^{\omega}2$, λ be a cardinal, $p \in \mathbb{S}^\lambda$ and $\dot{f}$ be a $\mathbb{S}^\lambda$-name for an element of ${}^{\omega}2$ such that for all $T \in \mathcal{T}$ we have*

$$p \Vdash \dot{f} \notin [T].$$

Then there is a nowhere dense tree S and $q \leq p$ such that $\mathcal{T} \cup \{S\}$ is almost disjoint and

$$q \Vdash \dot{f} \in [S].$$

Then we may obtain the following result with exactly the same proof:

Theorem 4.21. *Assume* CH*. Then there is a partition of ${}^{\omega}2$ into nowhere dense closed sets indestructible by any countably supported iteration or product of Sacks forcing of any length.*

Thus, we have a positive answer to Newelski's question under CH without requiring any forcing and even better for a partition into closed sets instead of just F_σ-sets. Notice that Lemma 4.20 has a similar proof as Lemma 4.8, where a routine fusion argument can be used to force the hull for the name of an element of ${}^{\omega}2$ to be a nowhere dense tree $T \subseteq {}^{<\omega}2$.

5. The consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$

In all models we considered so far we have that $\mathfrak{d} = \mathfrak{a}_{\mathrm{T}}$. In this section we will show that $\aleph_1 < \mathfrak{d} < \mathfrak{a} = \mathfrak{a}_{\mathrm{T}}$ holds in Shelah's ultrapower model for the consistency of $\aleph_1 < \mathfrak{d} < \mathfrak{a}$. Thus, $\aleph_1 < \mathfrak{d} < \mathfrak{a}_{\mathrm{T}}$ is consistent relative to a measurable. Throughout this section fix a measurable cardinal κ and a $<\kappa$-complete ultrafilter $\mathcal{U}$ on κ. First, we briefly recall the basic definitions and properties of the ultrapower forcing we will need from [15]:

Definition 5.1. Let $\mathbb{P}$ be a forcing. The ultrapower forcing of $\mathbb{P}$ by $\mathcal{U}$ is defined as the set of all equivalence classes

$$\mathbb{P}^\kappa/\mathcal{U} := \{[f]_{\mathcal{U}} \mid f : \kappa \to \mathbb{P}\}$$

where $[f]_{\mathcal{U}} = [g]_U$ iff $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in \mathcal{U}$. Usually, we will drop the subscript $\mathcal{U}$. Furthermore, we order $\mathbb{P}^\kappa/\mathcal{U}$ by $[f]_{\mathcal{U}} \leq [g]_{\mathcal{U}}$ iff $\{\alpha < \kappa \mid f(\alpha) \leq g(\kappa)\} \in \mathcal{U}$. It is easy to see that this defines a partial order. Furthermore, we have an embedding $\mathbb{P} \to \mathbb{P}^\kappa/\mathcal{U}$ of partial orders, where $p \in \mathbb{P}$ is mapped to the equivalence class of the constant map f_p, i.e. $f_p(\alpha) = p$ for all $\alpha \in \kappa$. Hence, we may identify p with $[f_p]$.

Lemma 5.2. *Let $\mathbb{P}$ be a forcing. Then the following statements hold:*

1. *$\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathcal{U}$ iff $\mathbb{P}$ is κ-cc.*
2. *If $\mu < \kappa$ and $\mathbb{P}$ is μ-cc, then also $\mathbb{P}^\kappa/\mathcal{U}$ is μ-cc.*

Proof. See [1]. □

From now on fix a c.c.c. forcing $\mathbb{P}$, so that both items of the previous lemma apply. Analogous to the average of $\mathbb{P}$-names of reals in [1] we consider the average of $\mathbb{P}$-names of finitely splitting trees. Let $\mathcal{A}(\mathbb{P})$ be the set of all maximal antichains in $\mathbb{P}$.

Definition 5.3. The pair $(A, T_\bullet)$ is a nice $\mathbb{P}$-name for a finitely splitting tree iff $A : \omega \to \mathcal{A}(\mathbb{P})$ and for every $n < \omega$ we have that $T_n : A(n) \to \mathcal{P}(^{\leq n}\omega)$ such that

(1) For all $n < \omega$ and $p \in A(n)$ we have that $T_n(p)$ is a finitely splitting n-tree.
(2) For all $n < m$ and $p \in A(n), q \in A(m)$ with $p \,||\, q$ we have that $T_m(q) \cap {}^{\leq n}\omega = T_n(p)$.

Remark 5.4. Given any $\mathbb{P}$-name $\dot{T}$ for a finitely splitting tree, for every $n < \omega$ we may choose an antichain $A(n)$ such that every element $p \in A(n)$ decides $\dot{T} \restriction n$ as $T_n(p)$. Then, the second item is also satisfied by these choices, so we have defined a nice $\mathbb{P}$-name $(A, T_\bullet)$ for a finitely splitting tree. On the other hand, from $(A, T_\bullet)$ we can define a $\mathbb{P}$-name $\dot{S}$ for a finitely splitting tree by

$$\dot{S} := \{(p, \check{s}) \mid \exists n < \omega \text{ such that } p \in A(n) \text{ and } s \in T_n(p)\}.$$

Then $\mathbb{P} \Vdash$ '$\dot{T} = \dot{S}$', so $\dot{T}$ can be represented by a nice $\mathbb{P}$-name for a finitely splitting tree. Hence, in the following we may always consider nice names for finitely splitting trees.

Next, we consider how nice $\mathbb{P}^\kappa/\mathcal{U}$-names for finitely splitting trees can be constructed from sequences of nice $\mathbb{P}$-names for finitely splitting trees and vice versa.

Remark 5.5. For every $\alpha < \kappa$ let $(A^\alpha, T^\alpha_\bullet)$ be nice $\mathbb{P}$-names for finitely splitting trees. Further, fix $\alpha < \kappa$ and enumerate $A^\alpha(n) = \{p^\alpha_{n,i} \mid i < \omega\}$ for $n < \omega$. Also, for fixed $n, i < \omega$ consider

$$[p_{n,i}] := \langle p^\alpha_{n,i} \mid \alpha < \kappa\rangle/\mathcal{U} \in \mathbb{P}^\kappa/\mathcal{U},$$
$$T_{n,i} = [T_{n,i}] := \langle T^\alpha_n(p^\alpha_{n,i}) \mid \alpha < \kappa\rangle/\mathcal{U} \in (\mathcal{P}(^{\leq n}\omega))^\kappa/\mathcal{U} = \mathcal{P}(^{\leq n}\omega).$$

By countable completeness of $\mathcal{U}$ we get that $A(n) = \{[p_{n,i}] \mid i < \omega\}$ is a maximal antichain for all $n < \omega$. Set $T_n([p_{n,i}]) := T_{n,i}$. We claim that $(A, T_\bullet)$ is a nice $\mathbb{P}^\kappa/\mathcal{U}$-name for a finitely splitting tree, which we call the average of $(A^\alpha, T^\alpha_\bullet)$. But (1) follows from the fact that $T^\alpha_n(p^\alpha_{n,i})$ is a finitely splitting tree for all $\alpha < \kappa$ and $n, i < \omega$. For item (2) let $n < m$ and assume $[p_{n,i}] \,||\, [p_{m,j}]$ for some $i, j < \omega$. But then

$$\{\alpha < \kappa \mid p^\alpha_{n,i} \,||\, p^\alpha_{m,j}\} \cap \{\alpha < \kappa \mid T_{n,i} = T^\alpha_n(p^\alpha_{n,i})\} \cap \{\alpha < \kappa \mid T_{m,j} = T^\alpha_m(p^\alpha_{m,j})\} \in \mathcal{U},$$

so choose such an $\alpha < \kappa$. Then we have that

$$T_m([p_{m,j}]) = T_{m,j} = T^\alpha_m(p^\alpha_{m,j}) \text{ and } T_n([p_{n,i}]) = T_{n,i} = T^\alpha_n(p^\alpha_{n,i}),$$

which implies $T_m([p_{m,j}]) \cap {}^{\leq n}\omega = T^\alpha_m(p^\alpha_{m,j}) \cap {}^{\leq n}\omega = T^\alpha_n(p^\alpha_{n,i}) = T_n([p_{n,i}])$, since $p^\alpha_{n,i} \,||\, p^\alpha_{m,j}$.

Conversely, assume we have a nice $\mathbb{P}^\kappa/\mathcal{U}$-name $(A, T_\bullet)$ for a finitely splitting tree. For $n < \omega$ enumerate $A(n) = \{[p_{n,i}] \mid i < \omega\}$. Note, that by countable completeness of $\mathcal{U}$ we have

$$D := \{\alpha < \kappa \mid \{p^\alpha_{n,i} \mid i < \omega\} \text{ is a maximal antichain for all } n < \omega\} \in \mathcal{D}.$$

Thus, by modifying the $p^\alpha_{n,i}$ on a small set with respect to the ultrafilter $\mathcal{D}$ we may also assume $A^\alpha(n) := \{p^\alpha_{n,i} \mid i < \omega\}$ is a maximal antichain for all $n < \omega$ and $\alpha < \kappa$. But then, defining $T^\alpha_n(p^\alpha_{n,i}) := T_n([p_{n,i}])$ yields a nice $\mathbb{P}$-name $(A^\alpha, T^\alpha_\bullet)$ for a finitely splitting tree for all $\alpha < \kappa$.

Similar to Lemma 0.3 in [1] for maximal almost disjoint families, we also have that ultrapowers of forcings destroys large witnesses for $\mathfrak{a}_{\mathrm{T}}$.

Lemma 5.6. *Assume $\dot{\mathcal{T}}$ is a $\mathbb{P}$-name for an a.d.f.s. family of size $\lambda \geq \kappa$. Then*

$$\Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \dot{\mathcal{T}} \text{ is not a maximal a.d.f.s. family.}$$

Proof. Choose $\mathbb{P}$-names for finitely splitting trees $\dot{T}^\alpha$ such that

$$\Vdash_{\mathbb{P}} \dot{\mathcal{T}} = \{\dot{T}^\alpha \mid \alpha < \lambda\}.$$

For $\alpha < \kappa$ choose nice names $(A^\alpha, T^\alpha_\bullet)$ for $\dot{T}^\alpha$ and enumerate $A^\alpha(n) = \{p^\alpha_{n,i} \mid i < \omega\}$. The average $(A, T_\bullet)$ of $\langle (A^\alpha, T^\alpha_\bullet) \mid \alpha < \kappa\rangle$ as defined in the previous remark is a nice $\mathbb{P}^\kappa/\mathcal{D}$-name for a finitely splitting tree. Let $\dot{T}$ be the corresponding $\mathbb{P}^\kappa/\mathcal{D}$-name for $(A, T_\bullet)$. We claim that

$$\Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \dot{T} \text{ is almost disjoint from } \dot{\mathcal{T}},$$

so let $\beta < \lambda$. By assumption for all $\alpha < \kappa$ with $\alpha \neq \beta$ there is a maximal antichain $\{q^\alpha_i \mid i < \omega\}$ and $\{n^\alpha_i \mid i < \omega\}$ such that

$$q^\alpha_i \Vdash_{\mathbb{P}} \dot{T}^\alpha \cap \dot{T}^\beta \subseteq {}^{\leq n^\alpha_i}\omega.$$

Consider $[q_i] := \langle q^\alpha_i \mid \alpha < \kappa\rangle/\mathcal{D} \in \mathbb{P}^\kappa/\mathcal{D}$ and $n_i = [n_i] := \langle n^\alpha_i \mid \alpha < \kappa\rangle/\mathcal{D} \in \omega^\kappa/\mathcal{D} = \omega$. Again, by countable completeness of $\mathcal{D}$ we have that $\{[q_i] \mid i < \omega\}$ is a maximal antichain. We prove that for every $i < \omega$ we have

$$[q_i] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} \dot{T}^\beta \cap \dot{T} \subseteq {}^{\leq n_i}\omega.$$

Assume not. Then there are $i < \omega$, $l > n_i$, $s \in {}^{\leq l}\omega \setminus {}^{\leq n_i}\omega$ and $[r] \leq [q_i]$ such that

$$[r] \Vdash_{\mathbb{P}^\kappa/\mathcal{D}} s \in \dot{T}^\beta \cap \dot{T}.$$

By possibly extending $[r]$ we may assume that there is a $j < \omega$ such that $[r] \leq [p_{l,j}]$ and also $s \in T_l([p_{l,j}])$. Furthermore, let $[r] = \langle r^\alpha \mid \alpha < \kappa\rangle/\mathcal{D}$. Then we have

$$\{\alpha < \kappa \mid r^\alpha \leq p^\alpha_{l,j}, r^\alpha \leq q^\alpha_i, T^\alpha_l(p^\alpha_{l,j}) = T_l([p_{l,j}]), n^\alpha_i = n_i \text{ and } r^\alpha \Vdash_{\mathbb{P}} s \in \dot{T}^\beta\} \in \mathcal{D},$$

so choose such an $\alpha < \kappa$. But $r^\alpha \leq p^\alpha_{l,j}$, $T^\alpha_l(p^\alpha_{l,j}) = T_l([p_{l,j}])$ and $s \in T_l([p_{l,j}])$ imply that

$$r^\alpha \Vdash_{\mathbb{P}} s \in \dot{T}^\alpha \cap \dot{T}^\beta.$$

On the other hand

$$q^\alpha_i \Vdash_{\mathbb{P}} \dot{T}^\alpha \cap \dot{T}^\beta \subseteq {}^{n^\alpha_i}\omega.$$

But this is a contradiction, since $r_\alpha \leq q^\alpha_i$ and $s \in {}^{\leq l}\omega \setminus {}^{\leq n_i}\omega$, but $n_i = n^\alpha_i$. □

Theorem 5.7. *Assume κ is measurable and $\kappa < \mu < \lambda$, $\lambda = \lambda^\omega$ are regular cardinals such that $\nu^\kappa < \lambda$ for all $\nu < \lambda$. Then there is a forcing extension satisfying $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = \mathfrak{a}_{\mathrm{T}} = \mathfrak{c} = \lambda$.*

Proof. This holds in Shelah's template iteration for iterating ultrapowers. There are no maximal a.d.f.s. families of size $<\mu$ as $\mathfrak{d} \leq \mathfrak{a}_{\mathrm{T}}$. Furthermore, the proof that there are no maximal a.d.f.s. families of size $\geq\mu$ works completely analogous as the proof of Theorem 2.3 in [1] where the use of Lemma 0.3 is replaced with the previous lemma. □

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UNIVERSALLY SACKS-INDESTRUCTIBLE COMBINATORIAL FAMILIES OF REALS

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Abstract. We introduce the notion of an arithmetical type of combinatorial family of reals, which serves to generalize different types of families such as mad families, maximal cofinitary groups, ultrafilter bases, splitting families and other similar types of families commonly studied in combinatorial set theory.

We then prove that every combinatorial family of reals of arithmetical type which is indestructible by the product of Sacks forcing $\mathbb{S}^{\aleph_0}$ is in fact universally Sacks-indestructible, i.e. it is indestructible by any countably supported iteration or product of Sacks-forcing of any length. Further, under CH we present a unified construction of universally Sacks-indestructible families for various arithmetical types of families. In particular we prove the existence of a universally Sacks-indestructible maximal cofinitary group under CH.

1. Introduction

One of the main objectives of combinatorial set theory is the study of subsets of reals with special additional (combinatorial) properties. Important examples are mad families, maximal cofinitary groups, maximal independent families, ultrafilter bases, unbounded families, splitting families, maximal eventually different families and many others. We refer to these as different types of combinatorial families of reals. Assume we fixed some type of combinatorial family and let $\mathcal{F}$ be a family of that type. Then, any forcing extension might add new reals, witnessing that $\mathcal{F}$ is not a family of our fixed type any more. For example, for a mad family $\mathcal{F}$ we might add a new real, which has finite intersection with every element of $\mathcal{F}$, so that $\mathcal{F}$ is not maximal in the forcing extension any more. We call such reals intruders for $\mathcal{F}$. For a forcing $\mathbb{P}$ we say that $\mathbb{P}$ preserves $\mathcal{F}$ iff forcing with $\mathbb{P}$ does not add any intruders for $\mathcal{F}$; we also say $\mathcal{F}$ is $\mathbb{P}$-indestructible.

Note that the notion of an intruder for $\mathcal{F}$ heavily depends on the type of family at hand. In many examples an intruder is a real witnessing non-maximality of the family $\mathcal{F}$, e.g. for mad families or independent families. However, in other contexts an intruder may also have different interpretations, for example:

- unbounded family $\mathcal{F} \longrightarrow$ a real dominating $\mathcal{F}$,
- splitting family $\mathcal{F} \longrightarrow$ a set not split by $\mathcal{F}$,
- ultrafilter basis $\mathcal{F} \longrightarrow$ a set A with both A and A^c not in the filter generated by $\mathcal{F}$.

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One important subarea of combinatorial set theory is the preservation of different types of combinatorial families under various forcings. The main results may then be categorized as constructions of families indestructible by different forcings under various assumptions, alternative (combinatorial) characterizations of the indestructibility of such families and implications between different types of forcing indestructibilities. We give a partial non-exhaustive overview of such results:

Maximal almost disjoint (mad) families are the most well studied types of families. In [14] Kunen constructed a Cohen-indestructible mad family under CH and Hrušák [11] and Kurilić [15] independently provided combinatorial characterizations of Cohen-indestructibility of mad families. These ideas have also been expanded to other types of forcings, such as Sacks, Miller, Laver and random forcing in [11][2]. Moreover, in [2] Brendle and Yatabe established implications between these different types of forcing indestructibilities and also considered characterizations of iterated Sacks-indestructibility of mad families.

For other types of families usually Cohen and Sacks-indestructibility are the most well-studied cases. Under CH in [8] Fischer, Schrittesser and Törnquist constructed a Cohen-indestructible maximal cofinitary group. The construction may be adapted to also obtain a Cohen-indestructible maximal eventually different (med) family. Moreover, in [7] Fischer and Schrittesser constructed a med family indestructible by any countably supported product or iteration of Sacks forcing. In [9] Fischer and Switzer introduced the notion of tightness to med families and showed that it implies Cohen-indestructibility. For independent families Shelah [18] implicitly proved the existence of a Sacks-indestructible maximal independent family, also see [1] and [4] for an explicit construction. In [17] Newelski forced the existence of a product Sacks-indestructible partition of Baire space into compact sets. The authors recently positively answered Newelski's question if there may be such a partition which is also indestructible by iterations of Sacks forcing [6]. For ultrafilter bases Laver generalized Halpern and Lauchli's results in [10] to prove that every selective ultrafilter is product Sacks-indestructible [16]; also see [3] for an analysis of Sacks-indestructibility of ultrafilters and reaping families. Of particular interest for this paper is the construction of a maximal eventually different family which is indestructible under any product or iteration of Sacks forcing of any length by Fischer and Schrittesser in [7]. Since this property is crucial for this paper we define:

Definition. A family $\mathcal{F}$ is called universally Sacks-indestructible iff $\mathcal{F}$ is indestructible under any product or iteration of Sacks forcing of any length.

In order to obtain such a universally Sacks indestructible med family, in [7] Fischer and Schrittesser constructed a $\mathbb{S}^{\aleph_0}$-indestructible med family and proved that this family is in fact universally Sacks-indestructible, where $\mathbb{S}^{\aleph_0}$ is the full support product of Sacks forcing. For the construction of a universally Sacks-indestructible partition of Baire space into compact sets the authors used a similar argument in [6]. In this paper, we will generalize these findings to various other types of combinatorial families. To this end, we introduce the notion of an arithmetical type of combinatorial family of reals and formally define the notion of an intruder (cf. Definition 3.2).

This gives us a common framework to prove theorems about forcing indestructibility for all the different types of combinatorial families of reals mentioned above at the same time:

Definition. An arithmetical type $\mathfrak{t}$ (of combinatorial families of reals) is a pair of sequences $\mathfrak{t} = ((\psi_n)_{n<\omega}, (\chi_n)_{n<\omega})$ such that both $\psi_n(w_0, w_1, \dots, w_n)$ and $\chi_n(v, w_1, \dots, w_n)$ are arithmetical formulas in $n+1$ real parameters. The domain of the type $\mathfrak{t}$ is the set

$$\mathrm{dom}(\mathfrak{t}) := \{\mathcal{F} \subseteq \mathcal{P}({}^{\omega}\omega) \mid \forall n < \omega \ \forall \{f_0, \dots, f_n\} \in [\mathcal{F}]^{n+1} \text{ we have } \psi_n(f_0, \dots, f_n)\}$$

If $\mathcal{F} \in \mathrm{dom}(\mathfrak{t})$ we say $\mathcal{F}$ is of type $\mathfrak{t}$. Further, for any $\mathcal{F}$ of type $\mathfrak{t}$ if a real g satisfies

$$\forall n < \omega \ \forall \{f_1, \dots, f_n\} \in [\mathcal{F}]^n \ \chi_n(g, f_1, \dots, f_n),$$

then we call g an intruder for $\mathcal{F}$.

Thus, in the notion of an arithmetical type we essentially require that what constitutes a suitable family and an intruder is definable by a sequence of arithmetical formulas in the above sense. We then prove the following Theorem 3.5:

Theorem. *Assume that* $\mathfrak{t}$ *is an arithmetical type and* $\mathcal{F}$ *is a* $\mathbb{S}^{\aleph_0}$*-indestructible family of type* $\mathfrak{t}$*. Then* $\mathcal{F}$ *is universally Sacks-indestructible.*

Thus, in a precise sense, indestructibility by $\mathbb{S}^{\aleph_0}$ is already the strongest form of Sacks-indestructibility one may hope for. Thus, amongst others we immediately obtain the following results (see Corollaries 5.4 and 5.33):

Corollary. *Every* $\mathbb{S}^{\aleph_0}$*-indestructible mad family is universally Sacks-indestructible.*

Corollary. *Every* $\mathbb{S}^{\aleph_0}$*-indestructible independent family and every* $\mathbb{S}^{\aleph_0}$*-indestructible ultrafilter is universally Sacks-indestructible.*

In particular, not only the med family constructed by Fischer and Schrittesser in [7] is universally Sacks-indestructible, but in fact every $\mathbb{S}^{\aleph_0}$-indestructible family already is. We also generalize the constructive part of their proof to obtain universally Sacks-indestructible families of various types under CH. However, in order to present a unified construction, we require the following additional property (see Definition 4.1):

Definition. Let $\mathfrak{t}$ be an arithmetical type. We say that $\mathfrak{t}$ satisfies elimination of intruders and write $\mathrm{EoI}(\mathfrak{t})$ holds iff the following property is satisfied: If $\mathcal{F}$ is a countable family of type $\mathfrak{t}$, $p \in \mathbb{S}^{\aleph_0}$ and $\dot{g}$ is a name for a real such that

$$p \Vdash \dot{g} \text{ is an intruder for } \mathcal{F}.$$

Then there is $q \leq p$ and a real f such that $\mathcal{F} \cup \{f\}$ is of type $\mathfrak{t}$ and

$$q \Vdash \dot{g} \text{ is not an intruder for } \mathcal{F} \cup \{f\}.$$

Now, if $\mathrm{EoI}(\mathfrak{t})$ is satisfied we prove a unified construction of a universally Sacks-indestructible witness under CH in Theorem 4.3:

Theorem. *Assume* CH *and* EoI($\mathfrak{t}$) *holds. Then there is a universally Sacks-indestructible family of type* $\mathfrak{t}$.

In Lemma 5.28 we prove that elimination of intruders indeed holds for maximal cofinitary groups. Thus, our framework yields the following new result (see Corollary 5.27):

Corollary. *Under* CH *there is a universally Sacks-indestructible maximal cofinitary group.*

Finally, since the definition of an arithmetical type requires us to work with arithmetical formulas, we prove the following technical Lemma 3.1, which is an interesting result on its own. Essentially, it allows us for any condition $p \in \mathbb{S}^{\aleph_0}$ to translate the statement "p forces an arithmetical property of the generic sequence $s_{\dot{G}}$" into an equivalent Π^1_3-statement (see Lemma 3.1):

Lemma. *Let* $\chi(v_1, \ldots, v_k, w_1, \ldots, w_l)$ *be an arithmetical formula in* $k+l$ *real parameters. Further, let* $p \in \mathbb{S}^{\aleph_0}$, $f_1, \ldots, f_l \in {}^{\omega}\omega$ *and* $g_1, \ldots, g_k$ *be codes. Then the following are equivalent:*

(1) $p \Vdash \chi(g_1^*(s_{\dot{G}}), \ldots, g_k^*(s_{\dot{G}}), f_1, \ldots, f_l)$,
(2) $\forall q \leq p\ \exists r \leq q\ \forall x \in [r]\ \chi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l)$.

Here, $s_{\dot{G}} \in {}^{\omega}({}^{\omega}2)$ is the name for the generic sequence of Sacks-reals, the codes g_i are interpreted as continuous functions $g_i^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$ and an arithmetical formula χ is a first-order formula with possibly real parameters, so that χ only contains integer quantifiers.

This paper is structured as follows: In the second section, we revisit all necessary preliminaries such as all important notions for Sacks forcing and its fusion, followed by a similar discussion for countably supported product/iteration of Sacks forcing. Furthermore, we will want to apply a nice version of continuous reading of names for countably supported products/iterations of Sacks forcing developed by Fischer and Schrittesser in [7], so in order to state their result we also go over some technicalities concerning coding of continuous functions.

In the third section, we prove the technical Lemma 3.1 just mentioned and the implication from $\mathbb{S}^{\aleph_0}$-indestructibility to universal Sacks-indestructibility (see Theorem 3.5). In the fourth section, we prove the existence of a universally Sacks-indestructible witness under CH given that elimination of intruders holds (see Theorem 4.3). In the fifth section, we show that various different types of combinatorial families of reals fit in our framework of arithmetical types. Mad families, med families and partition of Baire space are covered in Sections 5.1, 5.2 and 5.3, respectively. In Section 5.4 and 5.5 we consider maximal cofinitary groups. Finally, in Section 5.6 we cover independent families and ultrafilter bases and finish with other types of families such as unbounded, dominating, splitting and reaping families in the last Section 5.7. For all families except maximal independent families, we also provide a proof for elimination of intruders in their respective sections.

2. Preliminaries

First, we consider the basic notions and definitions used throughout this paper. We start with iterations and products of Sacks forcing and their fusion. The following lemmas are well-known, for a more detailed presentation see [12] for example.

Definition 2.1. Let $T \subseteq {}^{<\omega}2$ be a tree, i.e. T is non-empty and closed under initial subsequences.

(1) For $s, t \in {}^{<\omega}2$ we write $s \trianglelefteq t$ iff s is an initial subsequence of t.
(2) Let $s \in T$ then $T_s := \{t \in T \mid s \trianglelefteq t \text{ or } t \trianglelefteq s\}$.
(3) $\mathrm{spl}(T) := \{s \in T \mid s^\frown 0 \in T \text{ and } s^\frown 1 \in T\}$ is the set of all splitting nodes of T.
(4) T is perfect iff for all $s \in T$ there is $t \in \mathrm{spl}(T)$ such that $s \trianglelefteq t$.
(5) $\mathbb{S} := \{T \subseteq {}^{<\omega}2 \mid T \text{ is a perfect tree}\}$ ordered by inclusion is Sacks forcing.

Definition 2.2. Let $T \in \mathbb{S}$. We define the fusion ordering for Sacks forcing as follows:

(1) Let $s \in T$ then $\mathrm{succspl}_T(s)$ is the unique minimal splitting node in T extending s.
(2) $\mathrm{stem}(T) := \mathrm{succspl}_T(\emptyset)$.
(3) $\mathrm{spl}_0(T) := \{\mathrm{stem}(T)\}$ and for $n < \omega$ we set

$$\mathrm{spl}_{n+1}(T) := \{\mathrm{succspl}_T(s^\frown i) \mid s \in \mathrm{spl}_n(T), i \in 2\}.$$

$\mathrm{spl}_n(T)$ is called the n-th splitting level of T. Clearly, we have $\mathrm{spl}(T) = \bigcup_{n<\omega} \mathrm{spl}_n(T)$.
(4) Let $n < \omega$ and $S, T \in \mathbb{S}$. We write $S \leq_n T$ iff $S \subseteq T$ and $\mathrm{spl}_n(S) = \mathrm{spl}_n(T)$.

Lemma 2.3. *Let $\langle T_n \in \mathbb{S} \mid n < \omega\rangle$ be a sequence of trees such that $T_{n+1} \leq_n T_n$ for all $n < \omega$. Then $T := \bigcap_{n<\omega} T_n \in \mathbb{S}$ and $T \leq_n T_n$ for all $n < \omega$.*

We call such a sequence $\langle T_n \in \mathbb{S} \mid n < \omega\rangle$ a fusion sequence in $\mathbb{S}$ and the element $T \in \mathbb{S}$ its fusion.

Definition 2.4. Let λ be a cardinal. $\mathbb{S}^\lambda$ is the countable support product of Sacks forcing of size λ. Moreover,

(1) for $A \subseteq \mathbb{S}^\lambda$ let $\bigcap A$ be the function with $\mathrm{dom}(\bigcap A) := \bigcup_{p \in A} \mathrm{dom}(p)$ and for all $\alpha < \lambda$ we have $(\bigcap A)(\alpha) := \bigcap_{p\in A} p(\alpha)$. Notice that we do not necessarily have $\bigcap A \in \mathbb{S}^\lambda$.
(2) Let $n < \omega$, $p, q \in \mathbb{S}^\lambda$ and $F \in [\mathrm{dom}(q)]^{<\omega}$. Write $p \leq_{F,n} q$ iff $p \leq q$ and $p(\alpha) \leq_n q(\alpha)$ for all $\alpha \in F$. For $\lambda = \aleph_0$ we assume every condition has full support and write $\leq_n$ for $\leq_{n,n}$.

Lemma 2.5. *Let $\langle p_n \in \mathbb{S}^\lambda \mid n < \omega\rangle$ and $\langle F_n \in [\mathrm{dom}(p_n)]^{<\omega} \mid n < \omega\rangle$ be sequences such that*

(1) $p_{n+1} \leq_{F_n,n} p_n$ for all $n < \omega$.
(2) $F_n \subseteq F_{n+1}$ for all $n < \omega$ and $\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} \mathrm{dom}(p_n)$.

Then $p := \bigcap_{n<\omega} p_n \in \mathbb{S}^\lambda$ and $p \leq_{F_n,n} p_n$ for all $n < \omega$.

Again, we call such a sequence $\langle p_n \in \mathbb{S}^\lambda \mid n < \omega\rangle$ a fusion sequence in $\mathbb{S}^\lambda$ for $\langle F_n \mid n < \omega\rangle$ and the element $p \in \mathbb{S}^\lambda$ its fusion. In order to construct such fusion sequences we use the notion of suitable functions:

Definition 2.6. Let $p \in \mathbb{S}^\lambda$, $F \in [\mathrm{dom}(p)]^{<\omega}$, $n < \omega$ and $\sigma : F \to V$ be a suitable function for p, F and n, i.e. $\sigma(\alpha) \in \mathrm{spl}_n(p(\alpha))^\frown 2$ for all $\alpha \in F$. Then we define $p \restriction \sigma \in \mathbb{S}^\lambda$ by

$$(p \restriction \sigma)(\alpha) := \begin{cases} p(\alpha)_{\sigma(\alpha)} & \text{if } \alpha \in F, \\ p(\alpha) & \text{otherwise.} \end{cases}$$

Notice that for fixed $p \in \mathbb{S}^\lambda$, $n < \omega$ and $F \in [\mathrm{dom}(p)]^{<\omega}$ there are only finitely many σ which are suitable for p, F and n. Also, if $q \leq_{F,n} p$, then q and p have the same suitable functions for F and n. Furthermore, the set

$$\{p \restriction \sigma \mid \sigma : F \to V \text{ is a suitable function for } p, F \text{ and } n\}$$

is a maximal antichain below p. Again, if $\lambda = \aleph_0$ we just say σ is suitable for p and n in case that σ is suitable for p, n and n.

For most fusion arguments in the subsequent sections we will need the following well-known lemma. We provide a short proof for completeness.

Lemma 2.7. *Let $p \in \mathbb{S}^\lambda$, $F \in [\mathrm{dom}(p)]^{<\omega}$, $n < \omega$ and $D \subseteq \mathbb{S}^\lambda$ be dense open below p. Then there is $q \leq_{F,n} p$ such that $q \restriction \sigma \in D$ for all σ suitable for p, F and n.*

Proof. Let $\langle \sigma_i \mid i < N \rangle$ enumerate all suitable functions for p, F and n and set $q_0 := p$. We will define a $\leq_{F,n}$-decreasing sequence $\langle q_i \mid i \leq N \rangle$, so that all of the q_i have the same suitable functions as p for F and n. Assume $i < N$ and q_i is defined. Choose $r_i \leq q_i \restriction \sigma_i$ in D and define

$$q_{i+1}(\alpha) := \begin{cases} r_i(\alpha) \cup \bigcup \{q_i(\alpha)_s \mid s \in \mathrm{spl}_n(q_i(\alpha)) ^\frown 2 \text{ and } s \neq \sigma(\alpha)\} & \text{if } \alpha \in F, \\ r_i(\alpha) & \text{otherwise.} \end{cases}$$

Clearly, $q_{i+1} \leq_{F,n} q_i$ and $q_{i+1} \restriction \sigma = r_i$. Now, set $q := q_N$ and let σ be suitable for p, F and n. Choose $i < N$ such that $\sigma = \sigma_i$. Then we have $q \restriction \sigma \leq q_{i+1} \restriction \sigma = r_i \in D$, so $q \restriction \sigma \in D$ as D is open. □

Next, we briefly present a simplified version of the presentation of continuous reading of names for Sacks-forcing in [7] sufficient for our needs (see also [13]). First, we consider how to code continuous functions $f^* : {}^\omega({}^\omega 2) \to {}^\omega\omega$ by monotone and proper functions $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$:

Definition 2.8.

(1) For $s, t \in {}^{<\omega}({}^{<\omega}2)$ write $s \trianglelefteq t$ iff $\mathrm{dom}(s) \leq \mathrm{dom}(t)$ and for all $n \in \mathrm{dom}(s)$, $s(n) \trianglelefteq t(n)$.
(2) A function $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ is monotone iff for all $s \trianglelefteq t \in {}^{<\omega}({}^{<\omega}2)$, $f(s) \trianglelefteq f(t)$.
(3) A function $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ is proper iff for all $x \in {}^\omega({}^\omega 2)$:

$$|\mathrm{dom}(f(x \restriction n \times n))| \xrightarrow{n \to \infty} \infty.$$

(4) For a monotone and proper function $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ define a continuous function

$$f^* : {}^\omega({}^\omega 2) \to {}^\omega\omega \text{ via } f^*(x) := \bigcup_{n<\omega} f(x \restriction n \times n).$$

In this case f is called a code for f^*.

Remark 2.9. Conversely, for every continuous function $f^* : {}^\omega({}^\omega 2) \to {}^{<\omega}\omega$ there is a code for it. In the following, a code f will always refer to a monotone and proper function $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$.

Remark 2.10. For all $p, q \in \mathbb{S}$ there is a natural bijection $\pi : \mathrm{spl}(p) \to \mathrm{spl}(q)$, which for every $n < \omega$ restricts to bijections $\pi \restriction \mathrm{spl}_n(p) : \mathrm{spl}_n(p) \to \mathrm{spl}_n(q)$ and which preserves the lexicographical ordering. We can extend π to a monotone and proper function $\pi : p \to q$ in a similar sense as above. π then codes a homeomorphism $\pi : [p] \to [q]$ which we call the induced homeomorphism (of p and q). We usually identify both functions $\pi : p \to q$ and $\pi : [p] \to [q]$ with the same letter. Note that π is indeed a homeomorphism as its inverse is given by the induced homeomorphism from q to p.

Lemma 2.11. *Let $p, q, r \in \mathbb{S}$ with $r \leq p$ and let $\pi : [p] \to [q]$ be the induced homeomorphism. Then there is $s \in \mathbb{S}$ with $s \leq q$ and $\pi[[r]] = [s]$.*

Proof. We define

$$s := \{u \in {}^{<\omega}2 \mid \exists f \in [r]\ u \subseteq \pi(f)\}.$$

We have to show that $s \in \mathbb{P}$. Clearly, s is downwards closed, so let $u \in s$. Choose $f \in [r]$ such that $u \subseteq \pi(f)$. Since $\pi(f) \in [q]$ we have $u \in q$. By definition of the induced map choose $v \in p$ such that $v \subseteq f$ and $u \subseteq \pi(v)$. Then $v \in r$, so choose $w \in r$ with $v \subseteq w$ and $w \in \mathrm{spl}(r)$. Then, also $w \in \mathrm{spl}(p)$ and we have $\pi(w) \in \mathrm{spl}(q)$ and $\pi(v) \subseteq \pi(w)$. Since $r \in \mathbb{S}$ for $i \in 2$ we may choose $f_i \in [r]$ such that $w^\frown i \subseteq f_i$. But $\pi(w)^\frown i \subseteq \pi(f_i)$ implies $\pi(w)^\frown i \in s$ for $i \in 2$, i.e. $\pi(w) \in \mathrm{spl}(s)$. Further, $u \subseteq \pi(v) \subseteq \pi(w)$ completing the proof. □

Remark 2.12. Notice that s is uniquely determined by the property above. We call s the image of r under π. In the dual case where $p, q, s \in \mathbb{S}$ are such that $s \leq q$ and $\pi : [p] \to [q]$ is the induced homeomorphism we say that r is the preimage of s under π iff r is the image of s under π^{-1}. Here, we use that the inverse is given by the induced homeomorphism from q to p.

Definition 2.13. Let $\mathbb{P}$ be the countably supported iteration of Sacks forcing of length $\lambda \geq \omega$. Let $p \in \mathbb{P}$. By density we may always assume that $|\mathrm{dom}(p)| = \omega$ and $0 \in \mathrm{dom}(p)$.

(1) A standard enumeration of $\mathrm{dom}(p)$ is a sequence

$$\Sigma = \langle \sigma_k \mid k < \omega \rangle,$$

such that $\sigma_0 = 0$ and $\mathrm{ran}(\Sigma) = \mathrm{dom}(p)$.

(2) Let $[p]$ be a $\mathbb{P}$-name such that

$$p \Vdash [p] = \langle x \in {}^{\mathrm{dom}(p)}({}^{\omega}2) \mid \text{For all } \alpha \in \mathrm{dom}(p) \text{ we have } x(\alpha) \in [p(\alpha)] \rangle.$$

(3) Let Σ be a standard enumeration of $\mathrm{dom}(p)$. For $k < \omega$ let $\dot{e}_k^{p,\Sigma}$ be a $\mathbb{P} \restriction \sigma_k$-name such that

$$p \restriction \sigma_k \Vdash \dot{e}_k^{p,\Sigma} \text{ is the induced homeomorphism between } [p(\sigma_k)] \text{ and } {}^{\omega}2.$$

Moreover, let $\dot{e}^{p,\Sigma}$ be a $\mathbb{P}$-name such that

$$p \Vdash \dot{e}^{p,\Sigma} : [p] \to {}^{\omega}({}^{\omega}2) \text{ such that } \dot{e}^{p,\Sigma}(x) = \langle \dot{e}_k^{p,\Sigma}(x(\sigma_k)) \mid k < \omega \rangle \text{ for all } x \in [p].$$

(4) Given $s \in \mathbb{S}^{\aleph_0}$, we define the preimage r of s under $\dot{e}^{p,\Sigma}$ as follows. Let $r \in \mathbb{P}$ with $\mathrm{dom}(r) = \mathrm{dom}(p)$, where for $k < \omega$ we have that $r(\sigma_k)$ is a $\mathbb{P}\restriction\sigma_k$-name such that

$$p\restriction\sigma_k \Vdash r(\sigma_k) \text{ is the preimage of } s(k) \text{ under } \dot{e}_k^{p,\Sigma}.$$

In particular, we have

$$p\restriction\sigma_k \Vdash r(\sigma_k) \leq p(\sigma_k),$$

so that $r \leq p$. Furthermore, r satisfies for every $k < \omega$

$$r \Vdash s_{\dot{G}}(\sigma_k) \in [r(\sigma_k)],$$

so that by the previous discussion

$$r \Vdash \dot{e}_k^{p,\Sigma}(s_{\dot{G}}(\sigma_k)) \in [s(k)].$$

Thus, we obtain

$$r \Vdash \dot{e}^{p,\Sigma}(s_{\dot{G}}\restriction\mathrm{dom}(p)) \in [s].$$

Remark 2.14. For the countable support product of Sacks forcing we define the analogous notions. In fact, in this simpler case $[p]$, $\dot{e}_k^{p,\Sigma}$ and $\dot{e}^{p,\Sigma}$ can be defined as ground model objects. However, we will still treat them as names, so that we may consider both cases at the same time.

Definition 2.15. Let $\mathbb{P}$ be the countable support iteration or product of Sacks forcing of any length. Let $q \in \mathbb{P}$ and $\dot{f}$ be a $\mathbb{P}$-name for a real. Let $\Sigma = \langle \sigma_k \mid k < \omega \rangle$ be a standard enumeration of $\mathrm{dom}(q)$ and $f : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ be a code for a continuous function $f^* : {}^{\omega}({}^{\omega}2) \to {}^{\omega}\omega$ such that

$$q \Vdash \dot{f} = (f^* \circ \dot{e}^{q,\Sigma})(s_{\dot{G}}\restriction\mathrm{dom}(q)).$$

Then we say $\dot{f}$ is read continuously below q (by f and Σ).

Lemma 2.16 (Lemma 4 of [7]). *Let $\mathbb{P}$ be the countable support iteration or product of Sacks forcing of length λ. Suppose $p \in \mathbb{P}$ and $\dot{f}$ is a $\mathbb{P}$-name for a real. Then there is $q \leq p$ such that $\dot{f}$ is read continuously below q.*

Remark 2.17. For any $p \in \mathbb{P}$ and $\mathbb{P}$-name $\dot{f}$ for a real it is easy to see that if $\dot{f}$ is read continuously below p then for all $q \leq p$ also $\dot{f}$ is read continuously below q. Thus, the previous lemma shows that the set

$$\{q \in \mathbb{P} \mid \dot{f} \text{ is read continuously below } q\}$$

is dense open in $\mathbb{P}$.

3. $\mathbb{S}^{\aleph_0}$-indestructibility implies universal Sacks-indestructibility

Let us first consider the following application of Π^1_1-absoluteness. Given $p \in \mathbb{S}$, real parameters $f_1, \dots, f_n$ and a Π^1_1-formula $\chi(v, w_1, \dots, w_n)$ with $n+1$ real parameters assume the following holds

$$\forall q \leq p\ \exists r \leq q\ \forall x \in [r]\ \chi(x, f_1, \dots, f_n).$$

Then, we claim that also $p \Vdash \chi(s_{\dot{G}}, f_1, \dots, f_n)$ holds. Indeed, let $q \leq p$. By assumption choose $r \leq q$ such that

$$\forall x \in [r]\ \chi(x, f_1, \dots, f_n).$$

This is a Π^1_1-statement, so that by Π^1_1-absoluteness

$$\mathbb{S} \Vdash \forall x \in [r]\ \chi(x, f_1, \dots, f_n).$$

But we also have $r \Vdash$ "$s_{\dot G} \in [r]$", so that

$$r \Vdash \chi(s_{\dot G}, f_1, \dots, f_n),$$

proving the statement. The main goal of this chapter will be to show that if χ is an arithmetical formula, then we can also prove the converse, namely that $p \Vdash \chi(s_{\dot G}, f_1, \dots, f_n)$ implies

$$\forall q \leq p\ \exists r \leq q\ \forall x \in [r]\ \chi(x, f_1, \dots, f_n).$$

Even better, we will show that we also have an analogous equivalence for $\mathbb{S}^{\aleph_0}$ in place of $\mathbb{S}$. Thus, we are able to transfer forcing statements about arithmetical properties of the generic Sacks-sequence into Π^1_3-formulas and back, which will be one of the main ingredients for Theorem 3.5.

Lemma 3.1. *Let $\chi(v_1, \dots, v_k, w_1, \dots, w_l)$ be an arithmetical formula in $k + l$ real parameters. Further, let $p \in \mathbb{S}^{\aleph_0}$, $f_1, \dots, f_l \in {}^\omega\omega$ and $g_1, \dots, g_k$ be codes. Then the following are equivalent:*

(1) $p \Vdash \chi(g_1^*(s_{\dot G}), \dots, g_k^*(s_{\dot G}), f_1, \dots, f_l)$,

(2) $\forall q \leq p\ \exists r \leq q\ \forall x \in [r]\ \chi(g_1^*(x), \dots, g_k^*(x), f_1, \dots, f_l)$.

Proof. First, assume (2) and let $q \leq p$. By assumption choose $r \leq q$ such that

$$\forall x \in [r]\ \chi(g_1^*(x), \dots, g_k^*(x), f_1, \dots, f_l).$$

This is a Π^1_1-statement, so that Π^1_1-absoluteness implies

$$\mathbb{S}^{\aleph_0} \Vdash \forall x \in [r]\ \chi(g_1^*(x), \dots, g_k^*(x), f_1, \dots, f_l).$$

But we also have $r \Vdash$ "$s_{\dot G} \in [r]$", which implies

$$r \Vdash \chi(g_1^*(s_{\dot G}), \dots, g_k^*(s_{\dot G}), f_1, \dots, f_l).$$

Thus, we proved (1).

For the other direction, we may assume that all integer quantifiers are in the front of χ and do an induction over the number of quantifiers of χ. Let $q \leq p$. First, we have to consider the quantifier-free case. Then $\chi(v_1, \dots, v_k, w_1, \dots, w_l)$ only depends on finitely many function values of $v_1, \dots, v_k, w_1, \dots, w_l$. So choose N such that $\chi(v_1, \dots, v_k, w_1, \dots, w_l)$ only depends on the values of $v_1 \restriction N, \dots, v_k \restriction N, w_1 \restriction N, \dots, w_l \restriction N$. As $g_1, \dots, g_k$ are codes by Π^1_1-absoluteness we get

$$q \Vdash \exists K\ N \subseteq \operatorname{dom}(g_i(s_{\dot G} \restriction K \times K)) \text{ for all } i \in \{1, \dots, k\},$$

so choose $r \leq q$ and $K < \omega$ such that

$$r \Vdash N \subseteq \operatorname{dom}(g_i(s_{\dot G} \restriction K \times K)) \text{ for all } i \in \{1, \dots, k\}.$$

Now, choose any $x \in [r]$ and define $r_x \leq r$ by

$$r_x(n) := r(n)_{x(n) \restriction K},$$

which is well-defined since $x(n) \restriction K \in r(n)$ follows from $x \in [r]$. But then

$$r_x \Vdash s_{\dot{G}} \restriction K \times K = x \restriction K \times K,$$

so by choice of r and K we also have

$$r_x \Vdash g_i^*(s_{\dot{G}}) \restriction N = g_i^*(x) \restriction N \text{ for all } i \in \{1, \ldots, k\}.$$

But $r_x \Vdash \chi(g_1^*(s_{\dot{G}}), \ldots, g_k^*(s_{\dot{G}}), f_1, \ldots, f_l)$, so by choice of N we obtain

$$r_x \Vdash \chi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l).$$

Thus, we have proven $\forall x \in [r]\ \chi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l)$.

Next, we have to prove the induction step. We handle the two different quantifier cases separately. First, assume that $\chi \equiv \exists n \psi$, so by assumption

$$q \Vdash \exists n\ \psi(g_1^*(s_{\dot{G}}), \ldots, g_k^*(s_{\dot{G}}), f_1, \ldots, f_l, n).$$

Choose $r \leq q$ and $n < \omega$ such that

$$r \Vdash \psi(g_1^*(s_{\dot{G}}), \ldots, g_k^*(s_{\dot{G}}), f_1, \ldots, f_l, n).$$

By induction assumption choose $s \leq r$ such that

$$\forall x \in [s]\ \psi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l, n).$$

Then, we also have

$$\forall x \in [s]\ \exists n\ \psi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l, n).$$

Thus, we have proven $\forall x \in [s]\ \chi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l)$.

Finally, assume that $\chi \equiv \forall n \psi$. We construct a fusion sequence $\langle q_n \mid n < \omega \rangle$ below q as follows. Set $q_0 := q$. Assume q_n is defined. By induction assumption the set

$$D_n := \{r \leq q_n \mid \forall x \in [r]\ \psi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l, n)\}$$

is dense open below q_n. By Lemma 2.7 take $q_{n+1} \leq_n q_n$ such that $q_{n+1} \restriction \sigma \in D_n$ for all σ suitable for q_n and n. Notice that

$$[q_{n+1}] = \bigcup_{\sigma \text{ suitable for } q_n \text{ and } n} [q_{n+1} \restriction \sigma],$$

since $\{q_{n+1} \restriction \sigma \mid \sigma \text{ is suitable for } q_n \text{ and } n\}$ is a maximal antichain below q_{n+1}. But this implies

$$\forall x \in [q_{n+1}]\ \psi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l, n),$$

for if $x \in [q_{n+1}]$ choose σ suitable for q_n and n such that $x \in [q_{n+1} \restriction \sigma]$. Then, the desired conclusion follows from $q_{n+1} \restriction \sigma \in D_n$. Finally, let r be the fusion of $\langle q_n \mid n < \omega \rangle$. We claim that

$$\forall x \in [r]\ \forall n\ \psi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l, n),$$

so let $x \in [r]$ and $n < \omega$. Then $r \leq q_{n+1}$, so that $x \in [r] \subseteq [q_{n+1}]$. Hence, by construction of q_{n+1}

$$\psi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l, n).$$

Thus, we have proven $\forall x \in [r]\ \chi(g_1^*(x), \ldots, g_k^*(x), f_1, \ldots, f_l)$. □

Next, we will introduce the notion of an arithmetical type. The point of this definition is to provide a general framework for proving statements about forcing indestructibility for many different types of families at the same time. In Section 4 we will verify that indeed most types of combinatorial families studied in combinatorial set theory fit into this framework.

Definition 3.2. An arithmetical type $\mathfrak{t}$ (of combinatorial families of reals) is a pair of sequences $\mathfrak{t} = ((\psi_n)_{n<\omega}, (\chi_n)_{n<\omega})$ such that both $\psi_n(w_0, w_1, \dots, w_n)$ and $\chi_n(v, w_1, \dots, w_n)$ are arithmetical formulas in $n+1$ real parameters. The domain of the type $\mathfrak{t}$ is the set

$$\operatorname{dom}(\mathfrak{t}) := \{\mathcal{F} \subseteq \mathcal{P}({}^{\omega}\omega) \mid \forall n < \omega\ \forall \{f_0, \dots, f_n\} \in [\mathcal{F}]^{n+1} \text{ we have } \psi_n(f_0, \dots, f_n)\}$$

If $\mathcal{F} \in \operatorname{dom}(\mathfrak{t})$ we say $\mathcal{F}$ is of type $\mathfrak{t}$. Further, for any $\mathcal{F}$ of type $\mathfrak{t}$ if a real g satisfies

$$\forall n < \omega\ \forall \{f_1, \dots, f_n\} \in [\mathcal{F}]^n\ \chi_n(g, f_1, \dots, f_n),$$

then we call g an intruder for $\mathcal{F}$.

Thus, the sequence $(\psi_n)_{n<\omega}$ defines what constitutes a family of that type and the sequence $(\chi_n)_{n<\omega}$ defines which reals constitute intruders. Note that in some specific examples these two properties coincide, e.g. for eventually different families both $\psi_1(w_0, w_1)$ and $\chi_1(v, w_1)$ assert the eventual difference of w_0 (or v, resp.) and w_1 (cf. Section 5.2). Also, if we want no restriction of what constitutes a family of type $\mathfrak{t}$, we set $\psi_n :\equiv \top$ for all $n < \omega$ to obtain $\operatorname{dom}(\mathfrak{t}) = \mathcal{P}(\mathcal{P}({}^{\omega}\omega))$. This will be the case for the families considered in Section 5.7.

Lemma 3.3. *Let t be an arithmetical type. Then we have the following:*

(1) $\emptyset$ is of type $\mathfrak{t}$,
(2) If $\mathcal{G}$ is of type $\mathfrak{t}$ and $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F}$ is of type $\mathfrak{t}$,
(3) Let δ be a limit ordinal. If $\langle \mathcal{F}_\alpha \mid \alpha < \delta \rangle$ is an increasing sequence of families of type $\mathfrak{t}$, then also $\mathcal{F} := \bigcup_{\alpha<\delta} \mathcal{F}_\alpha$ is a family of type $\mathfrak{t}$.

Proof. (1) and (2) are obvious. For (3) let $\langle \mathcal{F}_\alpha \mid \alpha < \delta \rangle$ be an increasing sequence of families of type $\mathfrak{t}$, $n < \omega$ and $\{f_0, \dots, f_n\} \in [\mathcal{F}]^{n+1}$. Since δ is a limit, we may choose $\alpha < \delta$ such that $\{f_0, \dots, f_n\} \in [\mathcal{F}_\alpha]^{n+1}$. But then $\psi_n(f_0, \dots, f_n)$ holds since $\mathcal{F}_\alpha$ is of type $\mathfrak{t}$. □

Also, note that since the ψ_n are arithmetical formulas the notion of $\operatorname{dom}(\mathfrak{t})$ is absolute, i.e. for transitive models of set theory $M \subseteq N$ we have that $\operatorname{dom}(\mathfrak{t})^M = \operatorname{dom}(\mathfrak{t})^N \cap M$. Analogously, the notion of an intruder is absolute. However, a family may have no intruders in M, but some in the larger model N. Thus, we define the following:

Definition 3.4. Given a forcing $\mathbb{P}$ and a family $\mathcal{F}$ of type $\mathfrak{t}$ we say that $\mathcal{F}$ is $\mathbb{P}$-indestructible or $\mathbb{P}$ preserves $\mathcal{F}$ iff $\mathbb{P}$ forces that $\mathcal{F}$ has no intruders. In particular, $\mathcal{F}$ has no intruders in the ground model. If $\mathcal{F}$ is indestructible by any countably supported product or iteration of Sacks-forcing of any length, we say that $\mathcal{F}$ is universally Sacks-indestructible.

Now, equipped with Lemma 3.1 and these definitions we may now prove one of our main results, namely that for arithmetical types of combinatorial families indestructibility by $\mathbb{S}^{\aleph_0}$ is already the strongest form of Sacks-indestructibility one may hope for.

Theorem 3.5. *Assume* $\mathfrak{t}$ *is an arithmetical type and* $\mathcal{F}$ *is a* $\mathbb{S}^{\aleph_0}$*-indestructible family of type* $\mathfrak{t}$*. Then* $\mathcal{F}$ *is universally Sacks-indestructible.*

Proof. Let $\mathbb{P}$ be the countably supported product or iteration of Sacks-forcing of any length and assume that $\mathcal{F}$ is not preserved by $\mathbb{P}$. As the notion of an intruder is absolute, we may assume that the size of the product or the length of the iteration is at least $\aleph_0$. Choose $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{g}$ for a real such that

$$p \Vdash_{\mathbb{P}} \dot{g} \text{ is an intruder for } \mathcal{F}.$$

By Lemma 2.16 choose $q \leq p$, a standard enumeration Σ of $\mathrm{dom}(q)$ and a code g such that

$$q \Vdash_{\mathbb{P}} \dot{g} = g^*(\dot{e}^{q,\Sigma}(s_{\dot{G}} \restriction \mathrm{dom}(q))).$$

Since $\mathcal{F}$ is $\mathbb{S}^{\aleph_0}$-indestructible we have

$$\mathbb{S}^{\aleph_0} \Vdash_{\mathbb{S}^{\aleph_0}} g^*(s_{\dot{G}}) \text{ is not an intruder for } \mathcal{F},$$

which is by Definition 3.2 expressed by

$$\mathbb{S}^{\aleph_0} \Vdash_{\mathbb{S}^{\aleph_0}} \exists n < \omega\ \exists \{f_1, \dots, f_n\} \in [\mathcal{F}]^n\ \neg\chi_n(g^*(s_{\dot{G}}), f_1, \dots, f_n).$$

Thus, we may choose $s \in \mathbb{S}^{\aleph_0}$, $n < \omega$ and $\{f_1, \dots, f_n\} \in [\mathcal{F}]^n$ such that

$$s \Vdash_{\mathbb{S}^{\aleph_0}} \neg\chi_n(g^*(s_{\dot{G}}), f_1, \dots, f_n).$$

Since $\neg\chi_n$ is an arithmetical formula by Lemma 3.1 choose $t \leq s$ such that

$$\forall x \in [t]\ \neg\chi_n(g^*(x), f_1, \dots, f_n).$$

Now, this is a Π^1_1-formula, so we obtain

$$\mathbb{P} \Vdash_{\mathbb{P}} \forall x \in [t]\ \neg\chi_n(g^*(x), f_1, \dots, f_n).$$

As in Definition 2.13 let r be the preimage of t under $\dot{e}^{q,\Sigma}$, i.e. we have $r \leq q$ and

$$r \Vdash_{\mathbb{P}} \dot{e}^{q,\Sigma}(s_{\dot{G}} \restriction \mathrm{dom}(q)) \in [t].$$

Combining both statements yields

$$r \Vdash_{\mathbb{P}} \neg\chi_n(g^*(\dot{e}^{q,\Sigma}(s_{\dot{G}} \restriction \mathrm{dom}(q)), f_1, \dots, f_n).$$

But by choice of g we get

$$r \Vdash_{\mathbb{P}} \neg\chi_n(\dot{g}, f_1, \dots, f_n),$$

contradicting that r forces $\dot{g}$ to be an intruder for $\mathcal{F}$. □

4. Constructing Sacks-indestructible maximal combinatorial families under CH

In this section, we provide a unified construction of universally Sacks-indestructible families under CH. Note that by Theorem 3.5 we only need to construct $\mathbb{S}^{\aleph_0}$-indestructible families. Towards a unified construction, we will need the following additional property:

Definition 4.1. Let $\mathfrak{t}$ be an arithmetical type. We say that $\mathfrak{t}$ satisfies elimination of intruders and write $\mathrm{EoI}(\mathfrak{t})$ holds iff the following property is satisfied: If $\mathcal{F}$ is a countable family of type $\mathfrak{t}$, $p \in \mathbb{S}^{\aleph_0}$ and $\dot{g}$ is a name for a real such that

$$p \Vdash \dot{g} \text{ is an intruder for } \mathcal{F}.$$

Then there is $q \leq p$ and a real f such that $\mathcal{F} \cup \{f\}$ is of type $\mathfrak{t}$ and

$$q \Vdash \dot{g} \text{ is not an intruder for } \mathcal{F} \cup \{f\}.$$

As the name suggests, $\mathrm{EoI}(\mathfrak{t})$ essentially asserts that for every countable family $\mathcal{F}$ of type $\mathfrak{t}$ and every $\mathbb{S}^{\aleph_0}$-name $\dot{g}$ for a possible intruder for $\mathcal{F}$ we may extend $\mathcal{F}$ by one element f so that $\dot{g}$ is not an intruder for this extended family any more. Usually, in case that some $q \leq p$ forces "$\dot{g} = f \in V$" the conclusion of $\mathrm{EoI}(\mathfrak{t})$ holds trivially for $q := p$ and $\mathcal{F} \cup \{f\}$ or some other canonical extension of $\mathcal{F}$. Hence, in the following section we will additionally assume that $p \Vdash$ "$\dot{g} \notin V$" when verifying $\mathrm{EoI}(\mathfrak{t})$ for some arithmetical type $\mathfrak{t}$.

For the construction, we use CH and continuous reading of names to diagonalize against all possible intruders for our constructed family in length $\aleph_1$. Hence, we obtain a $\mathbb{S}^{\aleph_0}$-indestructible and thus universally Sacks-indestructibly family of type $\mathfrak{t}$. Note that in the following construction we make use of Lemma 3.3 multiple times.

Theorem 4.2. *Assume* CH *and* $\mathrm{EoI}(\mathfrak{t})$ *holds. Then there is a* $\mathbb{S}^{\aleph_0}$*-indestructible family of type* $\mathfrak{t}$.

Proof. By CH we may enumerate all pairs $\langle (p_\alpha, g_\alpha) \mid \alpha < \aleph_1 \rangle$ of elements $p \in \mathbb{S}^{\aleph_0}$ and codes g. We construct an increasing and continuous sequence $\langle \mathcal{F}_\alpha \mid \alpha < \aleph_1 \rangle$ of families of type $\mathfrak{t}$ as follows:

Set $\mathcal{F}_0 := \emptyset$. Now, assume $\mathcal{F}_\alpha$ is defined and (p_α, g_α) is given. If we have

$$p_\alpha \nVdash g^*_\alpha(s_{\dot{G}}) \text{ is an intruder for } \mathcal{F}_\alpha,$$

then set $\mathcal{F}_{\alpha+1} := \mathcal{F}_\alpha$. Otherwise, we have

$$p_\alpha \Vdash g^*_\alpha(s_{\dot{G}}) \text{ is an intruder for } \mathcal{F}_\alpha.$$

Thus, by $\mathrm{EoI}(\mathfrak{t})$ choose $q \leq p_\alpha$ and a real f such that the family $\mathcal{F}_{\alpha+1} := \mathcal{F}_\alpha \cup \{f\}$ is of type $\mathfrak{t}$ and such that

$$q \Vdash g^*_\alpha(s_{\dot{G}}) \text{ is not an intruder for } \mathcal{F}_{\alpha+1}.$$

Finally, we set $\mathcal{F} := \bigcup_{\alpha<\aleph_1} \mathcal{F}_\alpha$. We show that $\mathcal{F}$ is $\mathbb{S}^{\aleph_0}$-indestructible, so assume the contrary. Choose $p \in \mathbb{S}^{\aleph_0}$ and a $\mathbb{S}^{\aleph_0}$-name $\dot{g}$ for a real such that

$$p \Vdash \dot{g} \text{ is an intruder for } \mathcal{F}.$$

By Lemma 2.16 choose $q \leq p$ and a code $g : {}^{<\omega}({}^{<\omega}2) \to {}^{<\omega}\omega$ so that

$$q \Vdash \dot{g} = g^*(s_{\dot{G}}).$$

But then, the pair (q, g) appeared at some step α in our enumeration and we extended $\mathcal{F}_\alpha$ using $\mathrm{EoI}(\mathfrak{t})$ at that step. Thus, there is $r \leq q$ with

$$r \Vdash g^*(s_{\dot{G}}) \text{ is not an intruder for } \mathcal{F}_{\alpha+1},$$

which implies

$$r \Vdash \dot{g} \text{ is not an intruder for } \mathcal{F},$$

a contradiction to the choice of p and $\dot{g}$. □

Remember, that by Theorem 3.5 we have that $\mathbb{S}^{\aleph_0}$-indestructibility in fact implies universal Sacks-indestructibility, so we proved the following theorem:

Theorem 4.3. *Assume* CH *and* $\mathrm{EoI}(\mathfrak{t})$ *holds. Then there is a universally Sacks-indestructible family of type* $\mathfrak{t}$.

Proof. This follows directly by composing the previous theorem with Theorem 3.5. □

Thus, the remaining objective for this paper will the verification that many different types of combinatorial families fall into our framework of arithmetical types and proving that elimination of intruders holds for these types.

5. Applications of our framework

In this section, we provide multiple applications of the framework developed in the previous sections for universal Sacks-indestructibility.

5.1. **Mad families.** We begin our list of example applications with one of the most common type of combinatorial families, that is maximal almost disjoint families. For this case, we will explain a bit more explicitly how to phrase the definitions in a suitable way, so that we may apply Theorem 3.5. For the subsequent sections we will just mention the necessary coding arguments, but omit the analogous details. Remember the following definition:

Definition 5.1. A family $\mathcal{A}$ of infinite subsets of ω is almost disjoint (a.d.) iff $A \cap B$ is finite for all $A \neq B \in \mathcal{A}$ and for all $\mathcal{A}_0 \in [\mathcal{A}]^{<\omega}$ we have that $\omega \setminus \bigcup \mathcal{A}_0$ is infinite. $\mathcal{A}$ is called maximal (mad) iff it is maximal with respect to inclusion. The corresponding cardinal characteristic is the almost disjointness number $\mathfrak{a}$:

$$\mathfrak{a} := \min \{|\mathcal{A}| \mid \mathcal{A} \text{ is a mad family}\}.$$

Note that the second property in our (slightly unusual) definition of an a.d. family is vacuous if $\mathcal{A}$ is infinite, but in the finite case it allows us to exclude maximal finite a.d. families. Also, Theorem 3.5 is applied to combinatorial families on the Polish space ${}^{\omega}\omega$, so to be more precise we code a.d. families in that Polish space:

Definition 5.2. A family $\mathcal{F}$ of reals codes an a.d. family iff every $f \in \mathcal{F}$ codes an infinite subset of ω, i.e. f is a strictly increasing function, $\mathrm{ran}(f) \cap \mathrm{ran}(g)$ is finite for all $f \neq g \in \mathcal{F}$ and for all $\mathcal{F}_0 \in [\mathcal{F}]^{<\omega}$ we have that $\omega \setminus \bigcup_{f \in \mathcal{F}_0} \mathrm{ran}(f)$ is infinite. $\mathcal{F}$ is called maximal iff it is maximal with respect to inclusion.

Proposition 5.3. *Coded mad families are an arithmetical type.*

Proof. We define the formula $\psi_0(w_0)$ to be

$$\forall n \forall m (n < m \text{ implies } w_0(n) < w_0(m)),$$

expressing 'w_0 codes an infinite subset of ω'. Further, $\psi_1(w_0, w_1)$ is defined as

$$\exists N \forall n \forall m (n > N \text{ implies } w_0(n) \neq w_1(m)),$$

expressing '$\mathrm{ran}(w_0) \cap \mathrm{ran}(w_1)$ is finite'. Finally, for $n > 1$ we define $\psi_n(w_0, \dots, w_n)$ by

$$\forall N \exists n \forall m (n > N \text{ and } \bigwedge_{i=0}^{n} w_i(m) \neq n),$$

expressing '$\omega \setminus \bigcup_{i=0}^{n} \mathrm{ran}(w_i)$ is infinite'. Analogously, we define the formula $\chi_0(v)$ to be

$$\forall n \forall m (n < m \text{ implies } v(n) < v(m)),$$

expressing 'v codes an infinite subset of ω'. Finally, we define $\chi_1(v, w_1)$ by

$$\exists N \forall n \forall m (n > N \text{ implies } v(n) \neq w_1(m)),$$

expressing '$\mathrm{ran}(v) \cap \mathrm{ran}(w_1)$ is finite' and set $\chi_n :\equiv \top$ for all $n > 1$. Clearly, with respect to Definition 3.2 this exactly captures our definition of a coded mad family. Hence, coded mad families are an arithmetical type. □

Thus, explicitly for mad families Theorem 3.5 implies that $\mathbb{S}^{\aleph_0}$-indestructibility for families which code a mad family implies universal Sacks-indestructibility. Since the coding is absolute this is equivalent to the respective version without coding:

Corollary 5.4. *Every $\mathbb{S}^{\aleph_0}$-indestructible mad family is universally Sacks-indestructible.*

Next, we prove $\mathrm{EoI}(\mathfrak{a})$, i.e. elimination of intruders for mad families. To be more precise we would have to prove elimination of intruders for families which code mad families, but since our coding is absolute these are easily seen to be equivalent. The following proof is folklore, but we provide it for completeness.

Lemma 5.5. $\mathrm{EoI}(\mathfrak{a})$ *holds. That is, if $\mathcal{A}$ is a countable a.d. family, $p \in \mathbb{S}^{\aleph_0}$ and $\dot{B}$ is a name for an infinite subset of ω such that*

$$p \Vdash \dot{B} \notin V \text{ and } \mathcal{A} \cup \{\dot{B}\} \text{ is an a.d. family.}$$

Then there is $q \leq p$ and an infinite subset A of ω such that $\mathcal{A} \cup \{A\}$ is an a.d. family and

$$q \Vdash \mathcal{A} \cup \{A, \dot{B}\} \text{ is not an a.d. family.}$$

Proof. If $\mathcal{A}$ is finite we have that $\omega \setminus \bigcup \mathcal{A}$ is infinite, so let $D \cup E$ be a partition of $\omega \setminus \bigcup A$ into two infinite sets. By assumption for every $A \in \mathcal{A}$ we have

$$p \Vdash \exists k < \omega \; A \cap \dot{B} \subseteq k.$$

Since $\mathcal{A}$ is finite choose $q \leq p$ and $k < \omega$ such that for all $A \in \mathcal{A}$

$$q \Vdash A \cap \dot{B} \subseteq k.$$

This implies that

$$q \Vdash \dot{B} \setminus k \subseteq \omega \setminus \bigcup \mathcal{A} = D \cup E.$$

Since $\dot{B}$ is a name for an infinite subset of ω there is $r \leq q$ such that

$$r \Vdash D \cap \dot{B} \text{ is infinite} \quad \text{or} \quad r \Vdash E \cap \dot{B} \text{ is infinite}.$$

W.l.o.g. assume the first case holds. Then $\mathcal{A} \cup \{D\}$ is an a.d. family and

$$r \Vdash \mathcal{A} \cup \{D, \dot{B}\} \text{ is not an a.d. family}.$$

Now, assume that $\mathcal{A}$ is infinite, so enumerate $\mathcal{A} = \{A_n \mid n < \omega\}$. We construct a fusion sequence $\langle p_n \mid n < \omega \rangle$ below p and a sequence $\langle k_n < \omega \mid n < \omega \rangle$ as follows. Set $p_0 := p$. Now, assume p_n is defined. By assumption the set

$$D_n := \{q \leq p_n \mid \exists k < \omega \; q \Vdash \text{“} A_n \cap \dot{B} \subseteq k \text{”}\}$$

is dense open below p_n. By Lemma 2.7 take $p_{n+1} \leq_n p_n$ such that $p_{n+1} \restriction \sigma \in D_n$ for all σ suitable for p_n and n, witnessed by $k_\sigma < \omega$. Set $k_n := \max\{k_\sigma \mid \sigma \text{ suitable for } p_n \text{ and } n\}$, so that

$$p_{n+1} \Vdash A_n \cap \dot{B} \subseteq k_n.$$

Let q_0 be the fusion of $\langle p_n \mid n < \omega \rangle$. We define a second fusion sequence $\langle q_n \mid n < \omega \rangle$ below q_0 and a sequence $\langle a_n \in [\omega]^{<\omega} \mid n < \omega \rangle$ of finite disjoint sets. Assume q_n is defined and choose $K < \omega$ with $K > \max(a_m), k_m$ for all $m < n$. Since $q_n \Vdash$ “$\dot{B}$ is infinite” the set

$$E_n := \{q \leq q_n \mid \exists k > K \; q \Vdash \text{“} k \in \dot{B} \text{”}\}$$

is dense open below p_n. Again, by Lemma 2.7 take $q_{n+1} \leq_n q_n$ such that $q_{n+1} \restriction \sigma \in E_n$ for all σ suitable for q_n and n, witnessed by k_σ. Set $a_n := \{k_\sigma \mid \sigma \text{ suitable for } q_n \text{ and } n\}$, so that

$$q_{n+1} \Vdash \dot{B} \cap a_m \neq \emptyset.$$

By choice of k_m we also have that $a_n \cap A_m = \emptyset$ for all $m < n$. Finally, let q be the fusion of $\langle q_n \mid n < \omega \rangle$ and set $A := \bigcup_{n<\omega} a_n$. Then we have that $\mathcal{A} \cup \{A\}$ is an a.d. family and

$$q \Vdash A \cap \dot{B} \text{ is infinite}.$$

Since $q \Vdash$ “$\dot{B} \notin V$” we have $q \Vdash$ “$A \neq \dot{B}$”, so we obtain

$$q \Vdash \mathcal{A} \cup \{A, \dot{B}\} \text{ is not an a.d. family},$$

finishing the proof. □

Thus, we proved $\mathrm{EoI}(\mathfrak{a})$ and we get the following result as an instance of Theorem 4.3 for maximal almost disjoint families:

Corollary 5.6. *Assume* CH. *Then there is a universally Sacks-indestructible mad family.*

5.2. **Eventually different families.** Next, we will briefly consider maximal eventually different families. This case is especially noteworthy as the idea for the framework developed in this paper originates from Fischer's and Schrittesser's construction of a universally Sacks-indestructible eventually different family in [7]. The main definition is the following:

Definition 5.7. A family of reals $\mathcal{F} \subseteq {}^{\omega}\omega$ is eventually different (ed) iff for all $f \neq g \in \mathcal{F}$ there is $N < \omega$ such that $f(n) \neq g(n)$ for all $n > N$. $\mathcal{F}$ is called maximal (med) iff it is maximal with respect to inclusion. The corresponding cardinal characteristic is $\mathfrak{a}_{\mathrm{e}}$:

$$\mathfrak{a}_{\mathrm{e}} := \min \{|\mathcal{F}| \mid \mathcal{F} \text{ is a med family}\}.$$

Proposition 5.8. *Med families are an arithmetical type.*

Proof. We set $\psi_n :\equiv \top$ for $n \neq 1$ and define the formula $\psi_1(w_0, w_1)$ to be

$$\exists N \forall n (n > N \text{ implies } w_0(n) \neq w_1(n)),$$

expressing 'w_0 and w_1 are eventually different'. Analogously, we set $\chi_n \equiv \top$ for $n \neq 1$ and define the formula $\chi_1(v, w_1)$ to express 'v and w_1 are eventually different'. Thus, med families are an arithmetical type. □

Hence, Theorem 3.5 implies for this instance:

Corollary 5.9. *Every $\mathbb{S}^{\aleph_0}$-indestructible med family is universally Sacks-indestructible.*

Note that Fischer and Schrittesser essentially proved $\mathrm{EoI}(\mathfrak{a}_{\mathrm{e}})$ in [7]. More precisely, combining $\mathrm{EoI}(\mathfrak{a}_{\mathrm{e}})$ with Lemma 3.1 exactly yields their Lemma 7, which they used to construct their universally Sacks-indestructible med family. For completeness, we present a direct proof of $\mathrm{EoI}(\mathfrak{a}_{\mathrm{e}})$, which will also serve as a rough template for the corresponding lemma for maximal cofinitary groups in one of the following sections.

Lemma 5.10. $\mathrm{EoI}(\mathfrak{a}_{\mathrm{e}})$ *holds. That is, if $\mathcal{F}$ is a countable e.d. family, $p \in \mathbb{S}^{\aleph_0}$ and $\dot{g}$ is a name for a real such that*

$$p \Vdash \dot{g} \notin V \text{ and } \mathcal{F} \cup \{g\} \text{ is an e.d. family.}$$

Then there is $q \leq p$ and a real f such that $\mathcal{F} \cup \{f\}$ is an e.d. family and

$$q \Vdash \mathcal{F} \cup \{f, \dot{g}\} \text{ is not an e.d. family.}$$

Proof. Enumerate $\mathcal{F} = \{f_n \mid n < \omega\}$. We construct a fusion sequence $\langle p_n \mid n < \omega \rangle$ below p and a sequence $\langle k_n < \omega \mid n < \omega \rangle$ as follows. Set $p_0 := p$. Now, assume p_n is defined. By assumption the set

$$D_n := \{q \leq p_n \mid \exists k < \omega \; q \Vdash \text{"}\forall l > k \; f_n(l) \neq \dot{g}(l)\text{"}\}$$

is dense open below p_n. By Lemma 2.7 take $p_{n+1} \leq_n p_n$ such that $p_{n+1} \restriction \sigma \in D_n$ for all σ suitable for p_n and n, witnessed by $k_\sigma < \omega$. Set $k_n := \max \{k_\sigma \mid \sigma \text{ suitable for } p_n \text{ and } n\}$, so that

$$p_{n+1} \Vdash \forall l > k_n \; f_n(l) \neq \dot{g}(l).$$

Let q_0 be the fusion of $\langle p_n \mid n < \omega \rangle$. We define a second fusion sequence $\langle q_n \mid n < \omega \rangle$ below q_0 and a sequence of partial functions $\langle h_n \mid n < \omega \rangle$ with the following properties:

(1) The sequence $\langle \mathrm{dom}(h_n) \mid n < \omega \rangle$ is an increasing interval partition of ω,
(2) We have $k_n \leq \max(\mathrm{dom}(h_n))$,
(3) For all $m < n$ and $l \in \mathrm{dom}(h_n)$ we have $f_m(l) \neq h_n(l)$,
(4) $q_{n+1} \Vdash \exists l \in \mathrm{dom}(h_n)\ h_n(l) = \dot{g}(l)$.

This finishes the proof, for if q is the fusion of $\langle q_n \mid n < \omega \rangle$, then $f := \bigcup_{n<\omega} h_n$ is total function by (1). Furthermore, (3) implies that $\mathcal{F} \cup \{f\}$ is an e.d. family and (4) implies that

$$q \Vdash f \text{ and } \dot{g} \text{ are not eventually different.}$$

Since $q \Vdash$ "$\dot{g} \notin V$" we have $q \Vdash$ "$f \neq \dot{g}$, so we obtain

$$q \Vdash \mathcal{F} \cup \{f, \dot{g}\} \text{ is not an e.d. family.}$$

For the fusion construction, assume q_n is defined and enumerate with $\langle \sigma_i \mid i < N \rangle$ the set of all suitable functions σ for q_n and n. Let I be the interval above $\bigcup_{m<n} \mathrm{dom}(h_m)$ of size $\max(k_n, N)$. We construct h_n with $\mathrm{dom}(h_n) = I$, so that (1) to (4) hold. The choice of I already implies that (1) and (2) are satisfied. Note that the set

$$E_n := \{q \leq p_n \mid q \text{ decides } \dot{g} \text{ on } I\text{"}\}$$

is dense open below p_n. Again by Lemma 2.7 take $q_{n+1} \leq_n q_n$ such that $q_{n+1} \restriction \sigma \in E_n$ for all σ suitable for q_n and n, witnessed by the decision $g_\sigma : I \to \omega$. We define h_n for $i < N$ by

$$h_n(\min(I) + i) := g_{\sigma_i}(\min(I) + i).$$

For all other $l \in I$ with $l \geq \min(I) + N$ we define $h_n(l)$ arbitrarily so that (3) is satisfied. By construction of q_{n+1} and h_n we have for all $i < N$

$$q_{n+1} \restriction \sigma_i \Vdash h_n(\min(I) + i) = g_{\sigma_i}(\min(I) + i) = \dot{g}(\min(I) + i),$$

i.e. (4) is satisfied. For (3) let $m < n$ and $l \in I$. If $l \geq \min(I) + N$ there is nothing to show, so let $i < N$. By induction assumption for $m < n$ we have $k_m < \min(I) + i$, so that by choice of k_m

$$q_{n+1} \Vdash f_m(\min(I) + i) \neq \dot{g}(\min(I) + i).$$

On the other hand we have

$$q_{n+1} \restriction \sigma_i \Vdash \dot{g}(\min(I) + i) = h_n(\min(I) + i),$$

which implies $f_m(\min(I) + i) \neq h_n(\min(I) + i)$. □

Thus, we proved $\mathrm{EoI}(\mathfrak{a}_\mathrm{e})$ and as before, Theorem 4.3 yields the following theorem, which corresponds to Theorem 9 in [7]:

Corollary 5.11. *Assume* CH. *Then there is a universally Sacks-indestructible med family.*

5.3. **Partitions of Baire space into compact sets.** In this section, we want to consider partitions of Baire space into compact sets. Recently, the authors constructed a universally Sacks-indestructible such partition in [6] with the similar techniques, so that we only summarize how our results generalize that construction. Observe, that partitions of Baire space into compact sets are in one-to-one correspondence with the following type of families:

Definition 5.12. A family $\mathcal{T}$ of finitely splitting trees on ω is called an almost disjoint family of finitely splitting trees (or an a.d.f.s. family) iff S and T are almost disjoint, i.e. $S \cap T$ is finite for all $S \neq T \in \mathcal{T}$. It is called maximal iff it is maximal with respect to inclusion. Equivalently, for all $f \in {}^{\omega}\omega$ there is $T \in \mathcal{T}$ with $f \in [T]$. Here, $[T]$ denotes the set of branches trough T. The corresponding cardinal characteristic is $\mathfrak{a}_{\mathrm{T}}$:

$$\mathfrak{a}_{\mathrm{T}} := \min\{|\mathcal{T}| \mid \mathcal{T} \text{ is a maximal a.d.f.s. family}\}.$$

Remark 5.13. There is a one-to-one correspondence between non-empty compact subsets of ${}^{\omega}\omega$ and finitely splitting trees on ω given by the following maps:

Given a finitely splitting tree T on ω its set of branches $[T]$ is a non-empty compact subset of ${}^{\omega}\omega$. Conversely, given a non-empty compact subset C of ${}^{\omega}\omega$ we define a finitely splitting tree by

$$T_C := \{s \in {}^{\omega}\omega \mid \exists f \in C \ s \subseteq f\}.$$

It is easy to check that these maps are inverse to each other.

Notice that by König's lemma for finitely splitting trees S and T, we have that S and T are almost disjoint iff $[T] \cap [S] = \emptyset$. Thus, using the above identification of finitely splitting trees and non-empty compact subsets of ${}^{\omega}\omega$, we can also identify maximal a.d.f.s. families with partitions of ${}^{\omega}\omega$ into non-empty compact sets.

Proposition 5.14. *Maximal a.d.f.s. families are an arithmetical type.*

Proof. As for mad families, finitely splitting trees do not live in ${}^{\omega}\omega$, so we have to use an arithmetically definable coding of sequences of natural numbers (for example using prime decomposition). This means that there is an injection 'code' : ${}^{<\omega}\omega \to \omega$ such that the statements 'n is the code for some $s_n \in {}^{<\omega}\omega$' and '$s_n \trianglelefteq s_m$' are definable by arithmetical formulas $\varphi_0(n)$ and $\varphi_1(n, m)$. Then, we define $\psi_0(w_0)$ by

$$\begin{aligned}
&\exists n \ w_0(n) = 1 \\
\text{and } &\forall n (w_0(n) = 1 \text{ implies } \varphi_0(n)) \\
\text{and } &\forall n, m ((w_0(n) = 1 \text{ and } s_m \trianglelefteq s_n) \text{ implies } w_0(m) = 1) \\
\text{and } &\forall n (w_0(n) = 1 \text{ implies } \exists m (m \neq n, s_n \trianglelefteq s_m \text{ and } w_0(m) = 1)) \\
\text{and } &\forall n (w_0(n) = 1 \text{ implies } \exists M \forall m ((m > M, s_n \trianglelefteq s_m \text{ and } w_0(m) = 1) \\
&\qquad \text{implies } \exists k (k \neq n, k \neq m \text{ and } s_n \trianglelefteq s_k \trianglelefteq s_m))),
\end{aligned}$$

expressing 'code$^{-1}[w_0^{-1}[\{1\}]]$ is a non-empty finitely splitting tree'. Further, define $\psi_1(w_0, w_1)$ by

$$\exists N \forall N (n > N \text{ implies } (w_0(n) \neq 1 \text{ or } w_1(n) \neq 1)),$$

expressing ‘$\mathrm{code}^{-1}[w_0^{-1}[\{1\}]]$ and $\mathrm{code}^{-1}[w_1^{-1}[\{1\}]]$ are almost disjoint’. Set $\psi_n :\equiv \top$ for all $n > 1$. Similarly, set $\chi_n :\equiv \top$ for all $n \neq 1$ and let $\chi_1(v, w_1)$ express ‘$\mathrm{code}^{-1}[v^{-1}[\{1\}]]$ and $\mathrm{code}^{-1}[w_1^{-1}[\{1\}]]$ are almost disjoint’. So, maximal a.d.f.s. families are an arithmetical type. □

As usual, Theorem 3.5 yields the following result:

Corollary 5.15. *Every $\mathbb{S}^{\aleph_0}$-indestructible a.d.f.s. family (partition of Baire space into compact sets) is universally Sacks-indestructible.*

In order to construct a universally Sacks-indestructible partition of Baire space into compact sets, the authors proved $\mathrm{EoI}(\mathfrak{a}_\mathrm{T})$ in Lemma 4.8 in [6], that is:

Lemma 5.16. *Let $\mathcal{T}$ be a countable a.d.f.s. family, $p \in \mathbb{S}^{\aleph_0}$ and $\dot{g}$ be a name for a real such that*

$$p \Vdash \dot{g} \notin V \text{ and for all } T \in \mathcal{T} \text{ we have } \dot{g} \notin [T].$$

Then there is $q \leq p$ and a finitely splitting tree S such that $\mathcal{T} \cup \{S\}$ is an a.d.f.s. family and

$$q \Vdash \dot{g} \in [S].$$

Thus, together with Theorem 4.3 we also obtain Theorem 4.17 in [6]:

Corollary 5.17. *Assume* CH*. Then there is a universally Sacks-indestructible a.d.f.s. family (partition of Baire space into compact sets).*

Finally, by Lemma 4.20 in [6] exactly the same arguments also work for almost disjoint families of nowhere dense trees, which correspond to partitions of ${}^{\omega}2$ into closed sets.

5.4. **Maximal cofinitary groups.** Next on our list is a more algebraic example, namely maximal cofinitary groups (mcg). As with the other types of families we first show that also maximal cofinitary groups are an arithmetical type. The construction of a universally Sacks-indestructible cofinitary groups will follow a similar structure as the corresponding proof for maximal eventually different families. We will also carefully set up nice words again, clearing up some inaccuracies in the literature. For the remainder of this section fix a set A, which will serve as an index set.

5.4.1. *Definitions and notations.* We denote with W_A the set of all reduced words in the language $A^{\pm 1} := \{a^i \mid a \in A \text{ and } i = \pm 1\}$. W_A is a group with concatenate-and-reduce as group operation. W_A satisfies the universal property of the free group generated by A, i.e. for every group G any map $\rho : A \to G$ uniquely extends to a group homomorphism $\hat{\rho} : W_A \to G$.

Analogously, with M_A we denote the set of all words in the language $A^{\pm 1}$. M_A is a monoid with concatenate as monoid operation. M_A satisfies the universal property of the free monoid generated by $A^{\pm 1}$, i.e. for every monoid M any map $\rho : A^{\pm 1} \to M$ uniquely extends to a monoid homomorphism $\hat{\rho} : M_A \to M$.

S_∞ denotes the set of all permutations of ω and $\mathrm{S}_\infty^{\mathrm{fin}}$ the set of all finite partial injections $f : \omega \overset{\mathrm{part}}{\to} \omega$. For $f \in \mathrm{S}_\infty^{\mathrm{fin}}$ and $n < \omega$ we write $f(n)\downarrow$ iff $n \in \mathrm{dom}(f)$ and $f(n)\uparrow$ otherwise. Set $\mathrm{S}_\infty^+ := \mathrm{S}_\infty \cup \mathrm{S}_\infty^{\mathrm{fin}}$. Then S_∞ is a group with concatenation, whereas S_∞^+ is only a monoid. In fact, the elements of S_∞ are exactly the invertible elements of S_∞^+. For $f \in \mathrm{S}_\infty^+$ let

$$\mathrm{fix}(f) := \{n < \omega \mid f(n) = n\}$$

be the set of fixpoints of f. Further, define the set of all cofinitary permutations

$$\mathrm{cofin}(\mathrm{S}_\infty) := \{f \in \mathrm{S}_\infty \mid \mathrm{fix}(f) \text{ is finite}\}.$$

Definition 5.18. We say $\rho : A \to \mathrm{S}_\infty$ induces a cofinitary representation iff the induced map $\hat{\rho} : W_A \to \mathrm{S}_\infty$ satisfies $\mathrm{ran}(\hat{\rho}) \subseteq \mathrm{cofin}(\mathrm{S}_\infty) \cup \{\mathrm{id}\}$. Analogously, we call a subgroup G of S_∞ cofinitary iff $G \subseteq \mathrm{cofin}(\mathrm{S}_\infty) \cup \{\mathrm{id}\}$. Note that G is cofinitary iff there is cofinitary representation $\hat{\rho} : W_A \to \mathrm{S}_\infty$ with $\mathrm{ran}(\hat{\rho}) = G$. G is called maximal iff it is maximal with respect to inclusion. Analogously, a cofinitary representation ρ is called maximal iff $\mathrm{ran}(\hat{\rho})$ is a maximal cofinitary group. The corresponding cardinal characteristic is $\mathfrak{a}_\mathrm{g}$:

$$\mathfrak{a}_\mathrm{g} := \min\{|G| \mid G \text{ is a maximal cofinitary group}\}.$$

If $\rho : A \to \mathrm{S}^+_\infty$ its induced map $\hat{\rho} : M_A \to \mathrm{S}^+_\infty$ is defined as follows. Define $\rho^{\pm 1} : A^{\pm 1} \to \mathrm{S}^+_\infty$ by

$$\rho^{\pm 1}(a^i) := \begin{cases} \rho(a) & \text{if i = 1,} \\ \rho(a)^{-1} & \text{if i = -1.} \end{cases}$$

Then, let $\hat{\rho} : M_A \to \mathrm{S}^+_\infty$ be the induced map for $\rho^{\pm 1}$ given by the universal property of M_A. Note that this map coincides with the induced map given by the universal property of W_A in case that $\rho : A \to \mathrm{S}_\infty$.

From now on, we usually identify ρ and its induced map $\hat{\rho}$, write ϵ for the empty word and $|w|$ for the length of a word w. Further, we fix $x \notin A$ and if $\rho : A \to \mathrm{S}_\infty$ induces a cofinitary representation and $f \in \mathrm{S}^+_\infty$ we write $\rho[f]$ for $\rho \cup (x, f)$.

5.4.2. *Arithmetical definability.*

Proposition 5.19. *Maximal cofinitary groups are an arithmetical type.*

Proof. Define the formula $\psi_0(w_0)$ to be

$$(\forall n \forall m\ v(n) = v(m) \text{ implies } n = m) \text{ and } (\forall n \exists m\ v(m) = n),$$

expressing '$w_0 \in \mathrm{S}_\infty$'. Next, we fix an enumeration $\langle u_n \mid 1 < n < \omega\rangle$ of $W_\mathbb{N}$, so that u_n only contains natural numbers up to n as letters. For $n > 0$ we define $\psi_n(w_0, \dots, w_n)$ to be

$$\forall k_0 \exists k_1, \dots, \exists k_{|u_n|}\Big(\bigwedge_{i=0}^{|u_n|-1} \pi_i(v, w_1, \dots, w_n, k_i, k_{i+1}) \text{ and } k_0 = k_{|u_n|}\Big)$$

$$\text{or } \exists K \forall k_0, \exists k_1, \dots, \exists k_{|u_n|}\ \Big(K < k_0 \text{ implies } \Big(\bigwedge_{i=0}^{|u_n|-1} \pi_i(v, w_1, \dots, w_n, k_i, k_{i+1}) \text{ and } k_0 \neq k_{|u_n|}\Big)\Big),$$

expressing '$\rho(u_n) = \mathrm{id}$ or $\rho(u_n)$ has finitely many fixpoints', where ρ is defined by $m \mapsto w_m$. Here, for $u_n := y_{|u_n|-1} \dots y_0$ the formula $\pi_i(v, w_1, \dots, w_n, k_i, k_{i+1})$ is defined as

$$\begin{cases} k_{i+1} = w_m(k_i) & \text{if } y_i = m \text{ for } m \in \mathbb{N}, \\ k_i = w_m(k_{i+1}) & \text{if } y_i = m^{-1} \text{ for } m \in \mathbb{N}, \end{cases}$$

expressing '$\rho(y_i)(k_i) = k_{i+1}$'. Analogously, we define $\chi_0(v)$ as

$$(\exists n \exists m\ n \neq m \text{ and } v(n) = v(m)) \text{ or } (\exists n \forall m\ v(m) \neq n),$$

expressing $v \notin \mathrm{S}_\infty$. Analogously, for $n > 0$ there is an arithmetical formula $\chi_n(v, w_1, \dots, w_n)$ expressing

$$\rho(u_n) = \mathrm{id} \text{ or } \rho(u_n) \text{ has finitely many fixpoints},$$

where ρ is defined by $0 \mapsto v$ and $m \mapsto w_m$ for $m > 0$. Thus, maximal cofinitary groups are an arithmetical type. □

Hence, using Theorem 3.5 we obtain the following result:

Corollary 5.20. *Every $\mathbb{S}^{\aleph_0}$-indestructible mcg is universally Sacks-indestructible.*

5.4.3. *Nice words and range/domain extension.* In this subsection, we reintroduce nice words and reprove their corresponding range and domain extension lemmas, which are crucial tools to approximate elements of S_∞ by finite segments. First, we prove that if we are only interested in the number of fixpoints of $\rho(w)$, then we may also equivalently consider the number of fixpoints of any cyclic permutation of w.

Proposition 5.21. *Let $\rho : A \to \mathrm{S}_\infty$ and $u, v \in W_A$. Then $|\operatorname{fix}(\rho(uv))| = |\operatorname{fix}(\rho(vu))|$. In fact, there is a bijection given by $\pi : n \mapsto \rho(v)(n)$.*

Proof. π is injective as $\rho(v) \in \mathrm{S}_\infty$. Let $n \in \operatorname{fix}(\rho(uv))$. Then

$$\rho(vu)(\pi(n)) = \rho(vu)(\rho(v)(n)) = \rho(vuv)(n) = \rho(v)(\rho(uv)(n)) = \rho(v)(n) = \pi(n),$$

i.e. $\pi(n) \in \operatorname{fix}(vu)$. Now, let $n \in \operatorname{fix}(\rho(vu))$, then $\rho(v^{-1})(n) \in \operatorname{fix}(\rho(uv))$ since

$$\rho(uv)(\rho(v^{-1})(n)) = \rho(v^{-1}vuvv^{-1})(n) = \rho(v^{-1})(\rho(vu)(n)) = \rho(v^{-1})(n).$$

But $\pi(\rho(v^{-1}(n))) = \rho(v)(\rho(v^{-1})(n)) = \rho(vv^{-1})(n) = n$, thus π surjects onto $\operatorname{fix}(\rho(vu))$. □

From now on, we always assume that $\rho : A \to \mathrm{S}_\infty$ induces a cofinitary permutation.

Definition 5.22. We call two words $w, v \in W_{A \cup \{x\}}$ equivalent (with respect to ρ) and write $w \sim_\rho v$ iff $[w]_{\sim_\rho} = [v]_{\sim_\rho}$, where $[w]_{\sim_\rho}$ is the equivalence class of w in $W_A / \ker(\rho)$.

Definition 5.23. Define $W^0_{\rho,x} := W_A \setminus \ker(\rho)$. For $n > 0$ define $W^n_{\rho,x}$ to be the set of all reduced words $w \in W_{A \cup \{x\}}$ of the form $w = x^{\pm n}$ or

$$w = u_l x^{k_l} u_{l-1} x^{k_{l-1}} \dots u_1 x^{k_1} u_0 x^{k_0}$$

for some $l < \omega$ and $u_i \in W^0_{\rho,x}$, $k_i \in \mathbb{Z} \setminus \{0\}$ for $i \leq l$ and $\sum_{i=0}^{l} |k_i| = n$. Finally, we set $W_{\rho,x} := \bigcup_{n>0} W^n_{\rho,x}$. We call $W_{\rho,x}$ the set of all nice words (with respect to ρ).

Lemma 5.24. *Every word $w \in W_{A \cup \{x\}}$ can be split as $w = uv$ for $u, v \in W_{A \cup \{x\}}$ such that vu is equivalent to a word in W_A or equivalent to a nice word with respect to ρ.*

Proof. Let $w \in W_{A\cup\{x\}}$. If the set

$$\{w' \in [vu]_{\sim_\rho} \mid w \text{ is split as } w = uv \text{ for some } u, v \in W_{A\cup\{x\}}\}$$

contains a word $w' \in W_A$ we are done. Otherwise, choose w' from it of minimal length and such that q is minimal, where $w' = px^{\pm 1}q$ and $p \in W_{A\cup\{x\}}$, $q \in W_A$. Let $w = uv$ be the witnessing split for $w' \in [vu]_{\sim_\rho}$. In fact, this implies that q is the empty word, for if otherwise we may adjust the split $w = uv$ to move the q to the other side as this does not increase the length of w'.

If $w' = x^{\pm n}$ for some $n > \omega$ we are done. Otherwise let $w' = x^k q x^{\pm 1}$ where $q \in W_{A\cup\{x\}}$ and $k \in \mathbb{Z}$ with $|k|$ minimal. Hence, $k = 0$, for if otherwise we may adjust the split $w = uv$ to move the x^k to the right side. Note that this cannot lead to cancellations by minimality of w', so it does not increase the length of w' and does not introduce a non-empty $q \in W_A$ on the right side.

Finally, we may choose $l < \omega$, $u_i \in W_A$ and $k_i \in \mathbb{Z} \setminus \{0\}$

$$w' = u_l x^{k_l} u_{l-1} x^{k_{l-1}} \dots u_1 x^{k_1} u_0 x^{k_0}.$$

Then, we have $u_i \notin \ker(\rho)$, for if otherwise w' is equivalent to a shorter word by replacing u_i with ϵ, contradicting its minimality. Thus, w' is nice. □

Corollary 5.25. *Let $f \in \mathrm{S}_\infty$ and assume for all nice words $w \in W_{\rho,x}$ the set $\mathrm{fix}(\rho[f](w))$ is finite. Then $\rho[f]$ induces a cofinitary representation.*

Proof. Let $w \in W_{A\cup\{x\}}$. By the previous lemma write $w = uv$ where vu is equivalent to a word in W_A or equivalent to a nice word w' with respect to ρ. Then $\rho[f](vu) = \rho[f](w')$, so that Proposition 5.21 implies

$$|\,\mathrm{fix}(\rho[f](w))| = |\,\mathrm{fix}(\rho[f](uv))| = |\,\mathrm{fix}(\rho[f](vu))| = |\,\mathrm{fix}(\rho[f](w'))| < \omega,$$

which proves that $\rho[f]$ is a cofinitary representation. □

Thus, to construct a maximal cofinitary group, we may restrict ourselves to nice words. These have the advantage that they satisfy the following range and domain extension lemma:

Lemma 5.26. *Let $s \in \mathrm{S}_\infty^{\mathrm{fin}}$ and $W_0 \subseteq W_{\rho,x}$ be a finite subset. Then we have*

(1) *If $n \in \omega \setminus \mathrm{dom}(s)$ then for almost all $m \in \omega$ we have $t := s \cup (n, m) \in \mathrm{S}_\infty^{\mathrm{fin}}$ and for every word $w \in W_0$*
$$\mathrm{fix}(\rho[s](w)) = \mathrm{fix}(\rho[t](w)).$$
(2) *If $m \in \omega \setminus \mathrm{ran}(s)$ then for almost all $n \in \omega$ we have $t := s \cup (n, m) \in \mathrm{S}_\infty^{\mathrm{fin}}$ and for every word $w \in W_0$*
$$\mathrm{fix}(\rho[s](w)) = \mathrm{fix}(\rho[t](w)).$$

Proof. First, we show how (1) implies (2). Let $m \in \omega \setminus \mathrm{ran}(s)$ and let $W_0^\perp := \{w^\perp \mid w \in W_0\}$, where $w^\perp$ is constructed by replacing all occurrences of x by x^{-1} and vice versa. Note that w is nice iff $w^\perp$ is nice and for any $t \in \mathrm{S}_\infty^{\mathrm{fin}}$ we have

$$\rho[t](w) = \rho[t^{-1}](w^\perp).$$

Furthermore, $m \notin \mathrm{dom}(s^{-1})$, so by 1. for almost all $n \in \omega$ we have $t^{-1} := s^{-1} \cup (m,n) \in \mathrm{S}_\infty^{\mathrm{fin}}$ and for every word $w^\perp \in W_0^\perp$

$$\mathrm{fix}(\rho[s^{-1}](w^\perp)) = \mathrm{fix}(\rho[t^{-1}](w^\perp)).$$

But then, for every such $n < \omega$ we have $t \in \mathrm{S}_\infty^{\mathrm{fin}}$ and for every word $w \in W_0$

$$\mathrm{fix}(\rho[s](w)) = \mathrm{fix}(\rho[s^{-1}](w^\perp)) = \mathrm{fix}(\rho[t^{-1}](w^\perp)) = \mathrm{fix}(\rho[t](w)).$$

Next, we have to prove 1. It suffices to prove the statement for $W_0 = \{w\}$, the general case then follows iteratively. Consider the following cases:

Case 1: $w = x^n$ for some $n > 0$. We claim that every $m \in \omega \setminus (\mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n\})$ is suitable, for if $\rho[t](w)(k)\downarrow$ and $\rho[s](w)(k)\uparrow$ for some $k \in \mathrm{dom}(t)$, we can choose $i > 0$ minimal with $\rho[s](x^i)(k)\uparrow$. Then $\rho[s](x^{i-1})(k) = n$, so that $\rho[t](x^i)(k) = m \notin \mathrm{dom}(t)$. But $\rho[t](w)(k)\downarrow$, so $i = n$ and we get $\rho[t](w)(k) = m \neq k$. Thus, $k \notin \mathrm{fix}(\rho[t](w))$.

Case 2: $w = x^{-n}$ for some $n > 0$. We claim that every $m \in \omega \setminus (\mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n\})$ is suitable, for if $\rho[t](w)(k)\downarrow$ and $\rho[s](w)(k)\uparrow$ for some $k \in \mathrm{ran}(t)$, we can choose $i > 0$ minimal with $\rho[s](x^{-i})(k)\uparrow$. Then $\rho[s](x^{-i+1})(k) = m$, so that $i = 1$, for if otherwise $m \in \mathrm{ran}(\rho[s](x^{-1}))$ implies $m \in \mathrm{dom}(s)$, contradicting the choice of m. But then $k = m$ and we get $\rho[t](w)(m) \neq m$ as $m \notin \mathrm{dom}(t)$. Thus, $k \notin \mathrm{fix}(\rho[t](w))$.

For the remaining case, we may choose $l < \omega$ and $u_i \in W^0_{\rho,x}$, $k_i \in \mathbb{Z} \setminus \{0\}$ for $i \leq l$ such that

$$w = u_l x^{k_l} u_{l-1} x^{k_{l-1}} \dots u_1 x^{k_1} u_0 x^{k_0}.$$

Also, since $u_i \in W^0_{\rho,x}$ we may choose $M < \omega$ large enough such that for all $i \leq l$

(M1) $\mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n\} \subseteq M$.
(M2) $\rho(u_i)[\mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n\}] \subseteq M$.
(M3) $\rho(u_i)^{-1}[\mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n\}] \subseteq M$.
(M4) $\mathrm{fix}(\rho(u_i)) \subseteq M$.

We will show that every $m \geq M$ is suitable, so assume $\rho[t](w)(k)\downarrow$ and $\rho[s](w)(k)\uparrow$ for some $k \in \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n, m\}$. Choose $i \leq l$ minimal and then $j \leq |k_i|$ minimal such that

$$\rho[s](x^{\mathrm{sign}(k_i)j} u_{i-1} x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k)\uparrow.$$

Then $j > 0$ by minimality of i. We consider the following two cases:

Case 1: $k_i > 0$. Then

$$\rho[s](x^{j-1} u_{i-1} x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) = n,$$

so that by definition of t

$$\rho[t](x^j u_{i-1} x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) = m.$$

By (M1) $j < k_i$ contradicts $\rho[t](w)(k)\downarrow$, so $j = k_i$ and we get

$$\rho[t](x^{k_i} u_{i-1} x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) = m.$$

By (M3) and (M4) we get

$$\rho[t](u_i x^{k_i} u_{i-1} x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) \notin \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n, m\},$$

so $i < l$ contradicts $\rho[t](w)(k)\downarrow$. Thus $i = l$ and

$$\rho[t](w)(k) \notin \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n, m\},$$

which proves that $k \notin \mathrm{fix}(\rho[t](w))$.

Case 2: $k_i < 0$. Then

$$\rho[s](x^{-j+1}u_{i-1}x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) = m,$$

so that by definition of t

$$\rho[t](x^{-j}u_{i-1}x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) = n.$$

By (M1) we have $j = 1$ and we get

$$\rho[s](u_{i-1}x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) = m.$$

If $i > 0$ by minimality of i we would have

$$\rho[s](x^{k_{i-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(k) \in \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n\},$$

which contradicts (M2). Thus, $i = 0$, i.e. $k = \rho[s](\epsilon)(k) = m$. Further, we have

$$\rho[t](x^{k_l}u_{l-1}x^{k_{l-1}} \dots u_1 x^{k_1} u_0 x^{k_0})(m) \in \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup \{n, m\}.$$

But then (M2) and (M4) imply

$$\rho[t](w)(m) \neq m,$$

which proves that $k \notin \mathrm{fix}(\rho[t](w))$. □

5.5. **Elimination of intruders for maximal cofinitary groups.** Finally, we will prove $\mathrm{EoI}(\mathfrak{a}_{\mathrm{g}})$, so that Theorem 4.3 yields the following result:

Corollary 5.27. *Under* CH *there is a universally Sacks-indestructible maximal cofinitary group.*

For the proof, we will need to handle $W^1_{\rho,x}$, i.e. nice words with exactly one occurrence of x separately, so let us denote $W^{>1}_{\rho,x} := \bigcup_{n>1} W^n_{\rho,x}$.

Lemma 5.28. $\mathrm{EoI}(\mathfrak{a}_{\mathrm{g}})$ *holds. That is, if* A *is countable,* $\rho : A \to \mathrm{S}_\infty$ *induces a cofinitary representation,* $p \in \mathbb{S}^{\aleph_0}$ *and* $\dot{g}$ *be a name for an element of* S_∞ *such that*

$$p \Vdash \dot{g} \notin V \text{ and } \rho[\dot{g}] \text{ induces a cofinitary representation.}$$

Then there is $q \leq p$ *and* $f \in \mathrm{S}_\infty$ *such that* $\rho[f]$ *induces a cofinitary representation and*

$$q \Vdash f \text{ and } \dot{g} \text{ are not eventually different.}$$

Proof. First, we prove that for all $w \in W^1_{\rho,x}$

$$p \Vdash \mathrm{fix}(\rho[\dot{g}](w)) \text{ is finite.}$$

Otherwise, choose $w \in W^1_{\rho,x}$ and $q \leq p$ such that

$$q \Vdash \mathrm{fix}(\rho[\dot{g}](w)) \text{ is infinite.}$$

Then, by assumption on $\dot{g}$ we get

$$q \Vdash \rho[\dot{g}](w) = \mathrm{id}\,.$$

Write $w := ux^{\pm 1}$, where $u \in W_A$. Then

$$q \Vdash \dot{g}^{\pm 1} = \rho[\dot{g}](x^{\pm 1}) = \rho(u)^{-1},$$

so depending on the case we get

$$q \Vdash \dot{g} = \rho(u)^{-1} \quad \text{or} \quad q \Vdash \dot{g} = \rho(u).$$

Either way, we obtain $q \Vdash$ "$\dot{g} \in V$", a contradiction. Let $\langle w_n \mid n < \omega \rangle$ enumerate all nice words, so that w_m is a subword of w_n implies that $m \leq n$. We define a fusion sequence $\langle p_n \mid n < \omega \rangle$ below q_0, a sequence $\langle K_n < \omega \mid n < \omega \rangle$ and an increasing sequence $\langle f_n \in \mathrm{S}_\infty^{\mathrm{fin}} \mid n < \omega \rangle$ with the following properties:

(1) $n \in \operatorname{ran}(f_n) \cap \operatorname{dom}(f_n)$.
(2) If $w_n \in W^1_{\rho,x}$ then $p_{n+1} \Vdash \operatorname{fix}(\rho[\dot{g}](w_n)) \subseteq K_n$
(3) For all $m \leq n$ we have $\operatorname{fix}(\rho[f_n](w_m)) \subseteq K_m$.
(4) $p_{n+1} \Vdash \exists l \in (\operatorname{dom}(f_n) \setminus \bigcup_{m<n} \operatorname{dom}(f_m))\ f_n(l) = \dot{g}(l)$.

To see that this proves the theorem, let q be the fusion of $\langle p_n \mid n < \omega \rangle$ and define $f := \bigcup_{n<\omega} f_n$. By (1) and since $\langle f_n \mid n < \omega \rangle$ is an increasing sequence of partial injections we have that $f \in \mathrm{S}_\infty$. By (3) we have that $\rho[f]$ induces a cofinitary representation, since for every $m < \omega$ we have that $\operatorname{fix}(\rho[f](w_m)) \subseteq K_m$. Finally, (4) implies

$$q \Vdash f \text{ and } \dot{g} \text{ are not eventually different.}$$

So, set $p_0 := p$ and assume that q_n is defined - we then have to construct p_{n+1}, K_n and f_n. If $n \notin \bigcup_{m<n} \operatorname{dom}(f_m)$ or $n \notin \bigcup_{m<n} \operatorname{ran}(f_m)$ use Lemma 5.26 to extend f_n to $h_0 \in \mathrm{S}_\infty^{\mathrm{fin}}$ with $n \in \operatorname{dom}(h_0) \cap \operatorname{ran}(h_0)$ while preserving (3). In case that $w_n \in W^1_{\rho,x}$, by the previous observation the set

$$D_n := \{q \leq p_n \mid \exists K < \omega\ q \Vdash \operatorname{fix}(\rho[\dot{g}](w_n)) \subseteq K\}$$

is open dense below p_n. By Lemma 2.7 take $q_0 \leq_n p_n$ such that $q_0 \restriction \sigma \in D_n$ for all σ suitable for p_n and n. Thus, there is a $K < \omega$ such that

$$q_0 \Vdash \operatorname{fix}(\rho[\dot{g}](w_n)) \subseteq K.$$

Next, enumerate all suitable functions σ for q_0 and n by $\langle \sigma_i \mid i < N \rangle$. Inductively, we will define an $\leq_n$-decreasing sequence $\langle q_i \mid i \leq N \rangle$ and an increasing sequence $\langle h_i \in \mathrm{S}_\infty^{\mathrm{fin}} \mid i \leq N \rangle$ with

(a) For all $m < n$ we have $\operatorname{fix}(\rho[h_{i+1}](w_m)) \subseteq K_m$.
(b) $q_{i+1} \restriction \sigma_i \Vdash \exists l \in (\operatorname{dom}(h_{i+1}) \setminus \operatorname{dom}(h_i))\ f_n(l) = \dot{g}(l)$.

Assuming we are successful with this, we may set $f_n := h_N$ and choose $K' < \omega$ such that $\operatorname{fix}(\rho[f_n](w_n)) \subseteq K'$ and define

$$K_n := \max(K, K').$$

Then, we took care of (1) at the beginning of the construction and (2) follows from the definition of K. Furthermore, (3) follows from (a) for $m < n$ and by definition of K' for $m = n$. Finally, (4) follows directly from (b). For the construction, let $i < N$ and assume q_i and h_i are defined. Choose $M < \omega$ large enough such that

(M1) $\operatorname{ran}(h_i) \cup \operatorname{dom}(h_i) \subseteq M$, and for all $m < n$ if we can choose $l_m < \omega$ and $u_{i,m} \in W^0_{\rho,x}$, $k_{i,m} \in \mathbb{Z} \setminus \{0\}$ for $i \leq l$ such that

$$w = u_l x^{k_l} u_{l-1} x^{k_{l-1}} \dots u_1 x^{k_1} u_0 x^{k_0},$$

then we have for all $i \leq l$ that

(M2) $\rho(u_{i,m})[\mathrm{dom}(h_i) \cup \mathrm{ran}(h_i)] \subseteq M$,
(M3) $\rho(u_{i,m})^{-1}[\mathrm{dom}(h_i) \cup \mathrm{ran}(h_i)] \subseteq M$.
(M4) $\mathrm{fix}(\rho(u_{i,m})) \subseteq M$

Finally, we also require

(M5) for all $m < n$ we have $K_m \subseteq M$.

Choose $r \leq q_i \restriction \sigma_i$ which decides $M+1$ values of $\dot{g}$ above M. Since, $r \Vdash$ "$\dot{g}$ is injective" there are $n_0, m_0 \geq M$ such that $r \Vdash$ "$\dot{g}(n_0) = m_0$". Set $h_{i+1} := h_i \cup \langle n_0, m_0 \rangle$ and define $q_{i+1} \leq_n q_i$, so that $q_{i+1} \restriction \sigma_i = r_i$, by

$$q_{i+1}(\alpha) := \begin{cases} r_i(\alpha) \cup \bigcup \{q_i(\alpha)_s \mid s \in \mathrm{spl}_n(q_i(\alpha)) ^\frown 2 \text{ and } s \neq \sigma_i(\alpha)\} & \text{if } \alpha < n, \\ r_i(\alpha) & \text{otherwise.} \end{cases}$$

Clearly, $h_{i+1} \in \mathrm{S}_\infty$ by (M1) and q_{i+1} satisfies property (b) since

$$q_{i+1} \restriction \sigma_i \Vdash h_{i+1}(n_0) = m_0 = \dot{g}(n_0).$$

It remains to show that also property (a) is satisfied, so let $m < n$. First, we consider the case $w_m \in W^1_{\rho,x}$. But in this case we have $w_m = ux^{\pm 1}$ for $u \in W^0_{\rho,x} \cup \{\epsilon\}$, so that for every $l \in \mathrm{dom}(h_i) \cup \mathrm{ran}(h_i)$ we have

$$\rho[h_{i+1}](w_m)(l) = \rho[h_i](w_m)(l),$$

and for $l \in \{n_0, m_0\}$ we may apply (2) inductively to obtain

$$q_i \restriction \sigma_i \Vdash \rho[h_{i+1}](w_m)(l) = \rho[\dot{g}](w_m)(l) \text{ and } l \notin \mathrm{fix}(\rho[\dot{g}](w_m)),$$

since $n_0, m_0 \geq K_m$. Thus, $\mathrm{fix}(\rho[h_{i+1}](w_m)) = \mathrm{fix}(\rho[h_i](w_m)) \subseteq K_m$ inductively by (a).

Next, we consider the case $w_m \in W^{>1}_{\rho,x}$. Again, for all $l \in \mathrm{dom}(h_i) \cup \mathrm{ran}(h_i)$ we have

$$\rho[h_{i+1}](w_m)(l) = \rho[h_i](w_m)(l)$$

by properties (M1), (M2) and (M3). Thus, it remains to consider the cases $l \in \{n_0, m_0\}$. We show that $\rho[h_{i+1}](w_m)(l)\uparrow$, which finishes the proof. If $l = n_0$ we may write $w_m = vx^{\pm 1}ux$ for $v \in W_{A \cup \{x\}}$ and $u \in W^0_{\rho,x} \cup \{\epsilon\}$. By (M2) or (M3) we have

$$\rho[h_{i+1}](ux)(n_0) \notin \mathrm{dom}(h_i) \cup \mathrm{ran}(h_i).$$

Further, if $w_m = vx^{-1}ux$ we have $u \in W^0_{\rho,x}$, so that by (M4) $\rho(u)(m_0) \neq m_0$. Hence,

$$\rho[h_{i+1}](ux)(n_0) \neq m_0.$$

Otherwise, $w_m = vxux$. Then $ux \in W^1_{\rho,x}$ is a subword of w_m, so choose $m' < m$ with $w_{m'} = ux$. Thus, by (M5) and (2) we get

$$\rho[h_{i+1}](ux)(n_0) \neq n_0.$$

Thus, in both cases $\rho[h_{i+1}](x^{\pm 1}ux)\uparrow$.

Finally, for $l = m_0$ we may write $w_m = vx^{\pm 1}ux^{-1}$ for $v \in W_{A \cup \{x\}}$ and $u \in W^0_{\rho,x} \cup \{\epsilon\}$. By (M2) or (M3) we have that

$$\rho[h_{i+1}](ux^{-1})(m_0) \notin \operatorname{dom}(h_i) \cup \operatorname{ran}(h_i).$$

Further, if $w_m = vxux^{-1}$ we have $u \in W^0_{\rho,x}$, so that by (M4) $\rho(u)(n_0) \neq n_0$. Hence,

$$\rho[h_{i+1}](ux^{-1})(m_0) \neq n_0.$$

Otherwise, $w_m = vxux^{-1}$. Then $ux^{-1} \in W^1_{\rho,x}$ is a subword of w_m, so again choose $m' < m$ with $w_{m'} = ux^{-1}$. Finally, by (M5) and (2) we get

$$\rho[h_{i+1}](ux^{-1})(m_0) \neq m_0.$$

Thus, in both cases $\rho[h_{i+1}](x^{\pm 1}ux^{-1}) \uparrow$. □

5.6. **Independent families and ultrafilters.** In this section, we want to consider independent families and ultrafilters. We show that both of these types of families are arithmetically definable, so that we may apply Theorem 3.5. Further, we will prove EoI($\mathfrak{u}$), so by Theorem 4.3 we may easily construct a universally Sacks-indestructible ultrafilter under CH without referring to some kind of selectivity. At this time, we do not know if the same is possible for independent families. Recall the following two definitions:

Definition 5.29. Let $\mathcal{A}$ be a subset of $[\omega]^\omega$. Denote with $\mathrm{FF}(\mathcal{A})$ the set of all finite partial functions $f : \mathcal{A} \to 2$. Given $f \in \mathrm{FF}(A)$ we define

$$\mathcal{A}^f := \Big(\bigcap_{A \in f^{-1}[0]} A\Big) \cap \Big(\bigcap_{A \in f^{-1}[1]} A^c\Big).$$

The family $\mathcal{A}$ is independent iff for all $f \in \mathrm{FF}(A)$ we have that $\mathcal{A}^f$ is infinite. $\mathcal{A}$ is called maximal iff it is maximal with respect to inclusion. The corresponding cardinal characteristic is the independence number $\mathfrak{i}$:

$$\mathfrak{i} := \min\{|\mathcal{A}| \mid \mathcal{A} \text{ is a maximal independent family}\}.$$

Proposition 5.30. *Maximal independent family are an arithmetical type.*

Proof. Using the same coding of $[\omega]^\omega$ by reals as for mad families let $\psi_0(w_0)$ express 'w_0 codes an infinite subset of ω'. For $n > 0$ there is an arithmetical formula $\psi_n(w_0, \ldots, w_n)$ expressing

$$\begin{aligned} &(\operatorname{ran}(w_0) \cap \operatorname{ran}(w_1) \cap \cdots \cap \operatorname{ran}(w_n) \text{ is infinite}) \\ \text{and } &(\operatorname{ran}(w_0) \cap \operatorname{ran}(w_1) \cap \cdots \cap \operatorname{ran}(w_n)^c \text{ is infinite}) \\ \text{and } &\ldots \\ \text{and } &(\operatorname{ran}(w_0)^c \cap \operatorname{ran}(w_1)^c \cap \cdots \cap \operatorname{ran}(w_n)^c \text{ is infinite}). \end{aligned}$$

Analogously, we let $\chi_0(v)$ express 'v codes an infinite subset of ω' and for $n > 0$ choose $\chi_n(v, w_1, \dots, w_n)$ expressing

$$\begin{aligned} &(\mathrm{ran}(v) \cap \mathrm{ran}(w_1) \cap \dots \cap \mathrm{ran}(w_n) \text{ is infinite}) \\ \text{and } &(\mathrm{ran}(v) \cap \mathrm{ran}(w_1) \cap \dots \cap \mathrm{ran}(w_n)^c \text{ is infinite}) \\ \text{and } &\dots \\ \text{and } &(\mathrm{ran}(v)^c \cap \mathrm{ran}(w_1)^c \cap \dots \cap \mathrm{ran}(w_n)^c \text{ is infinite}). \end{aligned}$$

Thus, maximal independent families are an arithmetical type. □

Definition 5.31. We say a subset $\mathcal{A} \subseteq [\omega]^\omega$ satisfies the strong finite intersection property (SFIP) iff $\bigcap \mathcal{A}_0$ is infinite for all $\mathcal{A}_0 \in [\mathcal{A}]^{<\omega}$. In this case, we define the generated filter of $\mathcal{A}$ by

$$\langle \mathcal{A} \rangle := \{C \subseteq \omega \mid \exists \mathcal{A}_0 \in [\mathcal{A}]^{<\omega} \ \bigcap \mathcal{A}_0 \subseteq C\}.$$

We call $\mathcal{A}$ an ultrafilter subbasis iff $\langle \mathcal{A} \rangle$ is an ultrafilter. The corresponding cardinal characteristic is the ultrafilter number $\mathfrak{u}$:

$$\mathfrak{u} := \min \{|\mathcal{A}| \mid \mathcal{A} \text{ is an ultrafilter subbasis}\}.$$

Proposition 5.32. *Ultrafilter subbases are an arithmetical type.*

Proof. Using the same coding of $[\omega]^\omega$ by reals as for mad families let $\psi_0(w_0)$ express 'w_0 codes an infinite subset of ω'. For $n > 0$ there is an arithmetical formula $\psi_n(w_0, \dots, w_n)$ expressing

$$\bigcap_{i=0}^{n} \mathrm{ran}(w_i) \text{ is infinite.}$$

Analogously, let $\chi_0(v)$ express 'v codes an infinite subset of ω'. Furthermore, for $n > 0$ there is an arithmetical formula $\chi_n(v, w_1, \dots, w_n)$ expressing

$$\bigcap_{i=1}^{n} \mathrm{ran}(w_i) \not\subseteq \mathrm{ran}(v) \text{ and } \bigcap_{i=1}^{n} \mathrm{ran}(w_i) \not\subseteq \mathrm{ran}(v)^c$$

Thus, ultrafilter subbases are an arithmetical type. □

Applying Theorem 3.5 to these two cases yields:

Corollary 5.33. *Every $\mathbb{S}^{\aleph_0}$-indestructible independent family and every $\mathbb{S}^{\aleph_0}$-indestructible ultrafilter is universally Sacks-indestructible.*

In [18] Shelah constructed a selective independent family, which are also universally Sacks-indestructible. Further, in [16] Laver proved that every selective ultrafilter is preserved by any product of Sacks-forcing. Thus, together with the previous corollary we obtain another proof that every selective ultrafilter is universally Sacks-indestructible.

Hence, universally Sacks-indestructible independent families and ultrafilters may exist, but both of these constructions refer to some kind of selectivity. We show that the colouring principle HL_ω proven by Laver in [16] may be used to prove $\mathrm{EoI}(\mathfrak{u})$. Hence, we may also directly construct a universally Sacks-indestructible ultrafilter under CH without using selectivity. We use the following notion introduced by Laver in [16].

Definition 5.34. For $p \in \mathbb{S}$ let $p^{(n)}$ be the n-th level of p. For $A \subseteq \omega$ and $p \in \mathbb{S}^{\aleph_0}$ let

$$\bigotimes^A p := \bigcup_{n \in A} \bigotimes_{i<\omega} p(i)^{(n)},$$

where $\bigotimes_{i<\omega} p(i)^{(n)}$ is the cartesian product.

Lemma 5.35 (HL_ω, Laver, [16])**.** *Let $p \in \mathbb{S}^{\aleph_0}$ and $c : \bigotimes^\omega p \to 2$. Then there is $q \leq p$ and $A \subseteq \omega$ such that $c \restriction \bigotimes^A q$ is monochromatic.*

The following strengthening is implicit in [16], but for completeness we provide a proof:

Corollary 5.36. *Let $p \in \mathbb{S}^{\aleph_0}$, $A \subseteq \omega$, $k < \omega$ and $c : \bigotimes^A p \to k$. Then there is $q \leq p$ and $B \subseteq A$ such that $c \restriction \bigotimes^B q$ is monochromatic.*

Proof. First, we prove the statement for $k = 2$, so let $p \in \mathbb{S}^{\aleph_0}$, $A \subseteq \omega$ and $c : \bigotimes^A p \to 2$. Let $\langle a_n \mid n < \omega \rangle$ be the increasing enumeration of A. For every $i < \omega$ we may choose $p'(i) \in \mathbb{S}$ such that $p'(i) \leq p(i)$ and $\left|p(i)^{(a_n)}\right| \leq 2^n$ for all $n < \omega$. Then, we may choose an $\trianglelefteq$-order-preserving map $\Phi_i : p'(i) \to {}^{<\omega}2$ with $\Phi_i[p'(i)^{(a_n)}] \subseteq ({}^{<\omega}2)^{(n)}$ and Φ_i injective on $p'(i)^{(a_n)}$ for all $n < \omega$. Then, $\mathrm{ran}(\Phi_i) \in \mathbb{S}$, so define $r \in \mathbb{S}^{\aleph_0}$ by $r(i) := \mathrm{ran}(\Phi_i)$ for $i < \omega$. Next, define $d : \bigotimes^\omega r \to 2$ by

$$d(s) := c(t), \text{ where } t \text{ is the unique element of } \bigotimes^A p \text{ with } \Phi_i(t(i)) := s(i).$$

Now, by HL_ω choose $r' \leq r$ and $B \subseteq \omega$ such that $d \restriction \bigoplus^B r'$ is monochromatic. Finally, for every $i < \omega$ let $q(i)$ be the preimage of $r'(i)$ under Φ_i. Then $q(i) \in \mathbb{S}$ and $q(i) \leq p'(i)$. Further, let $C := \{a_n \mid n \in B\}$. Then, $q \in \mathbb{S}$, $q \leq p$, $C \subseteq A$ and $c \restriction \bigoplus^C q$ is monochromatic.

Next, we inductively prove the statement for higher k, so let $k + 1 > 2$, $p \in \mathbb{S}^{\aleph_0}$, $A \subseteq \omega$ and $c : \bigotimes^A p \to k + 1$. Define $c : \bigotimes^A p \to k$ by

$$d(s) := \max(c(s), k - 1).$$

By induction we may choose $q \leq p$ and $B \subseteq A$ such that $d \restriction \bigotimes^B q$ is monochromatic. In case that $d \restriction \bigotimes^B q = l$ for some $l < k$, we have that $c \restriction \bigotimes^B q = d \restriction \bigotimes^B q$ is monochromatic. Otherwise, we have $c \restriction \bigotimes^B q \to \{k, k - 1\}$, so by case $k = 2$ we may choose $C \subseteq B$ and $r \leq q$ such that $c \restriction \bigotimes^C r$ is monochromatic. □

Lemma 5.37. *For every $A \in [\omega]^\omega$ and $\mathbb{S}^{\aleph_0}$-name $\dot{B}$ for a subset of ω we have*

$$\mathbb{S}^{\aleph_0} \Vdash \exists B \in ([A]^\omega)^V \text{ with } B \subseteq \dot{B} \text{ or } B \subseteq \dot{B}^c$$

Proof. Let $p \in \mathbb{S}^{\aleph_0}$ and let $\langle a_n \mid n < \omega \rangle$ enumerate A. Using a fusion construction we may choose an increasing sequence $C = \{c_n \mid n < \omega\}$ and $q \leq p$ such that for every $s \in \bigotimes_{i<\omega} q(i)^{(c_n)}$ we have that q_s decides '$a_n \in \dot{B}$'. Define a colouring $c : \bigotimes^C q \to 2$ for $s \in \bigotimes_{i<\omega} q(i)^{(c_n)}$ by

$$c(s) := \begin{cases} 0 & \text{if } q_s \Vdash a_n \in \dot{B}, \\ 1 & \text{if } q_s \Vdash a_n \notin \dot{B}. \end{cases}$$

By the previous corollary choose $D \subseteq C$ and $r \leq q$ such that $c \restriction \bigotimes^D r$ is monochromatic and let $B := \{a_n \mid c_n \in D\}$, so that $B \in [A]^\omega$. Finally, if $c \restriction \bigotimes^D r \equiv 0$, then

$$q \Vdash B \subseteq \dot{B},$$

whereas if $c \restriction \bigotimes^D r \equiv 1$, then

$$q \Vdash B \subseteq \dot{B}^c$$

completes the proof. □

Notice that for $A = \omega$ the previous corollary precisely states that $\mathbb{S}^{\aleph_0}$ preserves $[\omega]^\omega$ as an unsplit/reaping family. Finally, we are in position to prove $\mathrm{EoI}(\mathfrak{u})$:

Corollary 5.38. $\mathrm{EoI}(\mathfrak{u})$ *holds. That is, if $\mathcal{A}$ is countable with SFIP, $p \in \mathbb{S}^{\aleph_0}$ and $\dot{B}$ a $\mathbb{S}^{\aleph_0}$-name for a subset of ω. Then there is $q \leq p$ and $B \subseteq \omega$ such that $\mathcal{A} \cup \{B\}$ satisfies the SFIP and*

$$q \Vdash B \subseteq \dot{B} \text{ or } B \subseteq \dot{B}^c$$

Proof. By assumption on $\mathcal{A}$ we may choose an infinite pseudo-intersection A of $\mathcal{A}$. By the previous lemma choose $q \leq p$ and $B \subseteq A$ such that

$$q \Vdash B \subseteq \dot{B} \text{ or } B \subseteq \dot{B}^c.$$

But A is a pseudo-intersection of $\mathcal{A}$ and $B \subseteq A$, so that $\mathcal{A} \cup \{B\}$ satisfies the SFIP. □

Hence, Theorem 4.3 yields another proof of the following corollary. As discussed before, selective ultrafilters may be used instead to also obtain this result:

Corollary 5.39. *Under* CH *there is a universally Sacks-indestructible ultrafilter.*

Question 5.40. Does $\mathrm{EoI}(\mathfrak{i})$ hold?

5.7. **Bounding, dominating and other types of families.** Finally, in this last section we take a look at a few other types of families which do not have any restrictions as to what constitutes a family of that type. Hence, for these families only the notion of intruders really matters.

Definition 5.41. Given $f, g \in {}^\omega\omega$ we write $f <^* g$ iff $f(n) < g(n)$ for all but finitely many $n < \omega$. A family $\mathcal{B} \subseteq {}^\omega\omega$ is called unbounded iff for all $g \in {}^\omega\omega$ there is $f \in \mathcal{B}$ such that $f \not<^* g$. The corresponding cardinal characteristic is the (un-)bounding number $\mathfrak{b}$:

$$\mathfrak{b} := \min\{|\mathcal{B}| \mid \mathcal{B} \text{ is an unbounded family}\}.$$

A family $\mathcal{D} \subseteq {}^\omega\omega$ is called dominating iff for all $g \in {}^\omega\omega$ there is $f \in \mathcal{D}$ such that $g <^* f$. The corresponding cardinal characteristic is the dominating number $\mathfrak{d}$:

$$\mathfrak{d} := \min\{|\mathcal{D}| \mid \mathcal{D} \text{ is a dominating family}\}.$$

Proposition 5.42. *Unbounded and dominating families are arithmetical types.*

Proof. Set $\psi_n :\equiv \top$ for all $n < \omega$. Analogously, set $\chi_n :\equiv \top$ for all $n \neq 1$. Finally, define the formula $\chi_1(v, w_1)$ to be

$$\exists N \forall n (n > N \text{ implies } w_1(n) < v(n)),$$

expressing '$w_1 <^* v$'. Thus, unbounded families are an arithmetical type. Dominating families may be defined analogously. □

Definition 5.43. Given $A, S \in [\omega]^\omega$ we say S splits A iff $A \cap S$ and $A \cap S^c$ are infinite. A family $\mathcal{S} \subseteq {}^\omega\omega$ is called splitting iff for all $A \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits A. The corresponding cardinal characteristic is the splitting number $\mathfrak{s}$:

$$\mathfrak{s} := \min \{|\mathcal{S}| \mid \mathcal{S} \text{ is a splitting family}\}.$$

A family $R \subseteq {}^\omega\omega$ is called reaping or unsplit iff for all $S \in [\omega]^\omega$ there is $R \in \mathcal{R}$ such that S does not split R. The corresponding cardinal characteristic is the reaping number $\mathfrak{r}$:

$$\mathfrak{r} := \min \{|\mathcal{R}| \mid \mathcal{R} \text{ is a reaping family}\}.$$

Proposition 5.44. *Reaping and splitting families are arithmetical types.*

Proof. Using the same coding of $[\omega]^\omega$ by reals as for mad families let $\psi_0(w_0)$ express 'w_0 codes an infinite subset of ω' and set $\psi_n :\equiv \top$ for $n > 0$. Analogously, we let $\chi_0(v)$ express 'v codes an infinite subset of ω'. Further, there is an arithmetical formula $\chi_1(v, w_1)$ expressing

$$\operatorname{ran}(v) \cap \operatorname{ran}(w_1) \text{ is finite or } \operatorname{ran}(v) \cap \operatorname{ran}(w_1)^c \text{ is finite}$$

and set $\chi_n :\equiv \top$ for all $n > 1$. Thus, splitting families are an arithmetical type. Reaping families can be defined analogously. □

Corollary 5.45. *Every $\mathbb{S}^{\aleph_0}$-indestructible unbounded/dominating/reaping/splitting family is universally Sacks-indestructible.*

For example, one special case is that ${}^\omega\omega$-bounding is equivalent to ${}^\omega\omega$ being preserved as a dominating family. Hence, one may prove that any countably supported product or iteration of Sacks-forcing of any length is ${}^\omega\omega$-bounding by just verifying that $\mathbb{S}^{\aleph_0}$ is ${}^\omega\omega$-bounding.

Similarly, it is easy to see that $\mathbb{S}^{\aleph_0}$ preserves $[\omega]^\omega$ as a splitting family, and in [16] Laver proved that $\mathbb{S}^{\aleph_0}$ preserves $[\omega]^\omega$ as a reaping family (we also reproved this in Lemma 5.37). Thus, in these special cases we obtain another proof that any countably supported product or iteration of Sacks-forcing of any length preserves $[\omega]^\omega$ as a splitting and reaping family.

6. Questions

First of all, we note that our framework may be applied to even more combinatorial families of reals than presented here, such as evading and predicting families, witnesses for $\mathfrak{p}$ and others. Also, let us restate the open question for elimination of intruders for independent families:

Question 6.1. Does EoI($\mathfrak{i}$) hold?

Finally, we discuss to which extend our framework may be extended or generalized. Note that there are a few constructions of generalized combinatorial families on κ indestructible by κ-Sacks forcing (for example see [5]). Naturally, we may ask:

Question 6.2. Is there a similar framework of Sacks-indestructibility for higher Sacks forcing at uncountable cardinals? In particular, is there a uniform construction for generalized combinatorial families indestructible by higher Sacks-forcing?

Finally, indestructibility for $\mathbb{S}^{\aleph_0}$ turned out to be the strongest form of Sacks-indestructibility. However, there is the following theorem:

Theorem (Brendle, Yatabe, 2005, [2])**.** *Assume that* $\operatorname{cov}(\mathcal{M}) = \mathfrak{c}$ *or* $\mathfrak{b} = \mathfrak{c}$*. Then there is a* $\mathbb{S}$*-indestructible, but* $\mathbb{S} * \mathbb{S}$*-destructible mad family.*

Hence, $\mathbb{S}$-indestructible families need not be universally Sacks-indestructible. One may ask the same question about finite products of Sacks-forcing $\mathbb{S}^2, \mathbb{S}^3$ and so on up to $\mathbb{S}^{\aleph_0}$:

Question 6.3. Consistently, is there a strictly increasing hierarchy of Sacks-indestructibility between $\mathbb{S}$ and $\mathbb{S}^{\aleph_0}$? What about iterations?

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REALIZING ARBITRARILY LARGE SPECTRA OF $\mathfrak{a}_{\mathrm{T}}$

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Abstract. We improve the state-of-the-art proof techniques for realizing various spectra of $\mathfrak{a}_{\mathrm{T}}$ in order to realize arbitrarily large spectra. Thus, we make significant progress in addressing a question posed by Brian in his work [4]. As a by-product, we obtain many complete subforcings and an algebraic analysis of the automorphisms of the forcing which adds a witness for the spectrum of $\mathfrak{a}_{\mathrm{T}}$ of desired size.

1. Introduction

Fundamentally, combinatorial set theory studies the possible sizes and relations between special subsets of reals. Usually, these special subsets are defined by some combinatorial property, e.g. mad families, independent families or partitions of Baire space into compact sets. Classically, the corresponding cardinal characteristics, i.e. the minimal sizes of such special subsets, and their relations are of main interest. However, a more recent approach is the study of their corresponding spectra, i.e. the set of all sizes of such special subsets. For some fixed type of combinatorial family of reals its spectrum may be studied from the following two angles:

On one hand, one may consider which properties of the spectrum are provable in ZFC. On the other hand, given a set of cardinals Θ with some additional assumptions one may construct forcing extensions in which Θ is precisely realized as the spectrum. Thus, the ultimate goal is to reduce the additional assumptions on Θ until they agree with the provable properties of the spectrum in ZFC, so that we obtain a complete classification of the possible spectra of some type of combinatorial family of reals.

Usually, the spectrum of some type of family may be rather arbitrary, so that there are not many provable properties in ZFC. However, recent progress suggests that the following properties are shared between different spectra. First, usually by some straightforward combinatorial argument the continuum $\mathfrak{c}$ is in the spectrum (a notable exception is the tower number $\mathfrak{t}$). By König's Theorem we obtain the following necessary restriction on Θ:

(I) $\max(\Theta)$ exists and has uncountable cofinality.

Secondly, there seems to be the following additional restriction on Θ:

(II) Θ is closed under singular limits.

For example, in [8] Hechler proved that $\mathrm{spec}(\mathfrak{a})$ is closed under singular limits. Similarly, recently Brian proved in [4] that also $\mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ (cf. Definition 2.1) is closed under singular limits. However, for most other types of families it is still not known if this restriction is necessary, i.e.:

Question. Are $\mathrm{spec}(\mathfrak{i})$, $\mathrm{spec}(\mathfrak{a}_{\mathrm{e}})$ and $\mathrm{spec}(\mathfrak{a}_{\mathrm{g}})$ closed under singular limits?

Finally, surprisingly specifically for the spectrum of $\mathfrak{a}_\mathrm{T}$ Brian recently provided yet another necessary assumption given by ZFC.

Theorem (Brian, 2022, [3]). *Assume* $0^\dagger$ *does not exist. If* θ *has countable cofinality and we have* $\theta \in \mathrm{spec}(\mathfrak{a}_\mathrm{T})$*, then also* $\theta^+ \in \mathrm{spec}(\mathfrak{a}_\mathrm{T})$.

In particular, a model in which $\theta \in \mathrm{spec}(\mathfrak{a}_\mathrm{T})$ and $\theta^+ \notin \mathrm{spec}(\mathfrak{a}_\mathrm{T})$ implies that $0^\dagger$ exists, so that there exists an inner model with a measurable cardinal. Hence, such a model cannot be constructed relative to ZFC. Note that this result is in stark contrast to the situation for the spectrum of $\mathfrak{a}$. In this, case Shelah and Spinas proved in [10] that consistently $\aleph_\omega \in \mathrm{spec}(\mathfrak{a})$, but $\aleph_{\omega+1} \notin \mathrm{spec}(\mathfrak{a})$. Hence, despite their similarities there are distinct discrepancies between the spectra of different types of families.

On the other hand, the realization of various spectra with the means of forcing was first studied for almost disjoint families. For mad families Hechler proved that any set of uncountable cardinals may be contained in the spectrum.

Theorem (Hechler, 1972, [8]). *Let* Θ *be any set of uncountable cardinals. Then, there is a c.c.c. forcing extension in which* $\Theta \subseteq \mathrm{spec}(\mathfrak{a})$ *holds.*

In order to exclude values from the spectrum and precisely realize Θ as some spectrum, one usually employs an isomorphism-of-names argument. For example, Blass proved that under the following additional assumptions on the set Θ, in Hechler's model the set Θ is already precisely realized as the spectrum of mad families:

Theorem (Blass, 1993, [1]). *Assume* GCH *and let* Θ *be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) $\aleph_1 \in \Theta$,
(IV) *If* $\theta \in \Theta$ *with* $\mathrm{cof}(\theta) = \omega$*, then* $\theta^+ \in \Theta$.

Then, there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}) = \Theta$ *holds.*

By employing a more sophisticated isomorphism-of-names argument, Shelah and Spinas later improved this result by weakening assumption (III) and removing assumption (IV):

Theorem (Shelah, Spinas, 2015, [10]). *Assume* GCH *and let* Θ *be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) $\min(\Theta)$ *is regular.*

Then, there is a c.c.c. forcing extension in which $\mathrm{spec}(\mathfrak{a}) = \Theta$ *holds.*

By the previous discussion (I) and (II) are necessary assumptions. However, $\mathfrak{a}$ may be singular, so that (III) is definitely not necessary. In fact, Brendle proved that $\mathfrak{a}$ may be any uncountable singular cardinal, even of countable cofinality [2]. Thus, an answer to the following question would yield a complete classification of all the possible spectra of $\mathfrak{a}$:

Question. Can assumption (III) be removed from the previous theorem?

Similar progress has been made for independent families by Fischer and Shelah [6] and partitions of Baire space into compact sets by Brian [4], which is the main focus of this paper:

Theorem (Brian, 2021, [4])**.** *Assume* GCH *and let* Θ *be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) *If* $\theta \in \Theta$ *with* $\operatorname{cof}(\theta) = \omega$*, then* $\theta^+ \in \Theta$*,*
(IV) $\min(\Theta)$ *is regular,*
(V) $|\Theta| < \min(\Theta)$*.*

Then, there is a c.c.c. forcing extension in which $\operatorname{spec}(\mathfrak{a}_{\mathrm{T}}) = \Theta$ *holds.*

Again, by the previous discussion (I), (II) and (III) are necessary assumptions. Assumption (IV) is not necessary as $\mathfrak{a}_{\mathrm{T}}$ may be any singular cardinal of uncountable cofinality. However, unlike $\mathfrak{a}$ it is still open if $\mathfrak{a}_{\mathrm{T}}$ may have countable cofinality. Assumption (V) is also not necessary as we may force any set of uncountable cardinals to be contained in $\operatorname{spec}(\mathfrak{a}_{\mathrm{T}})$ similar to Hechler's theorem for $\operatorname{spec}(\mathfrak{a})$. In other words, assumption (V) implies that once the minimum of Θ has been fixed, only a bounded set of cardinals may be realized with the methods developed by Brian. Thus, in [4] he asked if it is possible to remove assumption (V). Inspired by the methods of Shelah and Spinas for $\operatorname{spec}(\mathfrak{a})$ and towards obtaining a complete classification of the possible spectra of $\mathfrak{a}_{\mathrm{T}}$, we prove the following Main Theorem 3.1 and give a partial answer to Brian's question:

Main Theorem. *Assume* GCH *and let* Θ *be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) *If* $\theta \in \Theta$ *with* $\operatorname{cof}(\theta) = \omega$*, then* $\theta^+ \in \Theta$*,*
(IV) $\aleph_1 \in \Theta$*.*

Then, there is a c.c.c. forcing extension in which $\operatorname{spec}(\mathfrak{a}_{\mathrm{T}}) = \Theta$ *holds.*

Thus, we are indeed able to realize arbitrarily large spectra, however our current proof methods require us to strengthen assumption (IV). In Section 3 we outline the proof of Main Theorem 3.1 and discuss how to possibly avoid the strengthening of (IV) in order to obtain a full answer to Brian's question. Nevertheless, the following summarizes how the proof of Main Theorem 3.1 extends the current proof methods and techniques for realizing various spectra:

Generally, the forcing used to obtain our result is very similar to the forcing used in Brian's result above, but with a distinct modification in order to allow a more sophisticated isomorphism-of-names argument. Inspired by Shelah's and Spinas' result for $\operatorname{spec}(\mathfrak{a})$ the main feature of our argument is the restriction to isomorphic complete subforcings of the entire forcing. In contrast, Brian's argument only uses automorphism of the entire forcing, which leads to his restriction (V).

The main difficulty of our proof is showing that we indeed have many complete subforcings (see Theorem 7.1). In the situation for $\operatorname{spec}(\mathfrak{a})$ there is a product-like c.c.c. forcing, which adds a maximal almost disjoint family of desired size. Thus, the existence of complete subforcings is easy to prove in that case. In contrast, for $\mathrm{spec}(\mathfrak{a}_\mathrm{T})$ there is no known such product-like forcing and instead we have to use an iteration of c.c.c. forcings defined relative to some parameters in order to obtain a witness for $\mathfrak{a}_\mathrm{T}$ of desired size. To establish the existence of complete subforcings, we introduce the following novel advancements:

First, compared to Brian's forcing in [4], our forcing (see Definition 4.10) has a distinct modification, which allows for more automorphisms. In Section 5 we provide a very algebraic framework of these automorphisms. Nevertheless, we strive for a self-contained presentation. Secondly, since we do not have a parameter-less product forcing, we cannot simply use the standard notion of a canonical projection of a nice name of a real (cf. [7]). Instead, in Definition 7.4 we introduce the technical notion of a nice name for a finite set of reals with respect to a sequence of names for trees (these are the parameters of the iteration). The canonical projection of this technical nice name then has the desired properties in order to obtain complete subforcings.

Finally, in Section 9 we prove the isomorphism-of-names argument needed for our Main Theorem 3.1. However, again the situation is more complicated than for the spectrum of $\mathfrak{a}$ by Shelah and Spinas, because we are working with an iteration. To this end, in Section 8 we provide a very algebraic/categorical framework for the isomorphisms between the many complete subforcings just discussed. Lastly, we use these isomorphisms to show that the corresponding isomorphism-of-names argument can be carried out for the iteration. Thus, the main insight is that this more sophisticated isomorphism-of-names argument can not only be applied in a product-like context as for $\mathrm{spec}(\mathfrak{a})$, but also in a more intricate iteration-like context as for $\mathrm{spec}(\mathfrak{a}_\mathrm{T})$.

2. Preliminaries

In this section, we introduce the cardinal characteristic $\mathfrak{a}_\mathrm{T}$ and its associated spectrum $\mathrm{spec}(\mathfrak{a}_\mathrm{T})$. We will also define a c.c.c. forcing which forces the existence of witnesses in $\mathrm{spec}(\mathfrak{a}_\mathrm{T})$ of various sizes. In Definition 4.10 we define a slightly tweaked version of this forcing in order to realize arbitrarily large spectra of $\mathfrak{a}_\mathrm{T}$ in our Main Theorem 3.1.

Definition 2.1. We define the spectrum

$$\mathrm{spec}(\mathfrak{a}_\mathrm{T}) := \{\kappa > \aleph_0 \mid \text{There is a partition of } {}^\omega 2 \text{ into } \kappa\text{-many closed sets}\}.$$

and define the cardinal characteristic $\mathfrak{a}_\mathrm{T} := \min(\mathrm{spec}(\mathfrak{a}_\mathrm{T}))$.

We arbitrarily fixed ${}^\omega 2$ as our Polish space of choice here. However, Miller proved that a witness for $\aleph_1 \in \mathrm{spec}(\mathfrak{a}_\mathrm{T})$ does not depend on the underlying Polish space:

Theorem 2.2 (Miller, 1980, [9])**.** *There is a partition of ${}^\omega 2$ into $\aleph_1$-many closed sets iff there is a partition of some Polish space into $\aleph_1$-many closed sets iff every Polish space has a partition into $\aleph_1$-many closed sets.*

More generally, Spinas proved in [11] that $\mathfrak{a}_\mathrm{T}$ is independent of the underlying Polish space and that $\mathfrak{d} \leq \mathfrak{a}_\mathrm{T}$. Brian extended this result in the following way:

Theorem 2.3 ([4])**.** *Let κ be an uncountable cardinal. Then, all six statements of the following form are equivalent:*

Some/Every uncountable Polish space can be partitioned into κ compact/closed/F_σ-sets.

Hence, neither the cardinal characteristic $\mathfrak{a}_T$ nor its spectrum $\operatorname{spec}(\mathfrak{a}_T)$ depend on the underlying Polish space, or if partitions into compact, closed or F_σ-sets are considered. In order to force a desired constellation of $\operatorname{spec}(\mathfrak{a}_T)$, we will add partitions of ${}^\omega 2$ into F_σ-sets. To this end, we will use the usual identification of non-empty closed set of ${}^\omega 2$ and branches of trees:

Definition 2.4. A tree T is a non-empty subset of ${}^{<\omega}2$ such that

(1) for all $s \in {}^{<\omega}2$ and $t \in T$ with $s \trianglelefteq t$ we have $s \in T$,
(2) for all $s \in T$ we have $s^\frown 0 \in T$ or $s^\frown 1 \in T$ (or both).

We denote with $[T]$ the set of branches of T:

$$[T] := \{f \in {}^\omega 2 \mid \text{for all } n < \omega \text{ we have } f \restriction n \in T\}.$$

We call T nowhere dense if it additionally satisfies

(3) for all $s \in T$ there is a $t \in {}^{<\omega}2$ with $s \trianglelefteq t$ and $t \notin T$.

Remark 2.5. Given a tree T, the set $[T]$ is a non-empty closed set of ${}^\omega 2$. Conversely, given any non-empty closed set C the set

$$\operatorname{tree}(C) := \{s \in {}^{<\omega}2 \mid \text{there is an } f \in C \text{ with } s \trianglelefteq f\}$$

is a non-empty tree. Since, $[\operatorname{tree}(C)] = C$ and $\operatorname{tree}([T]) = T$ we may identify trees and non-empty closed sets of ${}^\omega 2$ under these bijections. Furthermore, if T is nowhere dense, then also $[T]$ is nowhere dense and conversely if C is nowhere dense, then also $\operatorname{tree}(C)$ is nowhere dense. Hence, this identification restricts to nowhere dense trees and nowhere dense closed subsets.

Definition 2.6. Let S, T be trees. We call S and T almost disjoint iff $S \cap T$ is finite.

Note that by König's lemma two trees S and T are almost disjoint exactly iff $[S] \cap [T] = \emptyset$. In order to force the existence of a witness for $\kappa \in \operatorname{spec}(\mathfrak{a}_T)$, we will add κ-many countable families $\{\mathcal{T}_\alpha \mid \alpha < \kappa\}$ of nowhere dense trees satisfying

(1) for all $\alpha < \beta < \kappa$ and $S \in \mathcal{T}_\alpha, T \in \mathcal{T}_\beta$ the trees S and T are almost disjoint,
(2) for all $f \in {}^\omega 2$ there is an $\alpha < \kappa$ with $f \in \bigcup_{T \in \mathcal{T}_\alpha} [T]$.

Notice that for $\alpha < \kappa$ and $S \neq T \in \mathcal{T}_\alpha$ we do not require that S and T are almost disjoint. However, the two conditions above imply that $\{\bigcup_{T \in \mathcal{T}_\alpha} [T] \mid \alpha < \kappa\}$ is a partition of ${}^\omega 2$ into κ-many F_σ-sets. Next, in order to approximate new nowhere dense trees with finite conditions we fix the following notions:

Definition 2.7. Let $n < \omega$. An n-tree T is a non-empty subset of ${}^{\leq n}2$ such that

(1) for all $s \in {}^{\leq n}2$ and $t \in T$ with $s \trianglelefteq t$ we have $s \in T$,
(2) for all $s \in T$ there is a $t \in T \cap {}^n 2$ with $s \trianglelefteq t$.

We denote with $[T]$ the set of leaves $T \cap {}^n2$ of T. Given $n \leq m$, an n-tree S and an m-tree T we write $S \trianglelefteq T$ iff T end-extends S, i.e. $T \cap {}^{\leq n}2 = S$.

Definition 2.8. Let $\mathcal{T}$ be a family of nowhere dense trees. We define the forcing $\mathbb{T}_0(\mathcal{T})$ to be the set of all pairs $p = (T_p, F_p)$, where T_p is an n_p-tree for some $n_p < \omega$ and $F_p \subseteq {}^\omega 2$ is finite such that for all $f \in F_p$ we have $f \notin \bigcup_{T \in \mathcal{T}}[T]$ and $f \restriction n_p \in [T_p]$.

Given two conditions $p, q \in \mathbb{T}_0(\mathcal{T})$ we define $q \leq p$ iff $n_p \leq n_q$, $F_p \subseteq F_q$ and $T_p \trianglelefteq T_q$. Further, we define $\mathbb{T}(\mathcal{T})$ to be the finitely supported product of size ω

$$\mathbb{T}(\mathcal{T}) := \prod_{\omega} \mathbb{T}_0(\mathcal{T}).$$

We give a brief overview of the crucial properties of $\mathbb{T}_0(\mathcal{T})$ and $\mathbb{T}(\mathcal{T})$ as they follow from standard density and forcing arguments. See [5] for more details for a very similar forcing.

Remark 2.9. $\mathbb{T}_0(\mathcal{T})$ is σ-centered, so also $\mathbb{T}(\mathcal{T})$ is σ-centered. Further, if G is $\mathbb{T}_0(\mathcal{T})$-generic in $V[G]$ the set

$$T^G := \bigcup \{T_p \mid p \in G\}$$

is a nowhere dense tree such that T^G and T are almost disjoint for all $T \in \mathcal{T}$. Analogously, if G is $\mathbb{T}(\mathcal{T})$-generic we denote with $\langle T_n^G \mid n < \omega \rangle$ the ω-many new nowhere dense trees by $\mathbb{T}(\mathcal{T})$. We have the following diagonalization properties:

- (D1) For all $n < \omega$ the tree T_n^G is almost disjoint from every $T \in \mathcal{T}$.
- (D2) For all $f \in ({}^\omega 2)^V$ with $f \notin \bigcup_{T \in \mathcal{T}}[T]$ we have $f \in \bigcup_{n<\omega}[T_n^G]$.

Note that in general T_n^G and T_m^G need not be almost disjoint for $n \neq m$. The diagonalization properties immediately yield the following lemma:

Lemma 2.10. *Let κ be an uncountable cardinal. Then, there is a c.c.c. forcing which forces the existence of a witness for $\kappa \in \operatorname{spec}(\mathfrak{a}_{\mathrm{T}})$.*

Proof. Sketch. Consider the following iteration: Start with the finitely supported product of Cohen forcing of size κ. In the generic extension, let $\mathcal{T}_1$ be the set $\langle T_\alpha \mid \alpha < \kappa \rangle$, where T_α is the nowhere dense tree with only branch the α-th Cohen real. Then, force with $\mathbb{T}(\mathcal{T}_1)$ to obtain ω-many new nowhere dense trees $\langle T_n \mid n < \omega \rangle$ with properties (D1) and (D2) in Remark 2.9. Extend $\mathcal{T}_1$ to $\mathcal{T}_2 := \mathcal{T}_1 \cup \{T_n \mid n < \omega\}$ and continue iterating $\mathbb{T}(\mathcal{T}_\alpha)$ the same way $\aleph_1$-many times with finite support. In the end we obtain $\kappa + \aleph_1 = \kappa$-many F_σ-sets which are disjoint by (D1) and cover ${}^\omega 2$ by (D2) and since $\aleph_1$ has uncountable cofinality. $\square$

In order to realize a whole spectrum of $\mathfrak{a}_{\mathrm{T}}$, in Definition 4.10 we define our forcing as a product of a slightly tweaked version of this iteration. A bookkeeping argument and Lemma 2.10 may be used to force $\mathfrak{a}_{\mathrm{T}} = \kappa$ and $\mathfrak{c} = \lambda$ for any regular κ and $\lambda > \kappa$ of uncountable cofinality [5]. Further, since $\mathfrak{d} \leq \mathfrak{a}_{\mathrm{T}}$, for κ of uncountable cofinality any model of $\mathfrak{d} = \kappa = \mathfrak{c}$ satifies $\mathfrak{a}_{\mathrm{T}} = \kappa$. However, this leaves open the following question:

Question 2.11. Can $\mathfrak{a}_{\mathrm{T}}$ be singular of uncountable cofinality and $\mathfrak{a}_{\mathrm{T}} < \mathfrak{c}$?

In [2] Brendle constructed a model of $\mathfrak{a} = \aleph_\omega$. While we may use Lemma 2.10 to force the existence of a witness for $\aleph_\omega \in \operatorname{spec}(\mathfrak{a}_{\mathrm{T}})$ the following question is still open:

Question 2.12. Can $\mathfrak{a}_{\mathrm{T}}$ be of countable cofinality? In particular is $\mathfrak{a}_{\mathrm{T}} = \aleph_\omega$ consistent?

Note that $\mathfrak{d} < \mathfrak{a}_{\mathrm{T}}$ must hold in such a model as $\mathfrak{d}$ can only have uncountable cofinality.

3. Realizing arbitrarily large spectra of $\mathfrak{a}_{\mathrm{T}}$

The culmination of this paper is the following Main Theorem. In this section, we will describe the proof ingredients and summarize the role of each section towards this goal.

Main Theorem 3.1. *Assume* GCH *and let* Θ *be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) *If* $\theta \in \Theta$ *with* $\operatorname{cof}(\theta) = \omega$*, then* $\theta^+ \in \Theta$*,*
(IV) $\aleph_1 \in \Theta$*.*

Then, there is a c.c.c. forcing extension in which $\operatorname{spec}(\mathfrak{a}_{\mathrm{T}}) = \Theta$ *holds.*

Our proof strategy is inspired by Shelah's and Spinas' work on the spectrum of mad families. They realize a desired spectrum Θ with a large product of Hechler forcing. In order to exclude cardinalities, their proof essentially hinges on the following fact:

Fact 3.2. *Let* $\mathbb{H}_I$ *be Hechler's forcing for adding an almost disjoint family indexed by* I*. If* $I \subseteq J$ *then* $\mathbb{H}_I \lessdot \mathbb{H}_J$*, i.e.* $\mathbb{H}_I$ *is a complete subforcing of* $\mathbb{H}_J$*.*

Their isomorphism-of-names argument then uses this fact in this product-like setting by reducing to countable subforcings of their whole forcing and using appropriate isomorphisms between these countable subforcings. In contrast, the isomorphism-of-names argument by Brian for the spectrum of $\mathfrak{a}_{\mathrm{T}}$ directly employs automorphisms of the whole forcing, which is less flexible. In our proof of our Main Theorem 3.1 we essentially adapt the proof strategy of Shelah and Spinas for the product-like context of $\mathfrak{a}$ to the iteration-like situation of $\mathfrak{a}_{\mathrm{T}}$. Consequently, this paper is structured as follows:

In Section 4 we define the c.c.c. forcing (see Definition 4.10) which yields Main Theorem 3.1. Similarly to Brian's forcing in [4], our forcing adds a witness for $\theta \in \operatorname{spec}(\mathfrak{a}_{\mathrm{T}})$ for every $\theta \in \Theta$. However, in contrast we define our forcing directly as an iteration. Moreover, we fix a larger family of trees after adding many Cohen reals in the first step of our iteration. Hence, the family of trees is closed under more automorphisms of the initial Cohen forcing. In Section 5 we provide a very algebraic framework for extending these Cohen automorphisms to automorphisms of the entire forcing (see Corollary 5.6). Next, throughout the paper we will need to work with nice conditions of our iteration, which describe all the forcing information in a given condition in a nice way. Hence, in Section 6 we inductively define the notion of a nice condition (see Definition 6.4) and prove their density in Lemma 6.7. We also define the hereditary support of a condition (see Definition 6.8) and study the behaviour of nice conditions under the automorphisms described in Section 5 (see Lemma 6.10 and Lemma 6.11).

Section 7 is the heart of the entire proof. We show that our forcing from Section 4 has enough complete subforcings to imitate the isomorphism-of-names argument by Shelah and Spinas. However, we do not obtain a direct analogue to Fact 3.2 above, but a weaker Theorem 7.1 instead:

Theorem. *Let $\Phi \subseteq \Psi$ be a Θ-subindexing function and assume Φ is countable. Then, $\mathbb{P}^{\Phi}_{\alpha} \lessdot \mathbb{P}^{\Psi}_{\alpha}$ for all $\alpha \leq \aleph_1$.*

Hence, with our current methods we can only show that we have complete subforcings if the index set is sufficiently small (countable in the sense of Definition 4.1) and the iteration is at most of length $\aleph_1$. This is precisely where we require the strengthening of (IV) in our Main Theorem 3.1. In other words, if Theorem 7.1 can be proven for longer iterations, requirement (IV) can again be relaxed to the requirement 'min(Θ) is regular', which would yield a full answer to Brian's question.

Theorem 7.1 is proved in an elaborate inductive fashion. First, in order to show that the embedding of some subforcing is well-defined, we will need the additional Cohen automorphisms our forcing possesses due to our modifications. Secondly, in order to show that these embeddings are indeed complete, we introduce the technical notion of a nice name for a finite set of reals with respect to a sequence of names for trees (see Definition 7.4). Then, the canonical projection of such a nice name (see Lemma 7.5) will have the desired properties in order to define a reduction of a condition in our forcing (see Lemma 7.6).

Finally, in Section 8 we give an algebraic/categorical analysis of the isomorphisms between the complete subforcings given by Theorem 7.1. Finally, we put everything together and provide the remaining isomorphism-of-names argument needed for Main Theorem 3.1 in Section 9.

4. Defining the iteration

In this section, we define the forcing used to prove Main Theorem 3.1. For the remainder of this paper let Θ be fixed as in Main Theorem 3.1.

Definition 4.1. A Θ-indexing function is a partial function $\Phi : \Theta \to V$. For two Θ-indexing functions Φ, Ψ, we write $\Phi \subseteq \Psi$ iff Φ is a Θ-subindexing function of Ψ, i.e. $\mathrm{dom}(\Phi) \subseteq \mathrm{dom}(\Psi)$ and for all $\theta \in \mathrm{dom}(\Phi)$ we have $\Phi(\theta) \subseteq \Psi(\theta)$. Finally, we call a Θ-indexing Φ function countable iff $\mathrm{dom}(\Phi)$ is countable and for every $\theta \in \mathrm{dom}(\Phi)$ we have that $\Phi(\theta)$ is countable.

Definition 4.2. Let Φ be a Θ-indexing function. Define $\mathbb{C}^{\Phi}$ to be the partial order adding new Cohen reals indexed by pairs (θ, i) where $\theta \in \mathrm{dom}(\Theta)$ and $i \in \Phi(\theta)$, i.e.

$$\mathbb{C}^{\Phi} := \{s : \bigcup_{\theta \in \mathrm{dom}(\Phi)} (\{\theta\} \times \Phi(\theta)) \to \mathbb{C} \mid \mathrm{supp}(s) \text{ is finite}\}.$$

Further, we write $\dot{c}^{\Phi,\theta}_i$ for the canonical $\mathbb{C}^{\Phi}$-name for the Cohen real indexed by (θ, i) and $\dot{T}^{\Phi,\theta}_i$ for the canonical $\mathbb{C}^{\Phi}$-name for the tree with only branch $\dot{c}^{\Phi,\theta}_i$.

Remark 4.3. Clearly, if $\Phi \subseteq \Psi$ we have $\mathbb{C}^{\Phi} \lessdot \mathbb{C}^{\Psi}$. In fact, there is a strong projection from $\mathbb{C}^{\Psi}$ onto $\mathbb{C}^{\Phi}$, which just forgets all Cohen information outside of Φ's indexing. We denote this complete embedding by $\iota^{\Phi,\Psi} : \mathbb{C}^\Phi \to \mathbb{C}^\Psi$. Notice that for $\theta \in \mathrm{dom}(\Phi)$ and $i \in \Phi(\theta)$ we have

$$\iota^{\Phi,\Psi}(\dot{c}_i^{\Phi,\theta}) = \dot{c}_i^{\Psi,\theta} \text{ and } \iota^{\Phi,\Psi}(\dot{T}_i^{\Phi,\theta}) = \dot{T}_i^{\Psi,\theta}.$$

$\mathbb{C}^\Phi$ will be the first step of our iteration. Note that $\mathbb{C}^\Phi$ has a vast amount of automorphisms and we need to extend some of these automorphisms through our iteration. In fact, we will need even more - we also need to preserve the group structure of the automorphisms. Hence, it is very natural to use the language of group actions and morphisms between group actions to express these properties.

Definition 4.4. Let Γ denote the group $\bigoplus_\omega \mathbb{Z}/2$ with group operation $+$. We define a group action $\Gamma \curvearrowright \mathbb{C}$ for $\gamma \in \Gamma$, $s \in \mathbb{C}$ by $\mathrm{dom}(\gamma.s) := \mathrm{dom}(s)$ and for $n \in \mathrm{dom}(s)$

$$(\gamma.s)(n) := \begin{cases} s(n) & \text{if } \gamma(n) = 0, \\ 1 - s(n) & \text{otherwise.} \end{cases}$$

Hence, an element $\gamma \in \Gamma$ flips the Cohen information at place n precisely iff $\gamma(n) = 1$.

Remark 4.5. Note that the action $\Gamma \curvearrowright \mathbb{C}$ preserves the order, that is $\gamma.s \leq \gamma.t$ for all $s, t \in \mathbb{C}$ with $s \leq t$. In other words, the action $\Gamma \curvearrowright \mathbb{C}$ is equivalent to a group homomorphism from $\pi : \Gamma \to \mathrm{Aut}(\mathbb{C})$. Also, remember that every automorphism of $\mathbb{C}$ is an involution.

Definition 4.6. Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$ and $i \in \Phi(\theta)$. Then, we have an induced group action of Γ acting on the (θ, i)-th component of $\mathbb{C}^\Phi$, which we denote with $\Gamma \overset{\theta,i}{\curvearrowright} \mathbb{C}^\Phi$. In other words, we have that the inclusion map $\iota_i^{\Phi,\theta} : \mathbb{C} \to \mathbb{C}^\Phi$ is a morphism of Γ-sets, i.e. the following diagram commutes for every $\gamma \in \Gamma$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\iota_i^{\Phi,\theta}} & \mathbb{C}^\Phi \\ \big\downarrow{\scriptstyle \pi(\gamma)} & & \big\downarrow{\scriptstyle \pi_i^{\Phi,\theta}(\gamma)} \\ \mathbb{C} & \xrightarrow{\iota_i^{\Phi,\theta}} & \mathbb{C}^\Phi \end{array}$$

where $\pi_i^{\Phi,\theta}$ is the group homomorphism corresponding to $\Gamma \overset{\theta,i}{\curvearrowright} \mathbb{C}^\Phi$.

Remark 4.7. Since we have various different group actions of Γ acting on $\mathbb{C}^\Phi$, we will usually use the corresponding group homomorphisms $\pi_i^{\Phi,\theta} : \Gamma \to \mathrm{Aut}(\mathbb{C}^\Phi)$ to avoid confusion. Also, note that more generally for any Θ-subindexing function $\Phi \subseteq \Psi$, $\theta \in \mathrm{dom}(\Phi)$ and $i \in \Phi(\theta)$ we have that $\iota^{\Phi,\Psi}$ is a morphisms of Γ-sets, i.e. the following diagram commutes for every $\gamma \in \Gamma$:

$$\begin{array}{ccc} \mathbb{C}^\Phi & \xrightarrow{\iota^{\Phi,\Psi}} & \mathbb{C}^\Psi \\ \big\downarrow{\scriptstyle \pi_i^{\Phi,\theta}(\gamma)} & & \big\downarrow{\scriptstyle \pi_i^{\Psi,\theta}(\gamma)} \\ \mathbb{C}^\Phi & \xrightarrow{\iota^{\Phi,\Psi}} & \mathbb{C}^\Psi \end{array}$$

Definition 4.8. Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$ and $i \in \Phi(\theta)$. Denote with $\dot{\mathcal{T}}_i^{\Phi,\theta}$ the canonical $\mathbb{C}^\Phi$-name for the set

$$\{\pi_i^{\Phi,\theta}(\gamma)(\dot{T}_i^{\Phi,\theta}) \mid \gamma \in \Gamma\}.$$

Similarly, we let $\dot{\mathcal{T}}^{\Phi,\theta}$ denote the canonical $\mathbb{C}^\Phi$-name for the set

$$\{\pi_i^{\Phi,\theta}(\gamma)(\dot{T}_i^{\Phi,\theta}) \mid i \in \Phi(\theta) \text{ and } \gamma \in \Gamma\}.$$

Remark 4.9. Since Γ is countable, also $\dot{\mathcal{T}}_i^{\Phi,\theta}$ is countable and $\dot{\mathcal{T}}^{\Phi,\theta}$ of size $|\Phi(\theta)| \cdot \aleph_0$, hence countable in case that Φ is countable. Further, using Remark 4.3 and 4.7 it is easy to verify the following properties for every Θ-subindexing function $\Phi \subseteq \Psi$, $\theta \in \mathrm{dom}(\Phi)$ and $i \in \Phi(\theta)$:

- $\dot{\mathcal{T}}^{\Phi,\theta}$ is the canonical $\mathbb{C}^\Phi$-name for $\bigcup_{i\in\Phi(\theta)} \dot{\mathcal{T}}_i^{\Phi,\theta}$,
- $\iota^{\Phi,\Psi}(\dot{\mathcal{T}}_i^{\Phi,\theta}) = \dot{\mathcal{T}}_i^{\Psi,\theta}$,
- $\mathbb{C}^\Phi \Vdash \bigcup_{T\in\dot{\mathcal{T}}_i^{\Phi,\theta}}[T] = \{f \in {}^\omega 2 \mid f =^* \dot{c}_i^{\Phi,\theta}\}$.

Next, given a Θ-indexing function Φ we define the forcing iteration realizing the desired spectrum of $\mathfrak{a}_{\mathrm{T}}$ for Main Theorem 3.1. The forcing is a finite support iteration of c.c.c. forcings of length $\aleph_1$:

Definition 4.10. Let Φ be a Θ-indexing function. We will define a finite support iteration $\langle \mathbb{P}_\alpha^\Phi, \dot{\mathbb{Q}}_\beta^\Phi \mid \alpha \leq \aleph_1, \beta < \aleph_1\rangle$, $\mathbb{P}_{\alpha+1}^\Phi$-names $\dot{T}_{\alpha,n}^{\Phi,\theta}$ for nowhere dense trees for $\theta \in \mathrm{dom}(\Phi)$, $0 < \alpha < \aleph_1$ and $n < \omega$, and $\mathbb{P}_\alpha^\Phi$-names $\dot{\mathcal{T}}_\alpha^{\Phi,\theta}$ for families of nowhere dense trees for $\theta \in \mathrm{dom}(\Phi)$ and $0 < \alpha \leq \aleph_1$:

- Let $\dot{\mathbb{Q}}_0^\Phi$ be the forcing $\mathbb{C}^\Phi$. Then, we already defined the $\mathbb{C}^\Phi$-names $\dot{\mathcal{T}}^{\Phi,\theta}$ in Definition 4.8 for every $\theta \in \mathrm{dom}(\Phi)$. Then, let $\dot{\mathcal{T}}_1^{\Phi,\theta}$ the corresponding canonical $\mathbb{P}_1^\Phi$-names.
- For $\alpha > 0$ let $\dot{\mathbb{Q}}_\alpha^\Phi$ be the canonical $\mathbb{P}_\alpha^\Phi$-name for the finitely supported product

$$\prod_{\theta\in\mathrm{supp}(\Phi)} \mathbb{T}(\mathcal{T}_\alpha^{\Phi,\theta}).$$

  Also, for every $\theta \in \mathrm{dom}(\Phi)$ let $\dot{T}_{\alpha,n}^{\Phi,\theta}$ be the canonical $\mathbb{P}_{\alpha+1}^\Phi$-names for the ω-many new nowhere dense trees added by $\mathbb{T}(\mathcal{T}_\alpha^{\Phi,\theta})$, where $n < \omega$. Finally, let $\dot{\mathcal{T}}_{\alpha+1}^{\Phi,\theta}$ be the canonical $\mathbb{P}_{\alpha+1}^\Phi$-name for $\dot{\mathcal{T}}_\alpha^{\Phi,\theta} \cup \{\dot{T}_{\alpha,n}^{\Phi,\theta} \mid n \in \omega\}$.
- At limit α for every $\theta \in \mathrm{dom}(\Phi)$ let $\dot{\mathcal{T}}_\alpha^{\Phi,\theta}$ be the canonical $\mathbb{P}_\alpha^\Phi$-name for $\bigcup_{\beta<\alpha} \dot{\mathcal{T}}_\beta^{\Phi,\theta}$.

Grouping together the ω-many new trees added at each successor step into one F_σ-set, we have that for every $\theta \in \mathrm{dom}(\Phi)$ the family $\dot{\mathcal{T}}_{\aleph_1}^{\Phi,\theta}$ will be witness of a partition of Cantor space into F_σ-sets of size $|\Phi(\theta)| \cdot \aleph_1$. Thus, if every $\Phi(\theta)$ is a set of size θ, then $\Theta \subseteq \mathrm{spec}(\mathfrak{a}_{\mathrm{T}})$ is forced by $\mathbb{P}_{\aleph_1}^\Phi$ as in Lemma 2.10. Thus, it only remains to prove the reverse inclusion.

5. Extending group actions through the iteration

Since $\mathbb{P}_1^\Phi \cong \mathbb{C}^\Phi$, in the last section we essentially considered group actions $\Gamma \overset{\theta,i}{\curvearrowright} \mathbb{P}_1^\Phi$. In this section, we will show that there is a canonical way to extend these group actions through the iteration, i.e. to group actions $\Gamma \overset{\theta,i}{\curvearrowright} \mathbb{P}_\alpha^\Phi$ for $0 < \alpha \leq \aleph_1$,. This process leads to the notion of an induced sequence of group actions in Corollary 5.6. We write $\iota_1^{\Phi,\Psi} : \mathbb{P}_1^\Phi \to \mathbb{P}_1^\Psi$ for the complete embedding corresponding to $\iota^{\Phi,\Psi} : \mathbb{C}^\Phi \to \mathbb{C}^\Psi$, $\pi_{1,i}^{\Phi,\theta} : \Gamma \to \mathrm{Aut}(\mathbb{P}_1^\Phi)$ for the group homomorphism corresponding to $\pi_i^{\Phi,\theta} : \Gamma \to \mathrm{Aut}(\mathbb{C}^\Phi)$ and $\dot{\mathcal{T}}_{1,i}^{\Phi,\theta}$ for the $\mathbb{P}_1^\Phi$-name corresponding to $\dot{\mathcal{T}}_1^{\Phi,\theta}$.

Definition 5.1. Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$ and $\epsilon \leq \aleph_1$. We say that

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \epsilon \rangle$$

is an increasing sequence of Γ-actions iff every $\pi^{\Phi,\theta}_{\alpha,i}$ is a group homomorphism (i.e. an action of Γ on $\mathbb{P}^{\Phi}_{\alpha}$), for all $0 < \alpha \leq \epsilon$, $\eta \in \mathrm{dom}(\Phi)$ and $\gamma \in \Gamma$ we have

$$\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_{\alpha}) = \dot{\mathcal{T}}^{\Phi,\eta}_{\alpha},$$

and for all $0 < \alpha \leq \beta \leq \epsilon$ the canonical embedding $\iota^{\Phi}_{\alpha,\beta} : \mathbb{P}^{\Phi}_{\alpha} \to \mathbb{P}^{\Phi}_{\beta}$ is a morphism of Γ-sets, i.e the following diagram commutes for every $\gamma \in \Gamma$:

$$\begin{array}{ccc} \mathbb{P}^{\Phi}_{\alpha} & \xrightarrow{\iota^{\Phi}_{\alpha,\beta}} & \mathbb{P}^{\Phi}_{\beta} \\ \big\downarrow{\scriptstyle \pi^{\Phi,\theta}_{\alpha,i}(\gamma)} & & \big\downarrow{\scriptstyle \pi^{\Phi,\theta}_{\beta,i}(\gamma)} \\ \mathbb{P}^{\Phi}_{\alpha} & \xrightarrow{\iota^{\Phi}_{\alpha,\beta}} & \mathbb{P}^{\Phi}_{\beta} \end{array}$$

Our goal for this section is to provide a canonical extension of $\pi^{\Phi,\theta}_{1,i}$ as defined in Definition 4.6 to an increasing sequence of Γ-actions of length $\aleph_1$. Since the iterands of the forcing in Definition 4.10 are definable from the parameters $\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}$, it is crucial that the group action fixes these parameters, which allows for an extension through the iteration. Before we consider the successor step, we show that for limit steps there is a unique way to extend an increasing sequence of Γ-actions by the universal property of the direct limit:

Lemma 5.2. *Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$ and let $\epsilon \leq \aleph_1$ be a limit. Assume*

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha < \epsilon \rangle$$

is an increasing sequence of Γ-actions. Then there is a unique group homomorphism $\pi^{\Phi,\theta}_{\epsilon,i}$ so that

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \epsilon \rangle$$

is an increasing sequences of Γ-actions.

Proof. By definition of an increasing sequence of Γ-actions (cf. Definition 5.1) for any $\gamma \in \Gamma$ we have a directed system of maps

$$\langle \iota^{\Phi}_{\alpha,\epsilon} \circ (\pi^{\Phi,\theta}_{\alpha,i}(\gamma)) : \mathbb{P}^{\Phi}_{\alpha} \to \mathbb{P}^{\Phi}_{\epsilon} \mid 0 < \alpha < \epsilon \rangle.$$

By the universal property of $\mathbb{P}^{\Phi}_{\epsilon}$ there is a unique map $\pi^{\Phi,\theta}_{\epsilon,i}(\gamma) : \mathbb{P}^{\Phi}_{\epsilon} \to \mathbb{P}^{\Phi}_{\epsilon}$, so that the following diagram commutes for every $0 < \alpha \leq \epsilon$ and $\gamma \in \Gamma$:

$$\begin{array}{ccc} \mathbb{P}^{\Phi}_{\alpha} & \xrightarrow{\iota^{\Phi}_{\alpha,\epsilon}} & \mathbb{P}^{\Phi}_{\epsilon} \\ \big\downarrow{\scriptstyle \pi^{\Phi,\theta}_{\alpha,i}(\gamma)} & & \big\downarrow{\scriptstyle \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)} \\ \mathbb{P}^{\Phi}_{\alpha} & \xrightarrow{\iota^{\Phi}_{\alpha,\epsilon}} & \mathbb{P}^{\Phi}_{\epsilon} \end{array}$$

Next, fix $\gamma, \delta \in \Gamma$. We need to verify that $\pi^{\Phi,\theta}_{\epsilon,i}(\gamma) \circ \pi^{\Phi,\theta}_{\epsilon,i}(\delta) = \pi^{\Phi,\theta}_{\epsilon,i}(\gamma + \delta)$, so let $p \in \mathbb{P}^{\Phi,\theta}_{\epsilon}$. Choose $\alpha < \epsilon$ such that $\iota^{\Phi}_{\alpha,\epsilon}(p \restriction \alpha) = p$. Then, we compute

$$\begin{aligned}
\pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\pi^{\Phi,\theta}_{\epsilon,i}(\delta)(p)) &= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\pi^{\Phi,\theta}_{\epsilon,i}(\delta)(\iota^{\Phi}_{\alpha,\epsilon}(p \restriction \alpha))) && \text{(choice of } \alpha\text{)}\\
&= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\iota^{\Phi}_{\alpha,\epsilon}(\pi^{\Phi,\theta}_{\alpha,i}(\delta)(p \restriction \alpha))) && \text{(choice of } \pi^{\Phi,\theta}_{\epsilon,i}(\delta)\text{)}\\
&= \iota^{\Phi}_{\alpha,\epsilon}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\pi^{\Phi,\theta}_{\alpha,i}(\delta)(p \restriction \alpha))) && \text{(choice of } \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)\text{)}\\
&= \iota^{\Phi}_{\alpha,\epsilon}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma + \delta)(p \restriction \alpha)) && (\pi^{\Phi,\theta}_{\alpha,i} \text{ is group homomorphism})\\
&= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma + \delta)(\iota^{\Phi}_{\alpha,\epsilon}(p \restriction \alpha)) && \text{(choice of } \pi^{\Phi,\theta}_{\epsilon,i}(\gamma+\delta)\text{)}\\
&= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma + \delta)(p) && \text{(choice of } \alpha\text{).}
\end{aligned}$$

Thus, $\pi^{\Phi,\theta}_{\epsilon,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\epsilon})$ is a group homomorphism. Finally, by Definition 4.10 $\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon}$ is the canonical name for $\bigcup_{\alpha<\epsilon} \iota^{\Phi}_{\alpha,\epsilon}(\dot{\mathcal{T}}^{\Phi,\eta}_{\alpha})$. Thus, for any $\gamma \in \Gamma$ and $\eta \in \mathrm{dom}(\Phi)$ we compute

$$\begin{aligned}
\pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon}) &= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\bigcup_{\alpha<\epsilon} \iota^{\Phi}_{\alpha,\epsilon}(\dot{\mathcal{T}}^{\Phi,\eta}_{\alpha})) && \text{(Definition 4.10)}\\
&= \bigcup_{\alpha<\epsilon} \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\iota^{\Phi}_{\alpha,\epsilon}(\dot{\mathcal{T}}^{\Phi,\eta}_{\alpha})) && \text{(canonical name)}\\
&= \bigcup_{\alpha<\epsilon} \iota^{\Phi}_{\alpha,\epsilon}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_{\alpha})) && \text{(choice of } \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)\text{)}\\
&= \bigcup_{\alpha<\epsilon} \iota^{\Phi}_{\alpha,\epsilon}(\dot{\mathcal{T}}^{\Phi,\eta}_{\alpha}) && \text{(Definition 5.1)}\\
&= \dot{\mathcal{T}}^{\Phi,\eta}_{\alpha} && \text{(Definition 4.10).}
\end{aligned}$$

□

Next, we consider the successor case. In this case, there is no unique extension of the increasing sequence of Γ-actions. However, we prove that there is a canonical one in the following sense:

Definition 5.3. Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$ and $\epsilon < \aleph_1$. Assume

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \epsilon \rangle$$

is an increasing sequence of Γ-actions. For every $\gamma \in \Gamma$ define $\pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma) : \mathbb{P}^{\Phi}_{\epsilon+1} \to \mathbb{P}^{\Phi}_{\epsilon+1}$ by

$$\pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma)(p) := \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(p \restriction \epsilon) ^\frown \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(p(\epsilon)).$$

Then, we call $\pi^{\Phi,\theta}_{\epsilon+1,i}$ the canonical extension of $\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \epsilon \rangle$.

Lemma 5.4. *Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$ and $\epsilon < \aleph_1$. Assume*

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \epsilon \rangle$$

is an increasing sequence of Γ-actions and let $\pi^{\Phi,\theta}_{\epsilon+1,i}$ be the canonical extension. Then

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \epsilon + 1 \rangle$$

is an increasing sequence of Γ-actions.

Proof. First, by definition of an increasing sequence of Γ-actions (cf. Definition 5.1) for every $\eta \in \mathrm{dom}(\Phi)$ and $\gamma \in \Gamma$ we have

$$\pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon}) = \dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon}.$$

By Definition 4.10 $\dot{\mathbb{Q}}^{\Phi}_{\epsilon}$ is the canonical $\mathbb{P}^{\Phi}_{\epsilon}$-name for $\prod_{\eta\in\mathrm{dom}(\Phi)} \mathbb{T}(\mathcal{T}^{\Phi,\eta}_{\epsilon})$. Thus, we obtain

$$\pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\dot{\mathbb{Q}}^{\Phi}_{\epsilon}) = \dot{\mathbb{Q}}^{\Phi}_{\epsilon},$$

as both $\prod_{\eta\in\mathrm{dom}(\Phi)} \mathbb{T}(\mathcal{T}^{\Phi,\eta}_{\epsilon})$ as well as the order $\leq$ are definable from the parameters $\mathcal{T}^{\Phi,\eta}_{\epsilon}$. Thus, we get $\pi^{\Phi,\theta}_{\epsilon+1,i} \in \mathrm{Aut}(\mathbb{P}^{\Phi}_{\epsilon+1})$. Next, we verify that for every $\gamma \in \Gamma$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^{\Phi}_{\epsilon} & \xrightarrow{\iota^{\Phi}_{\epsilon,\epsilon+1}} & \mathbb{P}^{\Phi}_{\epsilon+1} \\ \big\downarrow{\scriptstyle \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)} & & \big\downarrow{\scriptstyle \pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma)} \\ \mathbb{P}^{\Phi}_{\epsilon} & \xrightarrow{\iota^{\Phi}_{\epsilon,\epsilon+1}} & \mathbb{P}^{\Phi}_{\epsilon+1} \end{array}$$

Let $\gamma \in \Gamma$ and $p \in \mathbb{P}^{\Phi}_{\epsilon}$. Then, we compute

$$\begin{aligned} \pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma)(\iota^{\Phi}_{\epsilon,\epsilon+1}(p)) &= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\iota^{\Phi}_{\epsilon,\epsilon+1}(p)\restriction\epsilon)^\frown \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\iota^{\Phi}_{\epsilon,\epsilon+1}(p)(\epsilon)) && \text{(Definition 5.3)} \\ &= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(p)^\frown \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\mathbb{1}) && \text{(definition of } \iota^{\Phi}_{\epsilon,\epsilon+1}) \\ &= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(p)^\frown \mathbb{1} && (\pi^{\Phi,\theta}_{\epsilon,i}(\gamma) \in \mathrm{Aut}(\mathbb{P}^{\Phi}_{\epsilon})) \\ &= \iota^{\Phi}_{\epsilon,\epsilon+1}(\pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(p)) && \text{(definition of } \iota^{\Phi}_{\epsilon,\epsilon+1}). \end{aligned}$$

Now, let $\gamma, \delta \in \Gamma$. We need to verify that $\pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma) \circ \pi^{\Phi,\theta}_{\epsilon+1,i}(\delta) = \pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma+\delta)$, so let $p \in \mathbb{P}^{\Phi,\theta}_{\epsilon+1}$. Then, we compute

$$\begin{aligned} \pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma)(\pi^{\Phi,\theta}_{\epsilon+1,i}(\delta)(p)) &= \pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma)(\pi^{\Phi,\theta}_{\epsilon,i}(\delta)(p\restriction\epsilon)^\frown \pi^{\Phi,\theta}_{\epsilon,i}(\delta)(p(\epsilon))) && \text{(Definition 5.3)} \\ &= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\pi^{\Phi,\theta}_{\epsilon,i}(\delta)(p\restriction\epsilon))^\frown \pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\pi^{\Phi,\theta}_{\epsilon,i}(\delta)(p(\epsilon))) && \text{(Definition 5.3)} \\ &= \pi^{\Phi,\theta}_{\epsilon,i}(\gamma+\delta)(p\restriction\epsilon)^\frown \pi^{\Phi,\theta}_{\epsilon,i}(\gamma+\delta)(p(\epsilon)) && (\pi^{\Phi,\theta}_{\epsilon,i} \text{ is gr.hom.)} \\ &= \pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma+\delta)(p) && \text{(Definition 5.3).} \end{aligned}$$

Thus, $\pi^{\Phi,\theta}_{\epsilon+1,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\epsilon})$ is a group homomorphism. Finally, let $\eta \in \mathrm{dom}(\Phi)$ and $\gamma \in \Gamma$. By Definition 4.10 $\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon+1}$ is the canonical name for $\iota^{\Phi}_{\epsilon,\epsilon+1}(\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon}) \cup \{\dot{T}^{\Phi,\eta}_{\epsilon,n} \mid n \in \omega\}$. Since

$$\begin{aligned} \pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma)(\iota^{\Phi}_{\epsilon,\epsilon+1}(\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon})) &= \iota^{\Phi}_{\epsilon,\epsilon+1}(\pi^{\Phi,\theta}_{\epsilon,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon})) && \text{(by commutativity above)} \\ &= \iota^{\Phi}_{\epsilon,\epsilon+1}(\dot{\mathcal{T}}^{\Phi,\eta}_{\epsilon}) && \text{(by Definition 5.1),} \end{aligned}$$

it suffices to verify that for all $n < \omega$ we have

$$\pi^{\Phi,\theta}_{\epsilon+1,i}(\gamma)(\dot{T}^{\Phi,\eta}_{\epsilon,n}) = \dot{T}^{\Phi,\eta}_{\epsilon,n}.$$

But this follows since $\dot{T}^{\Phi,\eta}_{\epsilon,n}$ is the canonical $\mathbb{P}^{\Phi}_{\epsilon+1}$-name for the n-th new nowhere dense trees added by $\mathbb{T}(\mathcal{T}^{\Phi,\theta}_{\epsilon})$ and check-names are fixed by any automorphism; remember that $\dot{T}^{\Phi,\eta}_{\epsilon,n}$ is just canonical name the union of the finite approximations in the generic filter. $\square$

Lemma 5.5. *Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$. Then, for every $\eta \in \mathrm{dom}(\Phi)$ and $\gamma \in \Gamma$ we have $\pi^{\Phi,\theta}_{1,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_1) = \dot{\mathcal{T}}^{\Phi,\eta}_1$, where $\pi^{\Phi,\theta}_{1,i}$ is defined as in Definition 4.6. In other words*

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq 1 \rangle$$

is an increasing sequence of Γ-actions (of length 1).

Proof. Let $\eta \in \mathrm{dom}(\Phi)$ and $\gamma \in \Gamma$. By definition 4.8 $\dot{\mathcal{T}}^{\Phi,\eta}_1$ is the canonical $\mathbb{C}^{\Phi}$-name for the set

$$\bigcup_{j \in \Phi(\theta)} \dot{\mathcal{T}}^{\Phi,\eta}_{1,j},$$

so it suffices to check that for all $j \in \Phi(\eta)$ we have $\pi^{\Phi,\theta}_{1,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_{1,j}) = \dot{\mathcal{T}}^{\Phi,\eta}_{1,j}$, so fix some $j \in \Phi(\eta)$. By Definition 4.8 $\dot{\mathcal{T}}^{\Phi,\eta}_{1,j}$ is the canonical $\mathbb{C}^{\Phi}$-name for the set

$$\{\pi^{\Phi,\eta}_{j}(\delta)(\dot{T}^{\Phi,\eta}_{j}) \mid \delta \in \Gamma\}.$$

Thus, in case that $(\theta, i) = (\eta, j)$ we compute

$$\begin{aligned}
\pi^{\Phi,\theta}_{1,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\theta}_{1,i}) &= \pi^{\Phi,\theta}_{1,i}(\gamma)(\{\pi^{\Phi,\theta}_{i}(\delta)(\dot{T}^{\Phi,\theta}_{i}) \mid \delta \in \Gamma\}) && \text{(Definition 4.8)}\\
&= \{\pi^{\Phi,\theta}_{1,i}(\gamma)(\pi^{\Phi,\theta}_{i}(\delta)(\dot{T}^{\Phi,\theta}_{i})) \mid \delta \in \Gamma\} && \text{(canonical name)}\\
&= \{\pi^{\Phi,\theta}_{1,i}(\gamma + \delta)(\dot{T}^{\Phi,\theta}_{i}) \mid \delta \in \Gamma\} && (\pi^{\Phi,\theta}_{1,i} \text{ is gr.hom.})\\
&= \{\pi^{\Phi,\theta}_{1,i}(\delta)(\dot{T}^{\Phi,\theta}_{i}) \mid \delta \in \Gamma\} && (\Gamma \text{ is a group})\\
&= \dot{\mathcal{T}}^{\Phi,\theta}_{1,j} && \text{(Definition 4.8).}
\end{aligned}$$

Otherwise, $\pi^{\Phi,\eta}_{j}(\delta)(\dot{T}^{\Phi,\eta}_{j})$ has no information in the (θ, i)-th coordinate for every $\delta \in \Gamma$, so that

$$\begin{aligned}
\pi^{\Phi,\theta}_{1,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\eta}_{1,j}) &= \pi^{\Phi,\theta}_{1,i}(\gamma)(\{\pi^{\Phi,\eta}_{j}(\delta)(\dot{T}^{\Phi,\eta}_{j}) \mid \delta \in \Gamma\}) && \text{(Definition 4.8)}\\
&= \{\pi^{\Phi,\theta}_{1,i}(\gamma)(\pi^{\Phi,\eta}_{j}(\delta)(\dot{T}^{\Phi,\eta}_{j})) \mid \delta \in \Gamma\} && \text{(canonical name)}\\
&= \{\pi^{\Phi,\eta}_{1,j}(\delta)(\dot{T}^{\Phi,\eta}_{j}) \mid \delta \in \Gamma\} && ((\theta, i) \neq (\eta, j))\\
&= \dot{\mathcal{T}}^{\Phi,\eta}_{1,j} && \text{(Definition 4.8).}
\end{aligned}$$

□

Corollary 5.6. *Let Φ be a Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$. Then, there is an increasing sequence of Γ-actions*

$$\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \aleph_1 \rangle$$

such that $\pi^{\Phi,\theta}_{\epsilon+1,i}$ the canonical extension of $\langle \pi^{\Phi,\theta}_{\alpha,i} : \Gamma \to \mathrm{Aut}(\mathbb{P}^{\Phi}_{\alpha}) \mid 0 < \alpha \leq \epsilon \rangle$ for every $\epsilon < \aleph_1$. We call this sequence the induced sequence of group actions of $\pi^{\Phi,\theta}_{1,i}$ and will reserve the notions $\langle \pi^{\Phi,\theta}_{\alpha,i} \mid 0 < \alpha \leq \aleph_1 \rangle$ for the remainder of this paper.

Proof. We iteratively construct the desired sequence. By Lemma 5.5 we may start with $\pi^{\Phi,\theta}_{1,i}$ as in Definition 4.6, use Lemma 5.4 for the successor step and Lemma 5.2 for the limit step. □

6. A nice dense subset

In the following sections, we will need to work with a nice dense subset D^Φ_α of $\mathbb{P}^\Phi_\alpha$. A condition $p \in \mathbb{P}^\Phi_\alpha$ has finite support, where $p(0) \in \mathbb{C}^\Phi$ and for $\alpha \in \operatorname{supp}(p) \setminus \{0\}$ we have

$$p \restriction \alpha \Vdash p(\alpha) \in \dot{\mathbb{Q}}^\Phi_\alpha = \prod_{\theta \in \operatorname{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha).$$

We will inductively define D^Φ_α, so that as many parameters for $p(\alpha)$ as possible are decided as ground model objects. First, we will need the following definition of a nice name for a real.

Definition 6.1. Let $\mathbb{P}$ be a forcing and $p \in \mathbb{P}$. A nice $\mathbb{P}$-name for a real below p is a sequence $\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle$ such that

- for all $n < \omega$ the set $\mathcal{A}_n$ is a maximal antichain below p and $f_n : \mathcal{A}_n \to {}^{>n}2$,
- for all $n < m$ the set $\mathcal{A}_m$ refines $\mathcal{A}_n$, i.e. for every $b \in \mathcal{A}_m$ there is $a \in \mathcal{A}_n$ with $b \leq a$,
- for all $n < m$, $a \in \mathcal{A}_n$ and $b \in \mathcal{A}_m$ with $b \leq a$ we have $f_n(a) \trianglelefteq f_m(b)$.

Further, we write $\operatorname{name}(\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle)$ for the canonical $\mathbb{P}$-name of $\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle$, i.e.

$$\operatorname{name}(\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle) := \{(a, (n, f_n(a)(n))) \mid n < \omega \text{ and } a \in \mathcal{A}_n\}.$$

Remark 6.2. Remember, that for every $p \in \mathbb{P}$ and $\mathbb{P}$-name $\dot{g}$ for a real below p we may inductively define a nice $\mathbb{P}$-name $\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle$ for a real below p such that

$$p \Vdash \dot{f} = \operatorname{name}(\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle).$$

Further, if $\mathbb{P}$ is c.c.c., then for any $p \in \mathbb{P}$ there are at most $|\mathbb{P}|^{\aleph_0}$ many nice names for reals below p. We also have that nice names and their canonical names behave nicely under automorphisms in the following sense:

Remark 6.3. If $\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle$ is a nice $\mathbb{P}$-name for a real below p and $\pi \in \operatorname{Aut}(\mathbb{P})$, then $\pi(\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle) := \langle (\mathcal{B}_n, g_n) \mid n < \omega \rangle$, where $\mathcal{B}_n = \pi[\mathcal{A}_n]$ and

$$g_n(\pi(a)) := f_n(a),$$

is a nice $\mathbb{P}$-name for a real below $\pi(p)$ with

$$\pi(\operatorname{name}(\langle (\mathcal{A}_n, f_n) \mid n < \omega \rangle)) = \operatorname{name}(\langle (\mathcal{B}_n, g_n) \mid n < \omega \rangle).$$

Definition 6.4. Let Φ be a Θ-indexing function and $0 < \alpha \leq \aleph_1$. D^Φ_α is the set of all nice conditions in $\mathbb{P}^\Phi_\alpha$, where inductively $p \in \mathbb{P}^\Phi_\alpha$ is a nice condition

- for $\alpha = 1$: iff $p(0) = c^p$ for some $c^p \in \mathbb{C}^\Phi$,
- for $\alpha + 1 > 1$: iff $p \restriction \alpha \in D^\Phi_\alpha$ and
  - there is a finite set $\Theta^p_\alpha \subseteq \operatorname{dom}(\Phi)$,
  - for every $\theta \in \Theta^p_\alpha$ there is a finite set $I^p_{\alpha,\theta} \subseteq \omega$,
  - for every $i \in I^p_{\alpha,\theta}$ there is $n^p_{\alpha,\theta,i} < \omega$ and an $n^p_{\alpha,\theta,i}$-tree $s^p_{\alpha,\theta,i}$ and a finite set $F^p_{\alpha,\theta,i}$ of D^Φ_α-names, where every $\dot{f} \in F^p_{\alpha,\theta,i}$ is the canonical D^Φ_α-name of some nice D^Φ_α-name for a real below some $q \in D^\Phi_\alpha$ with $p \restriction \alpha \leq q$,

- ◦ such that $p(\alpha)$ is the canonical name for the condition in $\dot{\mathbb{Q}}^\Phi_\alpha = \prod_{\theta \in \mathrm{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha)$ with $\mathrm{supp}(p(\alpha)) = \Theta^p_\alpha$ and for every $\theta \in \Theta^p_\alpha$ with $\mathrm{supp}(p(\alpha)(\theta)) = I^p_{\alpha,\theta}$ and for every $i \in I^p_{\alpha,\theta}$ we have $p(\alpha)(\theta)(i) = (s^p_{\alpha,\theta,i}, F^p_{\alpha,\theta,i})$,
- for limit α: iff $p \restriction \beta \in D^\Phi_\beta$ for all $\beta < \alpha$.

Remark 6.5. Note that for any $p \in D^\Phi_\alpha$ the parameter c^p and for every $\beta < \alpha$ the parameters Θ^p_β, $I^p_{\beta,\theta}$, $n^p_{\beta,\theta,i}$, $s^p_{\beta,\theta,i}$ and $F^p_{\beta,\theta,i}$ are uniquely determined by p. Conversely, we may reconstruct p from these parameters. Further, by definition of $\dot{\mathbb{Q}}^\Phi_\alpha$ for every $\dot{f} \in F^p_{\alpha,\theta,i}$ as above we have that

$$p \restriction \alpha \Vdash \dot{f} \restriction n^p_{\alpha,\theta,i} \in s^p_{\alpha,\theta,i} \text{ and } \dot{f} \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Phi,\theta}_\alpha} [T].$$

Conversely, if $\dot{g}$ is the canonical D^Φ_α-name of some nice D^Φ_α-name for a real below some $q \in D^\Phi_\alpha$ with $p \restriction \alpha \leq q$ and for some $\eta \in \Theta^p_\alpha$ and $j \in I^p_{\alpha,\eta}$ we have

$$p \restriction \alpha \Vdash \dot{f} \restriction n^p_{\alpha,\eta,j} \in s^p_{\alpha,\eta,j} \text{ and } \dot{f} \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Phi,\eta}_\alpha} [T],$$

then we may extend $p \in D^\Phi_\alpha$ to a condition $r \in D^\Phi_\alpha$ by stipulating $r \restriction \alpha := p \restriction \alpha$ and

- $\Theta^r_\alpha := \Theta^p_\alpha$,
- $I^r_{\alpha,\theta} := I^p_{\alpha,\theta}$ for every $\theta \in \Theta^r_\alpha$,
- $n^r_{\alpha,\theta,i} := n^p_{\alpha,\theta,i}$, $s^r_{\alpha,\theta,i} := s^p_{\alpha,\theta,i}$ and

$$F^r_{\alpha,\theta,i} := \begin{cases} F^p_{\alpha,\theta,i} \cup \{\dot{g}\} & \text{if } (\theta, i) = (\eta, j), \\ F^p_{\alpha,\theta,i} & \text{otherwise.} \end{cases}$$

for every $\theta \in \Theta^r_\alpha$ and $i \in I^r_{\alpha,\theta}$.

Remark 6.6. For $0 < \beta \leq \alpha \leq \aleph_1$ we have $\iota^\Phi_{\beta,\alpha}(D^\Phi_\beta) \subseteq D^\Phi_\alpha$ and for limit $\alpha \leq \aleph_1$ we have

$$D^\Phi_\alpha = \bigcup_{\beta < \alpha} \iota^\Phi_{\beta,\alpha}(D^\Phi_\beta).$$

Lemma 6.7. *Let Φ be a Θ-indexing function and $0 < \alpha \leq \aleph_1$. Then, D^Φ_α is dense in $\mathbb{P}^\Phi_\alpha$.*

Proof. By induction. Case $\alpha = 1$ follows from $\mathbb{P}^\Phi_1 \cong \mathbb{C}^\Phi$. For limit α let $p \in \mathbb{P}^\Phi_\alpha$. Choose $\beta < \alpha$ and such that $\iota^\Phi_{\beta,\alpha}(p \restriction \beta) = p$. By induction choose $q \in D^\Phi_\beta$ with $q \leq p \restriction \beta$. By Remark 6.6 we have $\iota^\Phi_{\alpha,\beta}(q) \in D^\Phi_\alpha$ and $\iota^\Phi_{\beta,\alpha}(q) \leq \iota^\Phi_{\beta,\alpha}(p \restriction \beta) = p$.

Finally, for $\alpha + 1$ let $p \in \mathbb{P}^\Phi_{\alpha+1}$. Then

$$p \restriction \alpha \Vdash p(\alpha) \in \dot{\mathbb{Q}}^\Phi_\alpha = \prod_{\theta \in \mathrm{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha)$$

and by induction D^Φ_α is dense in $\mathbb{P}^\Phi_\alpha$ we may choose $q \in D^\Phi_\alpha$ which decides all necessary parameters of an element in $\prod_{\theta \in \mathrm{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha)$. By Remark 6.2 there a nice D^Φ_α-names for all D^Φ_α-names for reals below q which occur in some $F^p_{\alpha,\theta,i}$. Then, the canonical name $\dot{q}_\alpha$ for $p(\alpha)$ as defined in Definition 6.4 satisfies

$$q \Vdash p(\alpha) = \dot{q}_\alpha.$$

Hence, $q^\frown \dot{q}_\alpha \in D^\Phi_{\alpha+1}$ and $q^\frown \dot{q}_\alpha \leq p$. □

Definition 6.8. Let Φ be a Θ-indexing function, $0 < \alpha \leq \aleph_1$ and $p \in D^\Phi_\alpha$. We will inductively define countable subsets $\mathrm{hsupp}_\Theta(p) \subseteq \mathrm{dom}(\Phi)$ and $\mathrm{hsupp}(p) \subseteq \bigcup_{\theta \in \mathrm{hsupp}_\Theta(p)}(\{\theta\} \times \Phi(\theta))$ called the hereditary support of p.

Given this definition, we define for the canonical D^Φ_α-name $\dot{f}$ of a nice D^Φ_α name $\langle(\mathcal{A}_n, f_n) \mid n < \omega\rangle$ for a real below p the countable sets

$$\mathrm{hsupp}_\Theta(\dot{f}) := \bigcup_{n<\omega, a \in \mathcal{A}_n} \mathrm{hsupp}_\Theta(a),$$
$$\mathrm{hsupp}(\dot{f}) := \bigcup_{n<\omega, a \in \mathcal{A}_n} \mathrm{hsupp}(a).$$

For $\alpha = 1$, we define $\mathrm{hsupp}(p) := \mathrm{supp}(p(0))$ and let $\mathrm{hsupp}_\Theta(p)$ be the projection of $\mathrm{hsupp}(p)$ onto the first component. Next, for limit α we may choose $\beta < \alpha$ with $\iota^\Phi_{\beta,\alpha}(p \restriction \beta) = p$ and define $\mathrm{hsupp}_\Theta(p) := \mathrm{hsupp}_\Theta(p \restriction \beta)$ and $\mathrm{hsupp}(p) := \mathrm{hsupp}(p \restriction \beta)$. Finally, for $\alpha + 1 > 1$ we define

$$\mathrm{hsupp}_\Theta(p) := \mathrm{hsupp}_\Theta(p \restriction \alpha) \cup \Theta^p_\alpha \cup \bigcup\{\mathrm{hsupp}_\Theta(\dot{f}) \mid \theta \in \Theta^p_\alpha, i \in I^p_{\alpha,\theta} \text{ and } \dot{f} \in F^p_{\alpha,\theta,i}\},$$
$$\mathrm{hsupp}(p) := \mathrm{hsupp}(p \restriction \alpha) \cup \bigcup\{\mathrm{hsupp}(\dot{f}) \mid \theta \in \Theta^p_\alpha, i \in I^p_{\alpha,\theta} \text{ and } \dot{f} \in F^p_{\alpha,\theta,i}\}.$$

Lemma 6.9. *Assume* CH *and let Φ be a Θ-indexing function, $0 < \alpha \leq \aleph_1$ and assume that both $\Theta_0 \subseteq \Theta$ and $I_0 \subseteq \bigcup_{\theta \in \Theta_0}(\{\theta\} \times \Phi(\theta))$ are countable. Then, there are at most $\aleph_1$-many $p \in D^\Phi_\alpha$ with* $\mathrm{hsupp}_\Theta(p) \subseteq \Theta_0$ *and* $\mathrm{hsupp}(p) \subseteq I_0$. *Thus, for any $p \in D^\Phi_\alpha$ there are at most $\aleph_1$-many canonical D^Φ_α-names $\dot{f}$ of nice D^Φ_α-names for reals below p with* $\mathrm{hsupp}_\Theta(\dot{f}) \subseteq \Theta_0$ *and* $\mathrm{hsupp}(\dot{f}) \subseteq I_0$.

Proof. In order to see the second part of the statement, let $\dot{f}$ be the canonical D^Φ_α-name of a nice D^Φ_α-name $\langle(\mathcal{A}_n, f_n) \mid n < \omega\rangle$ for a real below $p \in D^\Phi_\alpha$ with $\mathrm{hsupp}_\Theta(\dot{f}) \subseteq \Theta_0$ and $\mathrm{hsupp}(\dot{f}) \subseteq I_0$. Then, for any $n < \omega$ and $a \in \mathcal{A}_n$ we also have $\mathrm{hsupp}(a) \subseteq I_0$ and $\mathrm{hsupp}_\Theta(a) \subseteq \Theta_0$. But by the first part of the statement

$$\big|\{a \in D^\Phi_\alpha \mid \mathrm{hsupp}(a) \subseteq I_0 \text{ and } \mathrm{hsupp}_\Theta(a) \subseteq \Theta_0\}\big| \leq \aleph_1,$$

so that Remark 6.2 using CH and the fact that $\mathbb{P}^\Phi_\alpha$ is c.c.c., we may compute the number of nice D^Φ_α-names for reals below p as at most

$$\aleph_1^{\aleph_0} = (\aleph_0^{\aleph_0})^{\aleph_0} = \aleph_0^{\aleph_0 \cdot \aleph_0} = \aleph_0^{\aleph_0} = \aleph_1.$$

We prove the first part of the statement by induction. For $\alpha = 1$, as $|\mathbb{C}| = \aleph_0$ and I_0 is countable there are at most $\aleph_0^{\aleph_0} = \aleph_1$-many conditions in $\mathbb{P}^\Phi_1 \cong \mathbb{C}^\Phi$ with $\mathrm{hsupp}(p) \subseteq I_0$. For limit α, note that by Remark 6.6 we have

$$\begin{aligned}&\{p \in D^{\Phi}_{\alpha} \mid \text{hsupp}(p) \subseteq I_0 \text{ and } \text{hsupp}_{\Theta}(p) \subseteq \Theta_0\}\\ &= \bigcup_{\beta<\alpha} \{\iota^{\Phi}_{\beta,\alpha}(p) \mid p \in D^{\Phi}_{\beta}, \text{hsupp}(p) \subseteq I_0 \text{ and } \text{hsupp}_{\Theta}(p) \subseteq \Theta_0\}\end{aligned}$$

Thus, by induction we compute

$$\left|\{p \in D^{\Phi}_{\alpha} \mid \text{hsupp}(p) \subseteq I_0 \text{ and } \text{hsupp}_{\Theta}(p) \subseteq \Theta_0\}\right| \leq |\alpha| \cdot \aleph_1 = \aleph_1.$$

Finally, for $\alpha+1>1$ and $p \in D^{\Phi}_{\alpha+1}$ we have $p \restriction \alpha \in D^{\Phi}_{\alpha}$ and $\text{hsupp}_{\Theta}(p \restriction \alpha) \subseteq \text{hsupp}_{\Theta}(p) \subseteq \Theta_0$ and $\text{hsupp}(p \restriction \alpha) \subseteq \text{hsupp}(p) \subseteq I_0$, so by induction there are at most $\aleph_1$-many choices for $p \restriction \alpha$. Also, $\Theta^{p}_{\alpha} \subseteq \text{hsupp}_{\Theta}(p) \subseteq \Theta_0$, so there are at most countably many choices for Θ^{p}_{α}. Further, for any of the finitely many $\theta \in \Theta^{p}_{\alpha}$ there are at most countably many choices $I^{p}_{\alpha,\theta}$ and for any of the finitely many $i \in I^{p}_{\alpha,\theta}$ there are at most countably many choices for $n^{p}_{\alpha,\theta,i}$ and $s^{p}_{\alpha,\theta,i}$. Finally, for any $\dot{f} \in F^{p}_{\alpha,\theta,i}$ choose $q \in D^{\Phi}_{\alpha}$ such that $\dot{f}$ is the canonical D^{Φ}_{α}-name of some nice D^{Φ}_{α}-name for a real below q with $p \restriction \alpha \leq q$. Then, we have $\text{hsupp}_{\Theta}(q) \subseteq \text{hsupp}_{\Theta}(p \restriction \alpha) \subseteq \text{hsupp}_{\Theta}(p) \subseteq \Theta_0$ and $\text{hsupp}(q) \subseteq \text{hsupp}(p \restriction \alpha) \subseteq \text{hsupp}(p) \subseteq I_0$, so by induction assumption there at most $\aleph_1$-many choices for q. Analogously, $\text{hsupp}_{\Theta}(\dot{f}) \subseteq \text{hsupp}_{\Theta}(p) \subseteq \Theta_0$ and $\text{hsupp}(\dot{f}) \subseteq \text{hsupp}(p) \subseteq I_0$, so by induction assumption there are at most $\aleph_1$-many choices for $\dot{f}$. Hence, there are at most $\aleph_1$-many choices for $F^{p}_{\alpha,\theta,i}$. By Remark 6.5 $p(\alpha)$ is uniquely determined by these parameters, so that there are at most $\aleph_1$-many choices for p. □

Next, we prove that the action of Γ on $\mathbb{P}^{\Phi}_{\alpha}$ restricts to actions on our nice dense set D^{Φ}_{α}.

Lemma 6.10. *Let Φ be an Θ-indexing function, $\theta \in \text{dom}(\Phi)$, $i \in \Phi(\theta)$, $\gamma \in \Gamma$ and $0 < \alpha \leq \aleph_1$. Then, $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(D^{\Phi}_{\alpha}) = D^{\Phi}_{\alpha}$.*

Proof. It suffices to verify that $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(D^{\Phi}_{\alpha}) \subseteq D^{\Phi}_{\alpha}$, which we prove by induction. For $\alpha = 1$, let $p \in D^{\Phi}_{1}$. Then, we compute

$$\pi^{\Phi,\theta}_{1,i}(\gamma)(p)(0) = \pi^{\Phi,\theta}_{1,i}(\gamma)(p(0)) = \pi^{\Phi,\theta}_{1,i}(\gamma)(c^{p}) \in \mathbb{C}^{\Phi},$$

so that $\pi^{\Phi,\theta}_{1,i}(\gamma)(p) \in D^{\Phi}_{1}$. For limit α, let $p \in D^{\Phi}_{\alpha}$ and choose $\beta < \alpha$ such that $\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta) = p$. By induction assumption $\pi^{\Phi,\theta}_{\beta,i}(\gamma)(p \restriction \beta) \in D^{\Phi}_{\beta}$. By Remark 6.6 we have $\iota^{\Phi}_{\beta,\alpha}(\pi^{\Phi,\theta}_{\beta,i}(\gamma)(p \restriction \beta)) \in D^{\Phi}_{\alpha}$. Hence, by Definition 5.1 we compute

$$\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p) = \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta)) = \iota^{\Phi}_{\beta,\alpha}(\pi^{\Phi,\theta}_{\beta,i}(\gamma)(p \restriction \beta)) \in D^{\Phi}_{\alpha}.$$

Finally, for $\alpha+1>1$ let $p \in D^{\Phi}_{\alpha+1}$. Then, $p \restriction \alpha \in D^{\Phi}_{\alpha}$ and by Definition 5.3

$$\pi^{\Phi,\theta}_{\alpha+1,i}(\gamma)(p) = \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p \restriction \alpha)^\frown \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p(\alpha)).$$

By induction assumption we obtain $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p \restriction \alpha) \in D^{\Phi}_{\alpha}$. By Remark 6.3 $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p(\alpha))$ is the canonical name for the condition in $\dot{\mathbb{Q}}^{\Phi}_{\alpha} = \prod_{\theta \in \text{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha})$ with $\text{supp}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p(\alpha))) = \Theta^{p}_{\alpha}$ and for every $\theta \in \Theta^{p}_{\alpha}$ with $\text{supp}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p(\alpha))(\theta)) = I^{p}_{\alpha,\theta}$ and for every $i \in I^{p}_{\alpha,\theta}$ we have $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p(\alpha))(\theta)(i) = (s^{p}_{\alpha,\theta,i}, \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(F^{p}_{\alpha,\theta,i}))$. Hence, $\pi^{\Phi,\theta}_{\alpha+1,i}(\gamma)(p) \in D^{\Phi}_{\alpha+1}$. □

Lemma 6.11. *Let Φ be an Θ-indexing function, $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$, $\gamma \in \Gamma$ and $p \in D^{\Phi}_{\alpha}$ for some $0 < \alpha \leq \aleph_1$ such that $\pi^{\Phi,\theta}_{1,i}(\gamma)(p \restriction 1) = p \restriction 1$. Then, there is $q \leq p$ in D^{Φ}_{α} with $q(0) = p(0)$ and $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(q) = q$.*

Proof. By induction. The case $\alpha = 1$ is exactly the assumption given on γ. For limit α choose $\beta < \alpha$ with $\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta) = p$. By induction assumption choose $q \leq p \restriction \beta$ in D^{Φ}_{β} such that $q(0) = p(0)$ and $\pi^{\Phi,\theta}_{\beta,i}(\gamma)(q) = q$. Then, we have $\iota^{\Phi}_{\beta,\alpha}(q)(0) = q(0) = p(0)$, by Remark 6.6 $\iota^{\Phi}_{\beta,\alpha}(q) \in D^{\Phi}_{\alpha}$ and by Definition 5.1 we compute

$$\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi}_{\beta,\alpha}(q)) = \iota^{\Phi}_{\beta,\alpha}(\pi^{\Phi,\theta}_{\beta,i}(\gamma)(q)) = \iota^{\Phi}_{\beta,\alpha}(q).$$

Finally, for $\alpha + 1 > 1$ let $p \in D^{\Phi}_{\alpha+1}$. Then, by induction assumption we may choose $q \leq p \restriction \alpha$ in D^{Φ}_{α} with $q(0) = p(0)$ and $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(q) = q$. We define

- $\Theta^{q}_{\alpha} := \Theta^{p}_{\alpha}$,
- $I^{q}_{\alpha,\theta} := I^{p}_{\alpha,\theta}$ for every $\theta \in \Theta^{q}_{\alpha}$,
- $n^{q}_{\alpha,\theta,i} := n^{p}_{\alpha,\theta,i}$ and $s^{q}_{\alpha,\theta,i} := s^{p}_{\alpha,\theta,i}$ for every $\theta \in \Theta^{q}_{\alpha}$ and $i \in I^{q}_{\alpha,\theta}$,
- $F^{q}_{\alpha,\theta,i} := F^{p}_{\alpha,\theta,i} \cup \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(F^{p}_{\alpha,\theta,i})$.

Let $\dot{q}_{\alpha}$ be the canonical name for the condition in $\dot{\mathbb{Q}}^{\Phi}_{\alpha} = \prod_{\theta \in \mathrm{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha})$ with $\mathrm{supp}(\dot{q}_{\alpha}) = \Theta^{q}_{\alpha}$, for every $\theta \in \Theta^{q}_{\alpha}$ with $\mathrm{supp}(\dot{q}_{\alpha}(\theta)) = I^{q}_{\alpha,\theta}$ and for every $i \in I^{q}_{\alpha,\theta}$ we have $\dot{q}_{\alpha}(\theta)(i) = (s^{q}_{\alpha,\theta,i}, F^{q}_{\alpha,\theta,i})$. We claim that $q ^\frown \dot{q}_{\alpha}$ is as desired. To obtain $q ^\frown \dot{q}_{\alpha} \in D^{\Phi}_{\alpha+1}$ by Remark 6.5 it suffices to verify that for every $\theta \in \Theta^{p}_{\alpha}$, $i \in I^{p}_{\alpha,\theta}$ and $\dot{f} \in F^{p}_{\alpha,\theta,i}$ we have

$$q \Vdash \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\dot{f}) \restriction n^{p}_{\alpha,\theta,i} \in s^{p}_{\alpha,\theta,i} \text{ and } \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\dot{f}) \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}} [T].$$

To this end, notice that $p \in D^{\Phi}_{\alpha+1}$ implies

$$p \restriction \alpha \Vdash \dot{f} \restriction n^{p}_{\alpha,\theta,i} \in s^{p}_{\alpha,\theta,i} \text{ and } \dot{f} \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}} [T],$$

so also $q \leq p \restriction \alpha$ forces this. Further, $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(q) = q$ and $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}) = \dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}$, so applying the automorphism theorem to the previous statement yields the desired conclusion. Next, we have $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(F^{q}_{\alpha,\theta,i}) = F^{q}_{\alpha,\theta,i}$ since $\pi^{\Phi,\theta}_{\alpha,i}(\gamma)$ is an involution. This implies

$$\pi^{\Phi,\theta}_{\alpha+1,i}(\gamma)(q ^\frown \dot{q}_{\alpha}) = \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(q) ^\frown \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\dot{q}_{\alpha}) = q ^\frown \dot{q}_{\alpha}.$$

Finally, by definition we have $q ^\frown \dot{q}_{\alpha} \leq p$ and $(q ^\frown \dot{q}_{\alpha})(0) = q(0) = p(0)$. □

7. Complete embeddings

In this section, we combine the results of the previous sections in order to prove that our forcing in Definition 4.10 has enough complete subforcings to carry out our isomorphism-of-names argument for Main Theorem 3.1. The whole section will be devoted towards the proof of the following Theorem 7.1 as it is an elaborate inductive construction of complete embeddings.

Theorem 7.1. *Let $\Phi \subseteq \Psi$ be a Θ-subindexing function and assume Φ is countable. Then, $\mathbb{P}^\Phi_\alpha \lessdot \mathbb{P}^\Psi_\alpha$ for all $\alpha \leq \aleph_1$.*

By induction over $\alpha \leq \aleph_1$ we define embeddings $\iota^{\Phi,\Psi}_\alpha : \mathbb{P}^\Phi_\alpha \to \mathbb{P}^\Psi_\alpha$ and prove that they admit reductions from $\mathbb{P}^\Psi_\alpha$ to $\mathbb{P}^\Phi_\alpha$. Thus, $\iota^{\Phi,\Psi}_\alpha$ will be a complete embedding. Additionally, we will verify the following properties along our iteration:

(A) For all $\beta \leq \alpha$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^\Phi_\beta & \xrightarrow{\iota^{\Phi,\Psi}_\beta} & \mathbb{P}^\Psi_\beta \\ \downarrow{\scriptstyle \iota^\Phi_{\beta,\alpha}} & & \downarrow{\scriptstyle \iota^\Psi_{\beta,\alpha}} \\ \mathbb{P}^\Phi_\alpha & \xrightarrow{\iota^{\Phi,\Psi}_\alpha} & \mathbb{P}^\Psi_\alpha \end{array}$$

(B) For all $\theta \in \mathrm{dom}(\Phi)$ and $i \in \Phi(\theta)$ the embedding $\iota^{\Phi,\Psi}_\alpha : \mathbb{P}^\Phi_\alpha \to \mathbb{P}^\Psi_\alpha$ is a morphism of Γ-sets, i.e. the following diagram commutes for every $\gamma \in \Gamma$:

$$\begin{array}{ccc} \mathbb{P}^\Phi_\alpha & \xrightarrow{\iota^{\Phi,\Psi}_\alpha} & \mathbb{P}^\Psi_\alpha \\ \downarrow{\scriptstyle \pi^{\Phi,\theta}_{\alpha,i}(\gamma)} & & \downarrow{\scriptstyle \pi^{\Psi,\theta}_{\alpha,i}(\gamma)} \\ \mathbb{P}^\Phi_\alpha & \xrightarrow{\iota^{\Phi,\Psi}_\alpha} & \mathbb{P}^\Psi_\alpha \end{array}$$

(C) For all $\theta \in \mathrm{dom}(\Phi)$ and $i \in \Phi(\theta)$ we have

$$\iota^{\Phi,\Psi}_1(\dot{c}^{\Phi,\theta}_i) = \dot{c}^{\Psi,\theta}_i \text{ and thus } \iota^{\Phi,\Psi}_1(\dot{T}^{\Phi,\theta}_i) = \dot{T}^{\Psi,\theta}_i.$$

(D) For all $\alpha + 1 > 1$, $\theta \in \mathrm{dom}(\Phi)$ and $n < \omega$ we have

$$\iota^{\Phi,\Psi}_{\alpha+1}(\dot{T}^{\Phi,\theta}_{\alpha,n}) = \dot{T}^{\Psi,\theta}_{\alpha,n}.$$

(E) If $\alpha > 0$, then for all $\theta \in \mathrm{dom}(\Phi)$, the name $\dot{\mathcal{T}}^{\Psi,\theta}_\alpha$ is the canonical $\mathbb{P}^\Psi_\alpha$-name for

$$\iota^{\Phi,\Psi}_\alpha(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha) \cup \bigcup_{i \in \Psi(\theta) \setminus \Phi(\theta)} \iota^\Psi_{1,\alpha}(\mathcal{T}^{\Psi,\theta}_{1,i}).$$

(F) For all $\theta \in \mathrm{dom}(\Phi)$, $i \in \Psi(\theta) \setminus \Phi(\theta)$, $\gamma \in \Gamma$ we have that $\pi^{\Psi,\theta}_{\alpha,i}(\gamma)$ acts trivially on $\iota^{\Phi,\Psi}_\alpha(\mathbb{P}^\Phi_\alpha)$.

(G) For all $\theta \in \mathrm{dom}(\Phi)$, $i \in \Psi(\theta) \setminus \Phi(\theta)$, $\gamma \in \Gamma$ and $\mathbb{P}^\Phi_\alpha$-name $\dot{f}$ for a real

$$\mathbb{P}^\Psi_\alpha \Vdash \iota^{\Phi,\Psi}_\alpha(\dot{f}) \neq \iota^\Psi_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i)).$$

First, we prove that (G) follows from (F), so we only need to verify (A) to (F) inductively:

Proof. Let $p \in \mathbb{P}^\Psi_\alpha$. By Lemma 6.7, we may assume $p \in D^\Psi_\alpha$. Choose $N \notin \mathrm{dom}(p(0)(\theta, i))$. Let $\delta \in \Gamma$ be defined by $\delta(N) = 1$ and 0 otherwise. Then, $\pi^{\Psi,\theta}_{1,i}(\delta)(p \restriction 1) = p \restriction 1$, so by Lemma 6.11 we may choose $q \leq p$ in D^Ψ_α such that $q(0) = p(0)$ and $\pi^{\Psi,\theta}_{\alpha,i}(\delta)(q) = q$. Thus, $N \notin \mathrm{dom}(q(0)(\theta, i))$ and we may define $q_j \leq q$ which replaces $p(0)(\theta, i)$ by $p(0)(\theta, i) \cup \langle N, j\rangle$ for $j \in 2$. Then, we have $\pi^{\Psi,\theta}_{\alpha,i}(\delta)(q_j) = q_{1-j}$ for $j \in 2$ and there is a $k \in 2$ with

$$q_0 \Vdash \iota^\Psi_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i))(N) = k.$$

Further, using $(\delta+\gamma)(N) = 1-\gamma(N)$ we compute

$$\begin{aligned}\pi^{\Psi,\theta}_{\alpha,i}(\delta)(\iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i)))(N) &= \iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\delta)(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i)))(N)\\ &= \iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\delta+\gamma)(\dot{c}^{\Psi,\theta}_i))(N)\\ &= 1-\iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i))(N).\end{aligned}$$

Thus, by the automorphism theorem we obtain

$$q_1 \Vdash \iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i))(N) = 1-k.$$

Choose $r_0 \le q_0$ such that $r_0 \Vdash \iota^{\Phi,\Psi}_{\alpha}(\dot{f})(N) = l$ for some $l \in 2$. Since $\dot{f}$ is a $\mathbb{P}^{\Phi}_{\alpha}$-name by (F) we have $\pi^{\Psi,\theta}_{\alpha,i}(\delta)(\iota^{\Phi,\Psi}_{\alpha}(\dot{f})) = \iota^{\Phi,\Psi}_{\alpha}(\dot{f})$. Thus, the automorphism theorem yields

$$\pi^{\Psi,\theta}_{\alpha,i}(\delta)(r_0) \Vdash \iota^{\Phi,\Psi}_{\alpha}(\dot{f})(N) = l.$$

But then either $r_0 \le q_0 \le q \le p$ and

$$r_0 \Vdash \iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i))(N) = k \neq l = \iota^{\Phi,\Psi}_{\alpha}(\dot{f})(N),$$

or $\pi^{\Psi,\theta}_{\alpha,i}(\delta)(r_0) \le \pi^{\Psi,\theta}_{\alpha,i}(\delta)(q_0) = q_1 \le q \le p$ and

$$\pi^{\Psi,\theta}_{\alpha,i}(\delta)(r_0) \Vdash \iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_i))(N) = 1-k \neq l = \iota^{\Phi,\Psi}_{\alpha}(\dot{f})(N). \qquad \square$$

Next, we inductively define $\iota^{\Phi,\Psi}_{\alpha}$ and verify properties (A) to (F), so consider $\alpha = 1$ first. In this case, we already defined $\iota^{\Phi,\Psi}_{i} : \mathbb{P}^{\Phi}_1 \to \mathbb{P}^{\Psi}_1$ as the complete embedding corresponding to $\iota^{\Phi,\Psi} : \mathbb{C}^{\Phi} \to \mathbb{C}^{\Psi}$.

(A) There is nothing to show.
(B) Follows immediately from Remark 4.7.
(C) By definition of $\dot{c}^{\Phi,\theta}_i$, $\dot{c}^{\Psi,\theta}_i$ and $\iota^{\Phi,\Psi}_1$.
(D) There is nothing to show.
(E) Let $\theta \in \mathrm{dom}(\Phi)$. Then, we compute

$$\begin{aligned}\dot{\mathcal{T}}^{\Psi,\theta}_{1,i} &= \bigcup_{i\in\Psi(\theta)} \dot{\mathcal{T}}^{\Psi,\theta}_{1,i} && \text{(Remark 4.9)}\\ &= \bigcup_{i\in\Phi(\theta)} \dot{\mathcal{T}}^{\Psi,\theta}_{1,i} \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} \dot{\mathcal{T}}^{\Psi,\theta}_{1,i}\\ &= \bigcup_{i\in\Phi(\theta)} \iota^{\Phi,\Psi}_1(\dot{\mathcal{T}}^{\Phi,\theta}_{1,i}) \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} \dot{\mathcal{T}}^{\Psi,\theta}_{1,i} && \text{(Remark 4.9)}\\ &= \iota^{\Phi,\Psi}_1\Big(\bigcup_{i\in\Phi(\theta)} \dot{\mathcal{T}}^{\Phi,\theta}_{1,i}\Big) \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} \dot{\mathcal{T}}^{\Psi,\theta}_{1,i} && \text{(canonical name)}\\ &= \iota^{\Phi,\Psi}_1(\dot{\mathcal{T}}^{\Phi,\theta}_{1,i}) \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} \dot{\mathcal{T}}^{\Psi,\theta}_{1,i} && \text{(Remark 4.9).}\end{aligned}$$

(F) Follows immediately from the fact that $\pi^{\Psi,\theta}_{\alpha,i}(\gamma)$ only acts on Cohen information outside of the indexing of Φ.

Next, we consider limit α. Then, by (A) for every $\beta' \leq \beta < \alpha$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{P}^{\Phi}_{\beta'} & \xrightarrow{\iota^{\Phi,\Psi}_{\beta'}} & \mathbb{P}^{\Psi}_{\beta'} \\ \downarrow{\scriptstyle \iota^{\Phi}_{\beta',\beta}} & & \downarrow{\scriptstyle \iota^{\Psi}_{\beta',\beta}} \\ \mathbb{P}^{\Phi}_{\beta} & \xrightarrow{\iota^{\Phi,\Psi}_{\beta}} & \mathbb{P}^{\Psi}_{\beta} \end{array}$$

By the universal property of the direct limit there is a unique map $\iota^{\Phi,\Psi}_{\alpha} : \mathbb{P}^{\Phi}_{\alpha} \to \mathbb{P}^{\Psi}_{\alpha}$ such that for every $\beta \leq \alpha$ the diagram in (A) commutes. Further, as a direct limit of complete embeddings, also $\iota^{\Phi,\Psi}_{\alpha}$ is a complete embedding. Note that (C) and (D) are vacuous at limits.

(A) Follows from the universal property of the direct limit.

(B) Let $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$, $\gamma \in \Gamma$ and $p \in \mathbb{P}^{\Phi}_{\alpha}$. Choose $\beta < \alpha$ such that $\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta) = p$. Then, we compute

$$\begin{aligned} \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p)) &= \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta))) && \text{(choice of } \beta) \\ &= \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Psi}_{\beta,\alpha}(\iota^{\Phi,\Psi}_{\beta}(p \restriction \beta))) && \text{(A)} \\ &= \iota^{\Psi}_{\beta,\alpha}(\pi^{\Psi,\theta}_{\beta,i}(\gamma)(\iota^{\Phi,\Psi}_{\beta}(p \restriction \beta))) && \text{(Definition 5.1)} \\ &= \iota^{\Psi}_{\beta,\alpha}(\iota^{\Phi,\Psi}_{\beta}(\pi^{\Phi,\theta}_{\beta,i}(\gamma)(p \restriction \beta))) && \text{((B) inductively)} \\ &= \iota^{\Phi,\Psi}_{\alpha}(\iota^{\Phi}_{\beta,\alpha}(\pi^{\Phi,\theta}_{\beta,i}(\gamma)(p \restriction \beta))) && \text{(A)} \\ &= \iota^{\Phi,\Psi}_{\alpha}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta))) && \text{(Definition 5.1)} \\ &= \iota^{\Phi,\Psi}_{\alpha}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p)) && \text{(choice of } \beta). \end{aligned}$$

(E) Let $\theta \in \mathrm{dom}(\Phi)$. Then, we compute

$$\begin{aligned} \dot{\mathcal{T}}^{\Psi,\theta}_{\alpha} &= \bigcup_{\beta<\alpha} \iota^{\Psi}_{\beta,\alpha}(\dot{\mathcal{T}}^{\Psi,\theta}_{\beta}) && \text{(Definition 4.10)} \\ &= \bigcup_{\beta<\alpha} \iota^{\Psi}_{\beta,\alpha}\left[\iota^{\Phi,\Psi}_{\beta}(\dot{\mathcal{T}}^{\Phi,\theta}_{\beta}) \cup \bigcup_{i \in \Psi(\theta)\setminus\Phi(\theta)} \iota^{\Psi}_{1,\beta}(\mathcal{T}^{\Psi,\theta}_{1,i})\right] && \text{((E) inductively)} \\ &= \bigcup_{\beta<\alpha}\left[\iota^{\Psi}_{\beta,\alpha}(\iota^{\Phi,\Psi}_{\beta}(\dot{\mathcal{T}}^{\Phi,\theta}_{\beta})) \cup \bigcup_{i \in \Psi(\theta)\setminus\Phi(\theta)} \iota^{\Psi}_{\beta,\alpha}(\iota^{\Psi}_{1,\beta}(\mathcal{T}^{\Psi,\theta}_{1,i}))\right] && \text{(canonical name)} \\ &= \bigcup_{\beta<\alpha}\left[\iota^{\Phi,\Psi}_{\alpha}(\iota^{\Phi}_{\beta,\alpha}(\dot{\mathcal{T}}^{\Phi,\theta}_{\beta})) \cup \bigcup_{i \in \Psi(\theta)\setminus\Phi(\theta)} \iota^{\Psi}_{1,\alpha}(\mathcal{T}^{\Psi,\theta}_{1,i})\right] && \text{(A)} \\ &= \iota^{\Phi,\Psi}_{\alpha}\left[\bigcup_{\beta<\alpha} \iota^{\Phi}_{\beta,\alpha}(\dot{\mathcal{T}}^{\Phi,\theta}_{\beta})\right] \cup \bigcup_{i \in \Psi(\theta)\setminus\Phi(\theta)} \iota^{\Psi}_{1,\alpha}(\mathcal{T}^{\Psi,\theta}_{1,i}) && \text{(canonical name)} \\ &= \iota^{\Phi,\Psi}_{\alpha}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}) \cup \bigcup_{i \in \Psi(\theta)\setminus\Phi(\theta)} \iota^{\Psi}_{1,\alpha}(\mathcal{T}^{\Psi,\theta}_{1,i}) && \text{(Definition 4.10).} \end{aligned}$$

(F) Let $\theta \in \mathrm{dom}(\Phi)$, $i \in \Psi(\theta) \setminus \Phi(\theta)$, $\gamma \in \Gamma$ and $p \in \mathbb{P}^{\Phi}_{\alpha}$. Choose $\beta < \alpha$ with $\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta) = p$. Then, we compute

$$\begin{aligned}
\pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p)) &= \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta))) && \text{(choice of } \beta) \\
&= \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Psi}_{\beta,\alpha}(\iota^{\Phi,\Psi}_{\beta}(p \restriction \beta))) && \text{(A)} \\
&= \iota^{\Psi}_{\beta,\alpha}(\pi^{\Psi,\theta}_{\beta,i}(\gamma)(\iota^{\Phi,\Psi}_{\beta}(p \restriction \beta))) && \text{(Definition 5.1)} \\
&= \iota^{\Psi}_{\beta,\alpha}(\iota^{\Phi,\Psi}_{\beta}(p \restriction \beta)) && \text{((F) inductively)} \\
&= \iota^{\Phi,\Psi}_{\alpha}(\iota^{\Phi}_{\beta,\alpha}(p \restriction \beta)) && \text{(A)} \\
&= \iota^{\Phi,\Psi}_{\alpha}(p) && \text{(choice of } \beta).
\end{aligned}$$

Finally, consider $\alpha + 1 > 1$. By induction we have that $\iota^{\Phi,\Psi}_{\alpha} : \mathbb{P}^{\Phi}_{\alpha} \to \mathbb{P}^{\Psi}_{\alpha}$ is a complete embedding. Thus, we may naturally define for $p \in \mathbb{P}^{\Phi}_{\alpha+1}$

$$\iota^{\Phi,\Psi}_{\alpha+1}(p) := \iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) {}^\frown \iota^{\Phi,\Psi}_{\alpha}(p(\alpha)).$$

However, we need to verify that

$$\iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) \Vdash \iota^{\Phi,\Psi}_{\alpha}(p(\alpha)) \in \dot{\mathbb{Q}}^{\Psi}_{\alpha}.$$

Since $\Phi \subseteq \Psi$, by definition of $\dot{\mathbb{Q}}^{\Psi}_{\alpha}$ it suffices to prove that if $\theta \in \mathrm{dom}(\Phi)$ and $\dot{f}$ is a $\mathbb{P}^{\Phi}_{\alpha}$-name with

$$p \restriction \alpha \Vdash \dot{f} \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}} [T],$$

then also

$$\iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) \Vdash \iota^{\Phi,\Psi}_{\alpha}(\dot{f}) \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Psi,\theta}_{\alpha}} [T].$$

By induction assumption of (E) we may distinguish the following three different types of trees in $\dot{\mathcal{T}}^{\Psi,\theta}_{\alpha}$. First, let $i \in \Phi(\theta)$ and $\gamma \in \Gamma$. By assumption on $\dot{f}$ we have

$$p \restriction \alpha \Vdash \dot{f} \neq \iota^{\Phi}_{1,\alpha}(\pi^{\Phi,\theta}_{1,i}(\gamma)(\dot{c}^{\Phi,\theta}_{i})),$$

so that

$$\iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) \Vdash \iota^{\Phi,\Psi}_{\alpha}(\dot{f}) \neq \iota^{\Phi,\Psi}_{\alpha}(\iota^{\Phi}_{1,\alpha}(\pi^{\Phi,\theta}_{1,i}(\gamma)(\dot{c}^{\Phi,\theta}_{i}))).$$

Secondly, let $\beta < \alpha$ and $n < \omega$. By assumption on $\dot{f}$ we have

$$p \restriction \alpha \Vdash \dot{f} \notin [\iota^{\Phi}_{\beta,\alpha}(\dot{T}^{\Phi,\theta}_{\beta,n})].$$

Thus, by induction assumption of (A) and (C) we get

$$\iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) \Vdash \iota^{\Phi,\Psi}_{\alpha}(\dot{f}) \notin [\iota^{\Phi,\Psi}_{\alpha}(\iota^{\Phi}_{\beta,\alpha}(\dot{T}^{\Phi,\theta}_{\beta,n}))] = [\iota^{\Psi}_{\beta,\alpha}(\iota^{\Phi,\Psi}_{\beta}(\dot{T}^{\Phi,\theta}_{\beta,n}))] = [\iota^{\Psi}_{\beta,\alpha}(\dot{T}^{\Psi,\theta}_{\beta,n})].$$

Finally, for $i \in \Psi(\theta) \setminus \Phi(\theta)$ by induction assumption of (G) we get

$$\iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) \Vdash \iota^{\Phi,\Psi}_{\alpha}(\dot{f}) \neq \iota^{\Psi}_{1,\alpha}(\pi^{\Psi,\theta}_{1,i}(\gamma)(\dot{c}^{\Psi,\theta}_{i})).$$

Next, given $p \in \mathbb{P}^{\Psi}_{\alpha+1}$ we have to find a reduction $q \in \mathbb{P}^{\Phi}_{\alpha+1}$ with respect to the embedding $\iota^{\Phi,\Psi}_{\alpha+1}$. By Lemma 6.7 we may assume $p \in D^{\Psi}_{\alpha+1}$. By induction, pick a reduction $q \in \mathbb{P}^{\Phi}_{\alpha}$ of $p \restriction \alpha \in D^{\Psi}_{\alpha}$ with respect to $\iota^{\Phi,\Psi}_{\alpha}$. Remember, that for every $\theta \in \Theta^p_\alpha$, $i \in I^p_{\alpha,\theta}$ and $\dot{f} \in F^p_{\alpha,\theta,i}$ we have

$$p \restriction \alpha \Vdash \dot{f} \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Psi,\theta}_{\alpha}} [T].$$

Thus, we will need to find a reduction $\dot{g}$ of $\dot{f}$ which satisfies

$$q \Vdash \dot{g} \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}} [T].$$

Note that the standard canonical projection of a real (cf. [7]) need not satisfy this requirement. Thus, we introduce the following technical notions. For technical reasons, we need to enumerate the finite set $\bigcup \{\{\theta\} \times \{i\} \times F^p_{\alpha,\theta,i} \mid \theta \in \Theta^p_\alpha, i \in I^p_{\alpha,\theta}\}$ by $\langle (\theta_k, i_k, \dot{f}_k) \mid k \in K \rangle$. In particular, we have $\theta_\bullet : K \to \Theta^p_\alpha$. For every $\theta \in \Theta^p_\alpha$ by assumption on Φ the family $\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}$ is countable, so we may enumerate it as $\langle \dot{S}^\theta_n \mid n < \omega \rangle$. Next, we will need the following refinement of the definition of a nice name for a real below p in Definition 6.1.

Definition 7.2. Let $\mathbb{P}$ be a forcing, $p \in \mathbb{P}$ and K a finite set. A nice $\mathbb{P}$-name for K-many reals below p is a sequence $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ such that

- for all $n < \omega$ the set $\mathcal{A}_n$ is a maximal antichain below p and $K_n : K \times \mathcal{A}_n \to {}^{>n}2$,
- for all $n < m$ the set $\mathcal{A}_m$ refines $\mathcal{A}_n$, i.e. for every $b \in \mathcal{A}_m$ there is $a \in \mathcal{A}_n$ with $b \leq a$,
- for all $n < m$, $k \in K$, $a \in \mathcal{A}_n$ and $b \in \mathcal{A}_m$ with $b \leq a$ we have $K_n(k,a) \trianglelefteq K_m(k,b)$.

Further, we write name$(\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle)$ for the canonical $\mathbb{P}$-name of $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$, i.e.

$$\mathrm{name}(\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle) := \{(a, ((k,n), K_n(k,a)(n))) \mid n < \omega \text{ and } a \in \mathcal{A}_n\} \in {}^{K \times \omega}2.$$

Remark 7.3. Notice that if $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ is a nice $\mathbb{P}$-name for K-many reals below p, then for every $k \in K$ the sequence $\langle (\mathcal{A}_n, K_n(k)) \mid n < \omega \rangle$ is a nice $\mathbb{P}$-name for a real below p with

$$\mathrm{name}(\langle (\mathcal{A}_n, K_n(k)) \mid n < \omega \rangle) = \mathrm{name}(\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle) \restriction (\{k\} \times \omega).$$

However, $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ is more than just the product of K-many nice $\mathbb{P}$-names for reals below p as all antichains have to coincide.

With respect to the fixed $p \restriction \alpha \in D^{\Psi}_{\alpha}$, $\theta_\bullet : K \to \Theta^p_\alpha$ and sequences $\langle \langle \dot{S}^\theta_n \mid n < \omega \rangle \mid \theta \in \Theta^p_\alpha \rangle$ above, we define the following notion:

Definition 7.4. Let $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ be a nice $\mathbb{P}^{\Psi}_{\alpha}$-name for K-many reals below $p \restriction \alpha$. Then, we say $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ is a nice $\mathbb{P}$-name for K-many reals below $p \restriction \alpha$ with respect to $\theta_\bullet$ and $\langle \langle \dot{S}^\theta_n \mid n < \omega \rangle \mid \theta \in \Theta^p_\alpha \rangle$ iff for all $n < \omega$, $k \in K$ and $a \in \mathcal{A}_n$ we have

$$a \Vdash K_n(k,a) \notin \iota^{\Phi,\Psi}_{\alpha}(\dot{S}^{\theta_k}_n).$$

First, we argue that there is such a nice $\mathbb{P}^{\Psi}_{\alpha}$-name $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ of K-many reals below $p \restriction \alpha$ with respect to $\theta_\bullet$ and $\langle \langle \dot{S}^\Phi_n \mid n < \omega \rangle \mid \theta \in \Theta^p_\alpha \rangle$, which also satisfies for every $k \in K$ that

$$p \restriction \alpha \Vdash \dot{f}_k = \mathrm{name}(\langle (\mathcal{A}_n, K_n(k)) \mid n < \omega \rangle).$$

Proof. We construct the nice name by induction on n. Set $\mathcal{A}_{-1} := \{p \restriction \alpha\}$. Now, assume $\mathcal{A}_n$ is defined. For every $a \in \mathcal{A}_n$ choose a maximal antichain $\mathcal{B}(a)$ below a such that for every $b \in \mathcal{B}(a)$ and $k \in K$ there is $K_n(k,a) \in 2^{>n}$ with $K_{n-1}(k,b) \trianglelefteq K_n(k,b)$ if $n \neq -1$ and such that

$$b \Vdash K_n(k,b) \trianglelefteq \dot{f}_k \text{ and } K_n(k,b) \notin \iota_\alpha^{\Phi,\Psi}(\dot{S}_n^{\theta_k}).$$

This is possible as $b \leq a$, K is finite and by assumption on $\dot{f}_k$ we have for every $k \in K$

$$p \restriction \alpha \Vdash \dot{f}_k \notin [\iota_\alpha^{\Phi,\Psi}(\dot{S}_n^{\theta_k})].$$

Finally, set $\mathcal{A}_{n+1} := \bigcup_{a \in \mathcal{A}_n} \mathcal{B}(a)$. Clearly, $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ then has the desired properties. □

The point of this definition, is that for every $k \in K$ the n-th antichain $\mathcal{A}_n$ already witnesses that $\dot{f}_k \notin [\iota_\alpha^{\Phi,\Psi}(\dot{S}_n^{\theta_k})]$. In Lemma 3.8 in [7] the existence of a reduction of a nice name for a real is proven. We will need an analogous result for nice names of K-many reals:

Lemma 7.5. *Let $\mathbb{Q}$ be a complete suborder of $\mathbb{P}$, $p \in \mathbb{P}$, $q \in \mathbb{Q}$ a reduction of p and assume that $\{(\mathcal{A}_n, K_n) \mid n < \omega\}$ a nice $\mathbb{P}$-name for K-many reals below p. Then, there is a nice $\mathbb{Q}$-name $\{(\mathcal{B}_n, L_n) \mid n < \omega\}$ for K-many reals below q such that for all $n < \omega$ and $b \in \mathcal{B}_n$ there is an $a \in \mathcal{A}_n$ such that b is a reduction of a and $K_n(k,a) = L_n(k,b)$ for all $k \in K$.*

Proof. Exactly the same proof as for Lemma 3.8 in [7]. □

Analogously to [7], we will call the nice $\mathbb{P}_\alpha^\Phi$-name $\{(\mathcal{B}_n, L_n) \mid n < \omega\}$ a canonical projection of the nice $\mathbb{P}_\alpha^\Psi$-name $\{(\mathcal{A}_n, K_n) \mid n < \omega\}$ below q.

Lemma 7.6. *Assume $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ is a nice $\mathbb{P}_\alpha^\Psi$-name for K-many reals below $p \restriction \alpha$ with respect to $\theta_\bullet$ and $\langle \langle \dot{S}_n^\theta \mid n < \omega \rangle \mid \theta \in \Theta_\alpha^p \rangle$. Further, assume that $\langle (\mathcal{B}_n, L_n) \mid n < \omega \rangle$ is a canonical projection of $\{(\mathcal{A}_n, K_n) \mid n < \omega\}$ below q. Then, for every $k \in K$*

$$q \Vdash \mathrm{name}(\{(\mathcal{B}_n, L_n(k)) \mid n < \omega\}) \notin \bigcup_{n<\omega} [\dot{S}_n^{\theta_k}].$$

Proof. Assume not, so choose $k \in K$, $n < \omega$ and $r_0 \leq q$ such that

$$r_0 \Vdash \mathrm{name}(\{(\mathcal{B}_n, L_n(k)) \mid n < \omega\}) \in [\dot{S}_n^{\theta_k}].$$

Choose $b \in \mathcal{B}_n$ such that $b \,||\, r_0$. Choose $r_1 \in \mathbb{P}_\alpha^\Phi$ with $r_1 \leq b, r_0$. Since $\{(\mathcal{B}_n, L_n) \mid n < \omega\}$ is a canonical projection below q of $\{(\mathcal{A}_n, K_n) \mid n < \omega\}$ choose $a \in \mathcal{A}_n$ such that b is a reduction of a and $K_n(k,a) = L_n(k,b)$. Thus, $\iota_\alpha^{\Phi,\Psi}(r_1) \,||\, a$. Then, by assumption we have

$$r_1 \Vdash L_n(k,b) \in \dot{S}_n^{\theta_k},$$

which implies

$$\iota_\alpha^{\Phi,\Psi}(r_1) \Vdash K_n(k,a) = L_n(k,b) \in \iota_\alpha^{\Phi,\Psi}(\dot{S}_n^{\theta_k}).$$

However, as $\langle (\mathcal{A}_n, K_n) \mid n < \omega \rangle$ is a nice name with respect to $\theta_\bullet$ and $\langle \langle \dot{S}_n^\theta \mid n < \omega \rangle \mid \theta \in \Theta_\alpha^p \rangle$

$$a \Vdash K_n(k,a) \notin \iota_\alpha^{\Phi,\Psi}(\dot{S}_n^{\theta_k}),$$

contradicting $\iota_\alpha^{\Phi,\Psi}(r_1) \,||\, a$. □

Finally, we define a reduction of p as follows: By the previous discussion choose a nice $\mathbb{P}^\Psi$-name $\langle(\mathcal{A}_n, K_n) \mid n<\omega\rangle$ of K-many reals below $p \restriction \alpha$ with respect to $\theta_\bullet$ and $\langle\langle\dot{S}^\theta_n \mid n<\omega\rangle \mid \theta\in\Theta^p_\alpha\rangle$, so that for every $k\in K$ we have

$$p\restriction\alpha \Vdash \dot{f}_k = \operatorname{name}(\langle(\mathcal{A}_n, K_n(k)) \mid n<\omega\rangle).$$

By Lemma 7.5 choose a canonical projection $\langle(\mathcal{B}_n, L_n) \mid n<\omega\rangle$ of $\langle(\mathcal{A}_n, K_n) \mid n<\omega\rangle$. Now, for $\theta\in\Theta^p_\alpha$ and $i\in I^p_{\alpha,\theta}$ we define $G_{\alpha,\theta,i}$ as

$$\{\operatorname{name}(\langle(\mathcal{B}_n, L_n(k)) \mid n<\omega\rangle) \mid k\in K \text{ with } \theta_k=\theta \text{ and } i_k=i\}.$$

Let $\dot{q}_\alpha$ be the canonical name for the condition in $\dot{\mathbb{Q}}^\Phi_\alpha = \prod_{\theta\in\operatorname{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha)$ with $\operatorname{supp}(\dot{q}_\alpha)=\Theta^p_\alpha$, for every $\theta\in\Theta_\alpha$ with $\operatorname{supp}(\dot{q}_\alpha(\theta)) = I^p_{\alpha,\theta}$ and for every $i\in I_{\alpha,\theta}$ we have $\dot{q}_\alpha(\theta)(i) = (s^p_{\alpha,\theta,i}, G_{\alpha,\theta,i})$. Since $\langle\dot{S}^{\theta_k}_n \mid n<\omega\rangle$ enumerates $\dot{\mathcal{T}}^{\Phi,\theta_k}_\alpha$ by Lemma 7.6 for every $k\in K$ we have

$$q \Vdash \operatorname{name}(\langle(\mathcal{B}_n, L_n(k)) \mid n<\omega\rangle) \notin \bigcup_{T\in\dot{\mathcal{T}}^{\Phi,\theta_k}_\alpha} [T].$$

Hence, we obtain

$$q\Vdash \dot{q}_\alpha \in \dot{\mathbb{Q}}^\Phi_\alpha = \prod_{\theta\in\operatorname{dom}(\Phi)} \mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha),$$

i.e. $q^\frown\dot{q}_\alpha \in \mathbb{P}^\Phi_{\alpha+1}$. It remains to show that $q^\frown\dot{q}_\alpha$ is indeed a reduction of p with respect to $\iota^{\Phi,\Psi}_{\alpha+1}$.

Proof. Let $r\leq q^\frown\dot{q}_\alpha$. We need to show that $\iota^{\Phi,\Psi}_{\alpha+1}(r) \,||\, p$. By extending r we may assume $r\in D^\Phi_{\alpha+1}$. Further, $r\restriction\alpha\leq q$. Since $r\restriction\alpha\Vdash r(\alpha)\leq\dot{q}_\alpha$ we have

- $\Theta^p_\alpha \subseteq \Theta^r_\alpha$,
- $I^p_{\alpha,\theta}\subseteq I^r_{\alpha,\theta}$ for every $\theta\in\Theta^p_\alpha$,
- $n^p_{\alpha,\theta,i}\leq n^r_{\alpha,\theta,i}$ for every $\theta\in\Theta^p_\alpha$ and $i\in I^p_{\alpha,\theta}$,
- $s^p_{\alpha,\theta,i}\trianglelefteq s^r_{\alpha,\theta,i}$ for every $\theta\in\Theta^p_\alpha$ and $i\in I^p_{\alpha,\theta}$,
- For every $k\in K$ there is $\dot{h}_k\in F^r_{\alpha,\theta_k,i_k}$ such that

$$r\restriction\alpha\Vdash \dot{h}_k = \operatorname{name}(\langle(\mathcal{B}_n, L_n(k)) \mid n<\omega\rangle).$$

Let $N := \max\{n^r_{\alpha,\theta,i} \mid \theta\in\Theta^p_\alpha, i\in I^p_{\alpha,\theta}\}$. Since $r\restriction\alpha\leq q$ and $\mathcal{B}_N$ is a maximal antichain below q, choose $b\in\mathcal{B}_N$ and $\bar{r}\in\mathbb{P}^\Phi_\alpha$ with $\bar{r}\leq r\restriction\alpha, b$. As $\langle(\mathcal{B}_n, L_n) \mid n<\omega\rangle$ is a canonical projection of $\langle(\mathcal{A}_n, K_n) \mid n<\omega\rangle$ choose $a\in\mathcal{A}_N$, so that b is a reduction of a and for all $k\in K$ we have $K_N(k,a) = L_N(k,b)$. Hence, $\iota^{\Phi,\Psi}_\alpha(\bar{r}) \,||\, a$, so choose $\bar{p}\in\mathbb{P}^\Psi_\alpha$ with $\bar{p}\leq\iota^{\Phi,\Psi}_\alpha(\bar{r}), a$. We define

- $\Theta^{\bar{p}}_\alpha := \Theta^r_\alpha$,
- $I^{\bar{p}}_{\alpha,\theta} := I^r_{\alpha,\theta}$ for every $\theta\in\Theta^{\bar{p}}_\alpha$,
- $n^{\bar{p}}_{\alpha,\theta,i} := n^r_{\alpha,\theta,i}$ and $s^{\bar{p}}_{\alpha,\theta,i} := s^r_{\alpha,\theta,i}$ for every $\theta\in\Theta^{\bar{p}}_\alpha$ and $i\in I^{\bar{p}}_{\alpha,\theta}$,
- $F^{\bar{p}}_{\alpha,\theta,i} := F^p_{\alpha,\theta,i}\cup\iota^{\Phi,\Psi}_\alpha(F^r_{\alpha,\theta,i})$ for every $\theta\in\Theta^{\bar{p}}_\alpha$ and $i\in I^{\bar{p}}_{\alpha,\theta}$,

where every undefined set is to be treated as the empty set. Let $\dot{\bar{p}}_\alpha$ be the canonical name for the condition in $\dot{\mathbb{Q}}^\Phi_\alpha = \prod_{\theta\in\operatorname{dom}(\Phi)}\mathbb{T}(\dot{\mathcal{T}}^{\Phi,\theta}_\alpha)$ with $\operatorname{supp}(\dot{\bar{p}}_\alpha) = \Theta^{\bar{p}}_\alpha$ and for every $\theta\in\Theta^{\bar{p}}_\alpha$ with $\operatorname{supp}(\dot{\bar{p}}_\alpha(\theta)) = I^{\bar{p}}_{\alpha,\theta}$ and for every $i\in I^{\bar{p}}_{\alpha,\theta}$ we have $\dot{\bar{p}}_\alpha(\theta)(i) = (s^{\bar{p}}_{\alpha,\theta,i}, F^{\bar{p}}_{\alpha,\theta,i})$. By definition of $\bar{p}^\frown\dot{\bar{p}}_\alpha$

we have $\bar{p}^\frown \dot{\bar{p}}_\alpha \leq p, \iota^{\Phi,\Psi}_{\alpha+1}(r)$, so we finish the proof by showing that $\bar{p}^\frown \dot{\bar{p}}_\alpha \in \mathbb{P}^\Psi_\alpha$. By definition of $F^{\bar{p}}_{\alpha,\theta,i}$ we distinguish the following two cases. First, let $k \in K$, by Remark 6.5 we have to prove

$$\bar{p} \Vdash \dot{f}_k \restriction n^{\bar{p}}_{\alpha,\theta_k,i_k} \in s^{\bar{p}}_{\alpha,\theta_k,i_k} \text{ and } \dot{f}_k \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Psi,\theta_k}_\alpha} [T].$$

Since $p \in D^\Psi_{\alpha+1}$ we have

$$p \restriction \alpha \Vdash \dot{f}_k \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Psi,\theta_k}_\alpha} [T],$$

so also $\bar{p} \leq a \leq p \restriction \alpha$ forces this. For the other property, choose $\dot{h}_k \in F^r_{\alpha,\theta_k,i_k}$ such that

$$r \restriction \alpha \Vdash \dot{h}_k = \text{name}(\langle (\mathcal{B}_n, L_n(k)) \mid n < \omega \rangle).$$

Since $r \in D^\Phi_{\alpha+1}$ we have

$$r \restriction \alpha \Vdash \dot{h}_k \restriction n^r_{\alpha,\theta_k,i_k} \in s^r_{\alpha,\theta_k,i_k}.$$

Furthermore, as $N \geq n^r_{\alpha,\theta_k,i_k}$ and $b \in \mathcal{B}_N$ we have

$$b \Vdash \text{name}(\langle (\mathcal{B}_n, L_n(k)) \mid n < \omega \rangle) \restriction n^r_{\alpha,\theta_k,i_k} = L_N(k,b) \restriction n^r_{\alpha,\theta_k,i_k}.$$

Hence, $\bar{r} \leq r \restriction \alpha, b$ implies that

$$\bar{r} \Vdash L_N(k,b) \restriction n^r_{\alpha,\theta_k,i_k} = \dot{h}_k \restriction n^r_{\alpha,\theta_k,i_k} \in s^r_{\alpha,\theta_k,i_k}.$$

Thus, we obtain $L_N(k,a) \restriction n^r_{\alpha,\theta_k,i_k} \in s^r_{\alpha,\theta_k,i_k}$. But $n^{\bar{p}}_{\alpha,\theta_k,i_k} = n^r_{\alpha,\theta_k,i_k}$, $s^{\bar{p}}_{\alpha,\theta_k,i_k} = s^r_{\alpha,\theta_k,i_k}$ and by choice of b we have $L_N(k,b) = K_N(k,a)$, so that

$$K_N(k,a) \restriction n^{\bar{p}}_{\alpha,\theta_k,i_k} \in s^{\bar{p}}_{\alpha,\theta_k,i_k}.$$

Finally,

$$a \Vdash \dot{f}_k = \text{name}(\langle (\mathcal{A}_n, K_n(k)) \mid n < \omega \rangle)$$

and $\bar{p} \leq a$ yield the desired

$$\bar{p} \Vdash \dot{f}_k \restriction n^{\bar{p}}_{\alpha,\theta_k,i_k} = K_N(k,a) \restriction n^{\bar{p}}_{\alpha,\theta_k,i_k} \in s^{\bar{p}}_{\alpha,\theta_k,i_k}.$$

Secondly, let $\theta \in \Theta^r_\alpha$, $i \in I^r_{\alpha,\theta}$ and $\dot{h} \in F^r_{\alpha,\theta,i}$. Then, $\bar{r} \leq r \restriction \alpha$ implies

$$\bar{r} \Vdash \dot{h} \restriction n^r_{\alpha,\theta,i} \in s^r_{\alpha,\theta,i} \text{ and } \dot{h} \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Phi,\theta}_\alpha} [T].$$

As before, by applying $\iota^{\Phi,\Psi}_\alpha$ we obtain

$$\iota^{\Phi,\Psi}_\alpha(\bar{r}) \Vdash \iota^{\Phi,\Psi}_\alpha(\dot{h}) \restriction n^r_{\alpha,\theta,i} \in s^r_{\alpha,\theta,i} \text{ and } \iota^{\Phi,\Psi}_\alpha(\dot{h}) \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Psi,\theta}_\alpha} [T].$$

Hence, $\bar{p} \leq \iota^{\Phi,\Psi}_\alpha(\bar{r})$ implies that

$$\bar{p} \Vdash \iota^{\Phi,\Psi}_\alpha(\dot{h}) \restriction n^{\bar{p}}_{\alpha,\theta,i} \in s^{\bar{p}}_{\alpha,\theta,i} \text{ and } \iota^{\Phi,\Psi}_\alpha(\dot{h}) \notin \bigcup_{T \in \dot{\mathcal{T}}^{\Psi,\theta}_\alpha} [T].$$

Thus, we finished proving $\bar{p}^\frown \dot{\bar{p}}_\alpha \in \mathbb{P}^\Psi_\alpha$. $\square$

To complete the induction, it remains to verify (A) to (F):

(A) By induction on (A) it suffices to verify the following. Let $p \in \mathbb{P}^{\Phi}_{\alpha}$. Then, we compute

$$\begin{aligned}\iota^{\Phi,\Psi}_{\alpha+1}(\iota^{\Phi}_{\alpha,\alpha+1}(p)) &= \iota^{\Phi,\Psi}_{\alpha+1}(p^\frown \mathbb{1})\\ &= \iota^{\Phi,\Psi}_{\alpha}(p)^\frown \mathbb{1}\\ &= \iota^{\Psi}_{\alpha,\alpha+1}(\iota^{\Phi,\Psi}_{\alpha}(p)).\end{aligned}$$

(B) Let $\theta \in \mathrm{dom}(\Phi)$, $i \in \Phi(\theta)$, $\gamma \in \Gamma$ and $p \in \mathbb{P}^{\Phi}_{\alpha+1}$. Then, we compute

$$\begin{aligned}\pi^{\Psi,\theta}_{\alpha+1,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha+1}(p)) &= \pi^{\Psi,\theta}_{\alpha+1,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p\restriction\alpha)^\frown \iota^{\Phi,\Psi}_{\alpha}(p(\alpha))) && \text{(definition of } \iota^{\Phi,\Psi}_{\alpha+1})\\ &= \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p\restriction\alpha))^\frown \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p(\alpha))) && \text{(Definition 5.3)}\\ &= \iota^{\Phi,\Psi}_{\alpha}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p\restriction\alpha))^\frown \iota^{\Phi,\Psi}_{\alpha}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p(\alpha))) && \text{((B) inductively)}\\ &= \iota^{\Phi,\Psi}_{\alpha+1}(\pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p\restriction\alpha)^\frown \pi^{\Phi,\theta}_{\alpha,i}(\gamma)(p(\alpha))) && \text{(definition of } \iota^{\Phi,\Psi}_{\alpha+1})\\ &= \iota^{\Phi,\Psi}_{\alpha+1}(\pi^{\Phi,\theta}_{\alpha+1,i}(\gamma)(p)) && \text{(Definition 5.3).}\end{aligned}$$

(C) There is nothing to show.
(D) Let $\theta \in \mathrm{dom}(\Phi)$ and $n < \omega$. Then, $\iota^{\Phi,\Psi}_{\alpha+1}(\dot{T}^{\Phi,\theta}_{\alpha,n}) = \dot{T}^{\Psi,\theta}_{\alpha,n}$ immediately follows, since $\iota^{\Phi,\Psi}_{\alpha}$ preserves check-names.
(E) Let $\theta \in \mathrm{dom}(\Phi)$. Then, we compute using Definition 4.10, (E) inductively, (D), (A) and the fact that every name is chosen as a canonical name:

$$\begin{aligned}\dot{\mathcal{T}}^{\Psi,\theta}_{\alpha+1} &= \iota^{\Psi}_{\alpha,\alpha+1}(\dot{\mathcal{T}}^{\Psi,\theta}_{\alpha}) \cup \{\dot{T}^{\Psi,\theta}_{\alpha,n} \mid n \in \omega\}\\ &= \iota^{\Psi}_{\alpha,\alpha+1}\left[\iota^{\Phi,\Psi}_{\alpha}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}) \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} \iota^{\Psi}_{1,\alpha}(\mathcal{T}^{\Psi,\theta}_{1,i})\right] \cup \{\iota^{\Phi,\Psi}_{\alpha+1}(\dot{T}^{\Phi,\theta}_{\alpha,n}) \mid n \in \omega\}\\ &= \iota^{\Psi}_{\alpha,\alpha+1}(\iota^{\Phi,\Psi}_{\alpha}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha})) \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} \iota^{\Psi}_{\alpha,\alpha+1}(\iota^{\Psi}_{1,\alpha}(\mathcal{T}^{\Psi,\theta}_{1,i})) \cup \iota^{\Phi,\Psi}_{\alpha+1}(\{\dot{T}^{\Phi,\theta}_{\alpha,n} \mid n \in \omega\})\\ &= \iota^{\Phi,\Psi}_{\alpha+1}(\iota^{\Phi}_{\alpha,\alpha+1}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha})) \cup \iota^{\Phi,\Psi}_{\alpha+1}(\{\dot{T}^{\Phi,\theta}_{\alpha,n} \mid n \in \omega\}) \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} (\iota^{\Psi}_{1,\alpha+1}(\mathcal{T}^{\Psi,\theta}_{1,i}))\\ &= \iota^{\Phi,\Psi}_{\alpha+1}\left[\iota^{\Phi}_{\alpha,\alpha+1}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha}) \cup \{\dot{T}^{\Phi,\theta}_{\alpha,n} \mid n \in \omega\}\right] \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} (\iota^{\Psi}_{1,\alpha+1}(\mathcal{T}^{\Psi,\theta}_{1,i}))\\ &= \iota^{\Phi,\Psi}_{\alpha+1}(\dot{\mathcal{T}}^{\Phi,\theta}_{\alpha+1}) \cup \bigcup_{i\in\Psi(\theta)\setminus\Phi(\theta)} (\iota^{\Psi}_{1,\alpha+1}(\mathcal{T}^{\Psi,\theta}_{1,i})).\end{aligned}$$

(F) Let $\theta \in \mathrm{dom}(\Phi)$, $i \in \Psi(\theta) \setminus \Phi(\theta)$, $\gamma \in \Gamma$ and $p \in \mathbb{P}^{\Phi}_{\alpha+1}$. Then, we compute

$$\begin{aligned}
\pi^{\Psi,\theta}_{\alpha+1,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha+1}(p)) &= \pi^{\Psi,\theta}_{\alpha+1,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) ^\frown \iota^{\Phi,\Psi}_{\alpha}(p(\alpha))) && \text{(definition of } \iota^{\Phi,\Psi}_{\alpha+1}\text{)}\\
&= \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha)) ^\frown \pi^{\Psi,\theta}_{\alpha,i}(\gamma)(\iota^{\Phi,\Psi}_{\alpha}(p(\alpha))) && \text{(Definition 5.3)}\\
&= \iota^{\Phi,\Psi}_{\alpha}(p \restriction \alpha) ^\frown \iota^{\Phi,\Psi}_{\alpha}(p(\alpha)) && \text{((F) inductively)}\\
&= \iota^{\Phi,\Psi}_{\alpha+1}(p) && \text{(definition of } \iota^{\Phi,\Psi}_{\alpha+1}\text{).}
\end{aligned}$$

This completes the induction and thus the proof of Theorem 7.1. □

8. Extending Isomorphisms through the iteration

In Section 5 we considered how to extend automorphisms of certain group actions through the iteration. Similarly, given bijections between the index sets of the Cohen reals of our iteration we will show how to extend these bijections to isomorphisms of the full iteration. These extensions have a very categorical flavour, nevertheless we provide a self-contained presentation.

Definition 8.1. Let Φ, Ψ be Θ-indexing functions. Then, we say $\mathbf{x} = (g, \{h^\theta \mid \theta \in \mathrm{dom}(\Phi)\})$ is an isomorphism from Φ to Ψ iff the following properties hold:

(1) $g : \mathrm{dom}(\Phi) \to \mathrm{dom}(\Psi)$ is a bijection,
(2) for every $\theta \in \mathrm{dom}(\Phi)$ also $h^\theta : \Phi(\theta) \to \Psi(g(\theta))$ is a bijection.

Definition 8.2. Let Φ be a Θ-indexing function. Then, we define the identity isomorphism from Φ to Φ by $\mathbf{1}_\Phi := (\mathrm{id}_{\mathrm{dom}(\Phi)}, \{\mathrm{id}_{\Phi(\theta)} \mid \theta \in \mathrm{dom}(\Phi)\})$.

Definition 8.3. Let Φ, Ψ, X be Θ-indexing functions, $\mathbf{x}_0 = (g_0, \{h^\theta_0 \mid \theta \in \mathrm{dom}(\Phi)\})$ an isomorphism from Φ to Ψ and $\mathbf{x}_1 = (g_1, \{h^\theta_1 \mid \theta \in \mathrm{dom}(\Phi)\})$ is an isomorphism from Ψ to Φ. Then, we define its composition $\mathbf{x}_1 \circ \mathbf{x}_0 := (g_2, \{h^\theta_2 \mid \theta \in \mathrm{dom}(\Phi)\})$ by

(1) $g_2 := g_1 \circ g_0$,
(2) for every $\theta \in \mathrm{dom}(\Phi)$ we define $h^\theta_2 := h^{g_0(\theta)}_1 \circ h^\theta_0$.

Clearly, $\mathbf{x}_1 \circ \mathbf{x}_0$ is an isomorphism from Φ to Ψ and it is easy to check, that composition is associative and the identity isomorphism satisfies left and right unit laws. In other words, the class of all Θ-indexing functions with isomorphisms as morphisms is a category.

Definition 8.4. Let Φ, Ψ be Θ-indexing functions and $\mathbf{x} = (g, \{h^\theta \mid \theta \in \mathrm{dom}(\Phi)\})$ an isomorphism from Φ to Ψ. Then, we define its inverse $\mathbf{x}^{-1} := (g_*, \{h^\theta_* \mid \theta \in \mathrm{dom}(\Psi)\})$ by

(1) $g_* := g^{-1}$,
(2) for every $\theta \in \Theta$ we define $h^\theta_* := (h^{g^{-1}(\theta)})^{-1}$.

Clearly, $\mathbf{x}^{-1}$ is an isomorphism from Ψ to Φ and it is easy to check, that it is the unique isomorphism which satisfies $\mathbf{x}^{-1} \circ \mathbf{x} = \mathbf{1}_\Phi$ and $\mathbf{x} \circ \mathbf{x}^{-1} = \mathbf{1}_\Psi$. In other words, we not only have a category but a groupoid.

Definition 8.5. Let Φ, Ψ be Θ-indexing functions and $\mathbf{x} = (g, \{h^\theta \mid \theta \in \operatorname{dom}(\Phi)\})$ an isomorphism from Φ to Ψ. Define $\kappa_{\mathbf{x}} : \mathbb{C}^\Phi \to \mathbb{C}^\Psi$ for $p \in \mathbb{C}^\theta, \theta \in \operatorname{dom}(\Psi)$ and $i \in \Psi(\theta)$ by

$$\kappa_{\mathbf{x}}(p)(\theta, i) := p(g^{-1}(\theta), (h^{g^{-1}(\theta)})^{-1}(i)).$$

In other words, the information of p is swapped around as given by the bijections g and h^θ. Clearly, $\kappa_{\mathbf{x}}$ is an isomorphism from the partial order $\mathbb{C}^\Phi$ to $\mathbb{C}^\Psi$.

Lemma 8.6. *Let Φ, Ψ, X be Θ-indexing functions, $\mathbf{x}_0 = (g_0, \{h_0^\theta \mid \theta \in \operatorname{dom}(\Phi)\})$ an isomorphism from Φ to Ψ and $\mathbf{x}_1 = (g_1, \{h_1^\theta \mid \theta \in \operatorname{dom}(\Psi)\})$ is an isomorphism from X to Ψ. Then, we have*

(1) $\kappa_{\mathbf{1}_\Phi} = \mathrm{id}_{\mathbb{C}^\Phi}$,
(2) $\kappa_{\mathbf{x}_1 \circ \mathbf{x}_0} = \kappa_{\mathbf{x}_1} \circ \kappa_{\mathbf{x}_0}$.

In other words, $\kappa_\bullet$ is a functor between groupoids.

Proof. For the first statement let $p \in \mathbb{C}^\Phi$, $\theta \in \operatorname{dom}(\Phi)$ and $i \in \Phi(\theta)$. Then, we compute

$$\begin{aligned}
\kappa_{\mathbf{1}_\Phi}(p)(\theta, i) &= p((\mathrm{id}^{-1}_{\operatorname{dom}(\Phi)}(\theta), (\mathrm{id}_{\Phi(\theta)})^{-1}(i)) && \text{(Definition 8.2 and 8.5)} \\
&= p(\theta, i).
\end{aligned}$$

Secondly, let $p \in \mathbb{C}^\Phi$, $\theta \in \operatorname{dom}(\mathrm{X})$ and $i \in \mathrm{X}(\theta)$. Then, we compute

$$\begin{aligned}
\kappa_{\mathbf{x}_1 \circ \mathbf{x}_0}(p)(\theta, i) &= p((g_1 \circ g_0)^{-1}(\theta), (h_1^{(g_0 \circ (g_1 \circ g_0)^{-1})(\theta)} \circ h_0^{(g_1 \circ g_0)^{-1}(\theta)})^{-1}(i)) && \text{(Definition 8.3 and 8.5)} \\
&= p(g_0^{-1}(g_1^{-1}(\theta)), (h_1^{g_1^{-1}(\theta)} \circ h_0^{g_0^{-1}(g_1^{-1}(\theta))})^{-1}(i)) \\
&= p(g_0^{-1}(g_1^{-1}(\theta)), (h_0^{g_0^{-1}(g_1^{-1}(\theta))})^{-1}((h_1^{g_1^{-1}(\theta)})^{-1}(i))) \\
&= \kappa_{\mathbf{x}_0}(p)(g_1^{-1}(\theta), (h_1^{g_1^{-1}(\theta)})^{-1}(i)) && \text{(Definition 8.5)} \\
&= \kappa_{\mathbf{x}_1}(\kappa_{\mathbf{x}_0}(p))(\theta, i) && \text{(Definition 8.5)} \\
&= (\kappa_{\mathbf{x}_1} \circ \kappa_{\mathbf{x}_0})(p)(\theta, i).
\end{aligned}$$

□

Next, we need to verify that the canonical $\mathbb{C}^\Phi$-names $\dot{\mathcal{T}}_i^{\Phi,\theta}$ are mapped to $\dot{\mathcal{T}}_{h^\theta(i)}^{\Psi,g(\theta)}$ by $\kappa_{\mathbf{x}}$. To this end, we prove that $\kappa_{\mathbf{x}}$ behaves nicely with respect to the Γ-actions.

Lemma 8.7. *Let Φ, Ψ be Θ-indexing functions and $\mathbf{x} = (g, \{h^\theta \mid \theta \in \operatorname{dom}(\Phi)\})$ is an isomorphism from Φ to Ψ. Let $\theta \in \operatorname{dom}(\Phi)$ and $i \in \Phi(\theta)$. Then, $\kappa_{\mathbf{x}} : \mathbb{C}^\Phi \to \mathbb{C}^\Psi$ is a morphism of Γ-sets, i.e. the following diagram commutes for every $\gamma \in \Gamma$:*

$$\begin{array}{ccc}
\mathbb{C}^\Phi & \xrightarrow{\kappa_{\mathbf{x}}} & \mathbb{C}^\Psi \\
\big\downarrow \pi_i^{\Phi,\theta}(\gamma) & & \big\downarrow \pi_{h^\theta(i)}^{\Psi,g(\theta)}(\gamma) \\
\mathbb{C}^\Phi & \xrightarrow{\kappa_{\mathbf{x}}} & \mathbb{C}^\Psi
\end{array}$$

Proof. Let $\gamma \in \Gamma$ and $p \in \mathbb{C}^\Phi$. Further, let $\eta \in \operatorname{dom}(\Psi)$ and $j \in \Psi(\eta)$. In case that $\theta = g^{-1}(\eta)$ and $i = (h^{g^{-1}(\eta)})^{-1}(j)$ we compute:

$$\begin{aligned}
\kappa_{\mathbf{x}}(\pi_i^{\Phi,\theta}(\gamma)(p))(\eta, j) &= \pi_i^{\Phi,\theta}(\gamma)(p)(g^{-1}(\eta), (h^{g^{-1}(\eta)})^{-1}(j)) && \text{(Definition 8.5)}\\
&= \pi_i^{\Phi,\theta}(\gamma)(p)(\theta, i) && \text{(case property of } \eta, i)\\
&= \pi(\gamma)(p(\theta, i)) && \text{(Definition 4.6)}\\
&= \pi(\gamma)(p(g^{-1}(\eta), (h^{g^{-1}(\eta)})^{-1}(j))) && \text{(case property of } \eta, i)\\
&= \pi(\gamma)(\kappa_{\mathbf{x}}(p)(\eta, j)) && \text{(Definition 8.5)}\\
&= \pi_j^{\Psi,\eta}(\gamma)(\kappa_{\mathbf{x}}(p))(\eta, j) && \text{(Definition 4.6)}\\
&= \pi_{h^\theta(i)}^{\Psi,g(\theta)}(\gamma)(\kappa_{\mathbf{x}}(p))(\eta, j) && \text{(case property of } \eta, i).
\end{aligned}$$

Otherwise, we have that $\pi_i^{\Phi,\theta}$ acts trivially on the $(g^{-1}(\eta), (h^{g^{-1}(\eta)})^{-1}(j))$-component of p and $\pi_{h^\theta(i)}^{\Psi,g(\theta)}(\gamma)$ acts trivially on the (η, j)-component of $\kappa_{\mathbf{x}}(p)$, so we compute

$$\begin{aligned}
\kappa_{\mathbf{x}}(\pi_i^{\Phi,\theta}(\gamma)(p))(\eta, j) &= \pi_i^{\Phi,\theta}(\gamma)(p)(g^{-1}(\eta), (h^{g^{-1}(\eta)})^{-1}(j)) && \text{(Definition 8.5)}\\
&= p(g^{-1}(\eta), (h^{g^{-1}(\eta)})^{-1}(j)) && (\pi_i^{\Phi,\theta} \text{ acts trivially})\\
&= \kappa_{\mathbf{x}}(p)(\eta, j) && \text{(Definition 8.5)}\\
&= \pi_{h^\theta(i)}^{\Psi,g(\theta)}(\gamma)(\kappa_{\mathbf{x}}(p))(\eta, j) && (\pi_{h^\theta(i)}^{\Psi,g(\theta)} \text{ acts trivially}). \qquad \square
\end{aligned}$$

Lemma 8.8. *Let Φ, Ψ be Θ-indexing functions and $\mathbf{x} = (g, \{h^\theta \mid \theta \in \operatorname{dom}(\Phi)\})$ is an isomorphism from Φ to Ψ. Let $\theta \in \operatorname{dom}(\Phi)$ and $i \in \Phi(\theta)$. Then, we have*

(1) $\kappa_{\mathbf{x}}(\dot{c}_i^{\Phi,\theta}) = \dot{c}_{h^\theta(i)}^{\Psi,g(\theta)}$ and thus $\kappa_{\mathbf{x}}(\dot{T}_i^{\Phi,\theta}) = \dot{T}_{h^\theta(i)}^{\Psi,g(\theta)}$,
(2) $\kappa_{\mathbf{x}}(\dot{\mathcal{T}}_i^{\Phi,\theta}) = \dot{\mathcal{T}}_{h^\theta(i)}^{\Psi,g(\theta)}$,
(3) $\kappa_{\mathbf{x}}(\dot{\mathcal{T}}^{\Phi,\theta}) = \dot{\mathcal{T}}^{\Psi,g(\theta)}$.

Proof. (1) immediately follows from the definition of $\kappa_{\mathbf{x}}$ and the definition of the canonical name for a Cohen real. For (2) by Definition 4.8 remember $\dot{\mathcal{T}}_i^{\Phi,\theta}$ is the canonical $\mathbb{C}^\Phi$-name for the set

$$\{\pi_i^{\Phi,\theta}(\gamma)(\dot{T}_i^{\Phi,\theta}) \mid \gamma \in \Gamma\}.$$

Hence, we compute

$$\begin{aligned}
\kappa_{\mathbf{x}}(\dot{\mathcal{T}}_i^{\Phi,\theta}) &= \kappa_{\mathbf{x}}(\{\pi_i^{\Phi,\theta}(\gamma)(\dot{T}_i^{\Phi,\theta}) \mid \gamma \in \Gamma\}) && \text{(Definition 4.8)}\\
&= \{\kappa_{\mathbf{x}}(\pi_i^{\Phi,\theta}(\gamma)(\dot{T}_i^{\Phi,\theta})) \mid \gamma \in \Gamma\} && \text{(canonical name)}\\
&= \{\pi_{h^\theta(i)}^{\Psi,g(\theta)}(\gamma)(\kappa_{\mathbf{x}}(\dot{T}_i^{\Phi,\theta})) \mid \gamma \in \Gamma\} && \text{(Lemma 8.7)}\\
&= \{\pi_{h^\theta(i)}^{\Psi,g(\theta)}(\gamma)(\dot{T}_{h^\theta(i)}^{\Psi,g(\theta)}) \mid \gamma \in \Gamma\} && (1)\\
&= \dot{\mathcal{T}}_{h^\theta(i)}^{\Psi,g(\theta)} && \text{(Definition 4.8)}.
\end{aligned}$$

Finally, for (3) we compute

$$\begin{aligned}
\kappa_{\mathbf{x}}(\dot{\mathcal{T}}^{\Phi,\theta}) &= \kappa_{\mathbf{x}}(\bigcup_{i\in\Phi(\theta)} \dot{\mathcal{T}}_i^{\Phi,\theta}) && \text{(Remark 4.9)}\\
&= \bigcup_{i\in\Phi(\theta)} \kappa_{\mathbf{x}}(\dot{\mathcal{T}}_i^{\Phi,\theta}) && \text{(canonical name)}\\
&= \bigcup_{i\in\Phi(\theta)} \dot{\mathcal{T}}_{h^\theta(i)}^{\Psi,g(\theta)} && (2)\\
&= \bigcup_{i\in\Psi(g(\theta))} \dot{\mathcal{T}}_i^{\Psi,g(\theta)} && (h^\theta : \Phi(\theta)\to\Psi(g(\theta)) \text{ is a bijection})\\
&= \dot{\mathcal{T}}^{\Psi,g(\theta)} && \text{(Remark 4.9).}
\end{aligned}$$

□

So far, we have constructed a functor $\kappa_\bullet$ mapping indexing functions Φ to posets of the form $\mathbb{C}^\Phi$. In terms of our iteration this corresponds to a functor $\kappa_\bullet^1$ mapping Θ-indexing functions to posets of the form $\mathbb{P}_1^\Phi$. We will extend these functors through the iteration to obtain an increasing sequence of functors in the following sense:

Definition 8.9. Let $\epsilon \leq \aleph_1$. We say that

$$\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \epsilon\rangle$$

is an increasing sequence of functors iff every $\kappa_\bullet^\alpha$ is a functor mapping Θ-indexing functions Φ to posets $\mathbb{P}_\alpha^\Phi$, for all $0 < \alpha \leq \epsilon$, Φ, Ψ Θ-indexing functions, $\mathbf{x} = (g, \{h^\theta \mid \theta \in \mathrm{dom}(\Phi)\})$ an isomorphism from Φ to Ψ and $\theta \in \mathrm{dom}(\Phi)$ we have

$$\kappa_{\mathbf{x}}(\dot{\mathcal{T}}^{\Phi,\theta}) = \dot{\mathcal{T}}^{\Psi,g(\theta)},$$

and for every $0 < \alpha \leq \beta \leq \aleph_1$, Θ-indexing functions Φ, Ψ and isomorphism $\mathbf{x}$ from Φ to Ψ the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{P}_\alpha^\Phi & \xrightarrow{\kappa_{\mathbf{x}}^\alpha} & \mathbb{P}_\alpha^\Psi\\
\big\downarrow \iota_{\alpha,\beta}^\Phi & & \big\downarrow \iota_{\alpha,\beta}^\Psi\\
\mathbb{P}_\beta^\Phi & \xrightarrow{\kappa_{\mathbf{x}}^\beta} & \mathbb{P}_\beta^\Psi
\end{array}$$

In other words, for every $0 < \alpha \leq \beta \leq \aleph_1$ the maps $\iota_{\alpha,\beta}^\bullet$ are a natural transformations from the functor $\kappa_\bullet^\alpha$ to the functor $\kappa_\bullet^\beta$.

Corollary 8.10. $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq 1\rangle$ *is an increasing sequence of functors (of length 1).*

Proof. By Lemma 8.6 $\kappa_\bullet^\alpha$ is a functor, the second property of Definition 8.9 holds by Lemma 8.8, and the third property is vacuous for a sequence of length 1. □

Note the similarity to Definition 5.1 and Lemma 5.5. In Section 5 we made sure to preserve some group structure of automorphisms through the iteration. Similarly, in this section we need to preserve the groupoid structure given by isomorphisms between Θ-indexing functions.

Proposition 8.11. *Let $\epsilon \leq \aleph_1$ be a limit. Assume*

$$\langle \kappa_\bullet^\alpha \mid 0 < \alpha < \epsilon \rangle$$

is an increasing sequence of functors. Then, there is a unique functor $\kappa_\bullet^\epsilon$ so that

$$\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \epsilon \rangle$$

is an increasing sequences of functors.

Proof. Define $\kappa_\bullet^\epsilon$ as the pointwise direct limit of $\langle \kappa_\bullet^\alpha \mid 0 < \alpha < \epsilon \rangle$. That is, for given Θ-indexing functions Φ, Ψ and an isomorphism $\mathbf{x} = (g, \{h^\theta \mid \theta \in \Theta\})$ from Φ to Ψ we define $\kappa_\mathbf{x}^\epsilon$ to be the direct limit of $\langle \kappa_\mathbf{x}^\alpha \mid \alpha < \epsilon \rangle$. Then, argue as in Lemma 5.2. □

Analogously to Definition 5.3 the extension at successor steps is not unique. However, there is a canonical way to extend an increasing sequence of functors.

Definition 8.12. Let $\epsilon \leq \aleph_1$. Assume $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \epsilon \rangle$ is an increasing sequence of functors. Let Φ, Ψ be Θ-indexing functions and $\mathbf{x} = (g, \{h^\theta \mid \theta \in \mathrm{dom}(\Phi)\})$ an isomorphism from Φ to Ψ. Then, we define $\kappa_\mathbf{x}^{\epsilon+1} : \mathbb{P}_{\epsilon+1}^\Phi \to \mathbb{P}_{\epsilon+1}^\Psi$ for $p \in \mathbb{P}_{\epsilon+1}^\Phi$ by

$$\kappa_\mathbf{x}^{\epsilon+1}(p) := \kappa_\mathbf{x}^\epsilon(p \restriction \alpha)^\frown \kappa_\mathbf{x}^\epsilon(p(\epsilon)).$$

Then, we call $\kappa_\bullet^{\epsilon+1}$ the canonical extension of $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \epsilon \rangle$.

Finally, analogous to Lemma 5.4 and Corollary 5.6 we obtain our desired induced sequence of with the following lemma.

Lemma 8.13. *Let $\epsilon < \aleph_1$. Assume $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \epsilon \rangle$ is an increasing sequence of functors and let $\kappa_\bullet^{\epsilon+1}$ be the canonical extension. Then $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \epsilon + 1 \rangle$ is an increasing sequence of functors.*

Corollary 8.14. *There is an increasing sequence of functors $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \aleph_1 \rangle$ such that $\kappa_\bullet^{\epsilon+1}$ the canonical extension of $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \epsilon \rangle$ for every $\epsilon < \aleph_1$. We call this sequence the induced sequence of functors and will reserve the notions $\langle \kappa_\bullet^\alpha \mid 0 < \alpha \leq \aleph_1 \rangle$ for it.*

Proof. We iteratively construct the desired sequence. By Lemma 8.8 we may start with $\kappa_\bullet^\alpha$ as in Definition 8.5, use Lemma 8.13 for the successor step and Lemma 8.11 for the limit step. □

The final ingredient we will need for the proof of Main Theorem 3.1 is a notion of restriction for isomorphisms between Θ-indexing functions. We also show inductively that our increasing sequence of functors in Corollary 8.14 maps restrictions to restrictions.

Definition 8.15. Let Φ, Ψ be Θ-indexing functions, $\mathbf{x} = (g, \{h^\theta \mid \theta \in \mathrm{dom}(\Phi)\})$ is an isomorphism from Φ to Ψ and $\Phi_0 \subseteq \Phi$ a Θ-subindexing function. Then, we define the image of Φ_0 under $\mathbf{x}$ denoted by $\mathbf{x}[\Phi_0]$ as the Θ-subindexing function of Ψ defined by $\mathrm{dom}(\mathbf{x}[\Phi_0]) := g[\mathrm{dom}(\Phi_0)]$ and for $\theta \in \mathrm{dom}(\mathbf{x}[\Phi_0])$ by

$$\mathbf{x}[\Phi_0](\theta) := \{h^{g^{-1}(\theta)}(i) \mid i \in \Phi_0(g^{-1}(\theta))\}.$$

The restriction of $\mathbf{x}$ to Φ_0, denoted by $\mathbf{x} \restriction \Phi_0$, is the isomorphism from Φ_0 to $\mathbf{x}[\Phi_0]$ is defined by

$$\mathbf{x} \restriction \Phi_0 := (g \restriction \mathrm{dom}(\Phi_0), \{h^\theta \restriction \Phi_0(\theta) \mid \theta \in \mathrm{dom}(\Phi_0)\}).$$

Lemma 8.16. *Let Φ, Ψ be Θ-indexing functions, $\mathbf{x} = (g, \{h^\theta \mid \theta \in \mathrm{dom}(\Phi)\})$ is an isomorphism from Φ to Ψ and $\Phi_0 \subseteq \Phi$ a Θ-subindexing function. Set $\Psi_0 := \mathbf{x}[\Phi_0]$. Then, the following diagram commutes*

$$\begin{array}{ccc} \mathbb{C}^{\Phi_0} & \xrightarrow{\kappa_{\mathbf{x}\restriction\Phi_0}} & \mathbb{C}^{\Psi_0} \\ \Big\downarrow \iota^{\Phi_0,\Phi} & & \Big\downarrow \iota^{\Psi_0,\Psi} \\ \mathbb{C}^{\Phi} & \xrightarrow{\kappa_{\mathbf{x}}} & \mathbb{C}^{\Psi} \end{array}$$

Proof. Let $p \in \mathbb{C}^{\Phi_0}$, $\theta \in \mathrm{dom}(\Psi)$ and $i \in \Psi(\theta)$. If $\theta \in \mathrm{dom}(\Psi_0)$ and $i \in \Psi_0(\theta)$, we compute

$$\begin{aligned} \iota^{\Psi_0,\Psi}(\kappa_{\mathbf{x}\restriction\Phi_0}(p))(\theta, i) &= \kappa_{\mathbf{x}\restriction\Phi_0}(p)(\theta, i) && (i \in \Psi_0(\theta)) \\ &= p(g^{-1}(\theta), (h^{g^{-1}(\theta)})^{-1}(i)) && \text{(Definition 8.5)} \\ &= \iota^{\Phi_0,\Phi}(p)(g^{-1}(\theta), (h^{g^{-1}(\theta)})^{-1}(i)) && ((h^{g^{-1}(\theta)})^{-1}(i) \in \Phi_0(\theta)) \\ &= \kappa_{\mathbf{x}}(\iota^{\Phi_0,\Phi}(p))(\theta, i) && \text{(Definition 8.5)} \end{aligned}$$

Otherwise, $\theta \in \mathrm{dom}(\Psi) \setminus \mathrm{dom}(\Psi_0)$ or $i \in \Psi(\theta) \setminus \Psi_0(\theta)$. Then, we have $g^{-1}(\theta) \in \mathrm{dom}(\Phi) \setminus \mathrm{dom}(\Phi_0)$ or $(h^{g^{-1}(\theta)})^{-1}(i) \in \Phi(\theta) \setminus \Phi_0(\theta)$, respectively. Then, we compute

$$\begin{aligned} \iota^{\Psi_0,\Psi}(\kappa_{\mathbf{x}\restriction\mathbf{\Phi_0}}(p))(\theta, i) &= \mathbb{1} \\ &= \iota^{\Phi_0,\Phi}(p)(g^{-1}(\theta), (h^{g^{-1}(\theta)})^{-1}(i)) \\ &= \kappa_{\mathbf{x}}(\iota^{\Phi_0,\Phi}(p))(\theta, i) && \text{(Definition 8.5).} \end{aligned}$$ □

Inductively, we show that this commutative diagram not only holds for $\kappa^1_\bullet$, but for the entire increasing sequence of functors $\langle \kappa^\alpha_\bullet \mid 0 < \alpha \leq \aleph_1 \rangle$.

Lemma 8.17. *Let $\epsilon < \aleph_1$. Let Φ, Ψ be Θ-indexing functions, $\mathbf{x} = (g, \{h^\theta \mid \theta \in \Theta\})$ is an isomorphism from Φ to Ψ and $\Phi_0 \subseteq \Phi$ a Θ-subindexing function. Set $\Psi_0 := \mathbf{x}[\Phi_0]$. Then, the following diagram commutes*

$$\begin{array}{ccc} \mathbb{P}^{\Phi_0}_{\epsilon+1} & \xrightarrow{\kappa^{\epsilon+1}_{\mathbf{x}\restriction\Phi_0}} & \mathbb{P}^{\Psi_0}_{\epsilon+1} \\ \Big\downarrow \iota^{\Phi_0,\Phi}_{\epsilon+1} & & \Big\downarrow \iota^{\Psi_0,\Psi}_{\epsilon+1} \\ \mathbb{P}^{\Phi}_{\epsilon+1} & \xrightarrow{\kappa^{\epsilon+1}_{\mathbf{x}}} & \mathbb{P}^{\Psi}_{\epsilon+1} \end{array}$$

Proof. Let $p \in \mathbb{P}^{\Phi_0}_{\epsilon+1}$. Then, we compute

$$\begin{aligned} \iota^{\Psi_0,\Psi}_{\epsilon+1}(\kappa^{\epsilon+1}_{\mathbf{x}\restriction\Phi_0}(p)) &= \iota^{\Psi_0,\Psi}_{\epsilon}(\kappa^{\epsilon+1}_{\mathbf{x}\restriction\Phi_0}(p) \restriction \epsilon) ^\frown \iota^{\Psi_0,\Psi}_{\epsilon}(\kappa^{\epsilon+1}_{\mathbf{x}\restriction\Phi_0}(p)(\epsilon)) && (\text{definition of } \iota^{\Psi_0,\Psi}_{\epsilon+1}) \\ &= \iota^{\Psi_0,\Psi}_{\epsilon}(\kappa^{\epsilon}_{\mathbf{x}\restriction\Phi_0}(p \restriction \epsilon)) ^\frown \iota^{\Psi_0,\Psi}_{\epsilon}(\kappa^{\epsilon}_{\mathbf{x}}(p(\epsilon))) && \text{(Definition 8.12)} \\ &= \kappa^{\epsilon}_{\mathbf{x}}(\iota^{\Phi_0,\Phi}_{\epsilon}(p \restriction \epsilon)) ^\frown \kappa^{\epsilon}_{\mathbf{x}}(\iota^{\Phi_0,\Phi}_{\epsilon}(p(\epsilon))) && \text{(induction)} \\ &= \kappa^{\epsilon}_{\mathbf{x}}(\iota^{\Phi_0,\Phi}_{\epsilon+1}(p) \restriction \epsilon) ^\frown \kappa^{\epsilon}_{\mathbf{x}}(\iota^{\Phi_0,\Phi}_{\epsilon+1}(p)(\epsilon)) && (\text{definition of } \iota^{\Psi_0,\Psi}_{\epsilon+1}) \\ &= \kappa^{\epsilon+1}_{\mathbf{x}}(\iota^{\Phi_0,\Phi}_{\epsilon+1}(p)) && \text{(Definition 8.12).} \end{aligned}$$ □

Lemma 8.18. *Let $\epsilon \leq \aleph_1$ be a limit. Let Φ, Ψ be Θ-indexing functions, $\mathbf{x} = (g, \{h^\theta \mid \theta \in \Theta\})$ is an isomorphism from Φ to Ψ and $\Phi_0 \subseteq \Phi$ a Θ-subindexing function. Set $\Psi_0 := \mathbf{x}[\Phi_0]$. Then, the following diagram commutes*

$$\begin{array}{ccc} \mathbb{P}_\epsilon^{\Phi_0} & \xrightarrow{\kappa^\epsilon_{\mathbf{x}\restriction\Phi_0}} & \mathbb{P}_\epsilon^{\Psi_0} \\ \Big\downarrow \iota_\epsilon^{\Phi_0,\Phi} & & \Big\downarrow \iota_\epsilon^{\Psi_0,\Psi} \\ \mathbb{P}_\epsilon^{\Phi} & \xrightarrow{\kappa^\epsilon_{\mathbf{x}}} & \mathbb{P}_\epsilon^{\Psi} \end{array}$$

Proof. Let $p \in \mathbb{P}_\epsilon^{\Phi_0}$. Choose $\alpha < \epsilon$ such that $\iota_{\alpha,\epsilon}^{\Phi_0}(p \restriction \alpha) = p$. Then, we compute

$$\begin{aligned}
\iota_\epsilon^{\Psi_0,\Psi}(\kappa^\epsilon_{\mathbf{x}\restriction\Phi_0}(p)) &= \iota_\epsilon^{\Psi_0,\Psi}(\kappa^\epsilon_{\mathbf{x}\restriction\Phi_0}(\iota_{\alpha,\epsilon}^{\Phi_0}(p\restriction\alpha))) && \text{(choice of } \alpha) \\
&= \iota_\epsilon^{\Psi_0,\Psi}(\iota_{\alpha,\epsilon}^{\Psi_0}(\kappa^\alpha_{\mathbf{x}\restriction\Phi_0}(p\restriction\alpha))) && \text{(Definition 8.9)} \\
&= \iota_{\alpha,\epsilon}^{\Psi}(\iota_\alpha^{\Psi_0,\Psi}(\kappa^\alpha_{\mathbf{x}\restriction\Phi_0}(p\restriction\alpha))) && \text{((A) in Section 7)} \\
&= \iota_{\alpha,\epsilon}^{\Psi}(\kappa^\alpha_{\mathbf{x}}(\iota_\alpha^{\Phi_0,\Phi}(p\restriction\alpha))) && \text{(induction)} \\
&= \kappa^\epsilon_{\mathbf{x}}(\iota_{\alpha,\epsilon}^{\Phi}(\iota_\alpha^{\Phi_0,\Phi}(p\restriction\alpha))) && \text{(Definition 8.9)} \\
&= \kappa^\epsilon_{\mathbf{x}}(\iota_\epsilon^{\Phi_0,\Phi}(\iota_{\alpha,\epsilon}^{\Phi_0}(p\restriction\alpha))) && \text{((A) in Section 7)} \\
&= \kappa^\epsilon_{\mathbf{x}}(\iota_\epsilon^{\Phi_0,\Phi}(p)) && \text{(choice of } \alpha). \qquad \square
\end{aligned}$$

9. Proof of the Main Theorem

Finally, we prove the our Main Theorem 3.1. The main part of the proof is an isomorphism-of-names argument to exclude values from $\operatorname{spec}(\mathfrak{a}_{\mathrm{T}})$. For similar arguments, also see [4], [10].

Main Theorem 3.1. *Assume* GCH *and let Θ be a set of uncountable cardinals such that*

(I) $\max(\Theta)$ *exists and has uncountable cofinality,*
(II) Θ *is closed under singular limits,*
(III) *If $\theta \in \Theta$ with $\operatorname{cof}(\theta) = \omega$, then $\theta^+ \in \Theta$,*
(IV) $\aleph_1 \in \Theta$.

Then, there is a c.c.c. forcing extension in which $\operatorname{spec}(\mathfrak{a}_{\mathrm{T}}) = \Theta$ holds.

Proof. For technical reasons we assume that $\max(\Theta)$ appears $\max(\Theta)$ many times in Θ, so that Θ has size $\max(\Theta)$ and we add $\max(\Theta)$ many partitions of ${}^\omega 2$ into F_σ-sets of size $\max(\Theta)$. Let Ψ be the Θ-indexing function defined by $\Psi(\theta) := \theta$ for every $\theta \in \Theta$. We show that $\mathbb{P}^\Psi_{\aleph_1} \Vdash \operatorname{spec}(\mathfrak{a}_{\mathrm{T}}) = \Theta$. Since $\mathbb{P}^\Psi_{\aleph_1}$ is c.c.c. no cardinals are collapsed and since $\left|D^\Psi_{\aleph_1}\right| = \max(\Theta)$ and $\max(\Theta)^{\aleph_0} = \max(\Theta)$ we have $\mathbb{P}^\Psi_{\aleph_1} \Vdash \mathfrak{c} = \max(\Theta)$. Further, as in Lemma 2.10 we have

$$\mathbb{P}^\Psi_{\aleph_1} \Vdash \Theta \subseteq \operatorname{spec}(\mathfrak{a}_{\mathrm{T}}),$$

so we only have to prove the reverse inclusion. Let $\lambda \notin \Theta$, $p \in \mathbb{P}^\Psi_{\aleph_1}$ and $\langle \dot{T}_\alpha \mid \alpha < \lambda \rangle$ be a family of $\mathbb{P}^\Psi_{\aleph_1}$-names such that

$$p \Vdash \langle \dot{T}_\alpha \mid \alpha < \lambda \rangle \text{ is an almost disjoint family trees.}$$

Since trees can be coded by reals we may assume that $\dot{T}_\alpha$ is a nice $\mathbb{P}^{\Psi}_{\aleph_1}$-name as in Definition 6.1. By assumption on Θ and GCH there is a regular uncountable cardinal $\sigma \leq \lambda$ with $[\mu, \lambda] \cap \Theta = \emptyset$ and such that for all $\mu < \sigma$ we have $\mu^{\aleph_0} < \sigma$. Now, fix $\alpha < \lambda$. We define $\Theta_\alpha := \operatorname{hsupp}_\Theta(\dot{T}_\alpha)$, $D_\alpha := \operatorname{hsupp}(\dot{T}_\alpha)$ and for every $\theta \in \Theta$ let $D_\alpha(\theta) := D_\alpha \cap (\{\theta\} \times V)$. By possibly extending Θ_α and D_α we may assume $\operatorname{hsupp}_\Theta(p) \subseteq \Theta_\alpha$ and $\operatorname{hsupp}(p) \subseteq D_\alpha$. Then, $\langle \Theta_\alpha \mid \alpha < \sigma \rangle$ satisfies the assumptions of the generalized Δ-system lemma:

- $\langle \Theta_\alpha \mid \alpha < \sigma \rangle$ is a family of size σ,
- $|\Theta_\alpha| < \aleph_1$ for all $\alpha < \sigma$,
- $\aleph_1 < \sigma$ and for all $\mu < \sigma$ we have $\mu^{<\aleph_1} = \mu^{\aleph_0} < \sigma$.

Choose $I_0 \in [\sigma]^\sigma$ and Θ_R such that $\{\Theta_\alpha \mid \alpha \in I_0\}$ is a Δ-system lemma with root Θ_R. Since $|\Theta| = \max(\Theta) > \sigma$, we may assume that we extended every Θ_α for $\alpha \in I_0$ such that

(1) Θ_α is still countable and $\{\Theta_\alpha \mid \alpha \in I_0\}$ is still a Δ-system with root Θ_R,
(2) For every $\alpha \in I_0$ we have $|\Theta_\alpha \setminus \Theta_R| = \aleph_0$.

Next, also $\{D_\alpha \mid \alpha \in I_0\}$ satisfies the assumptions of the generalized Δ-system lemma:

- $\{D_\alpha \mid \alpha \in I_0\}$ is a family of size σ,
- $|D_\alpha| < \aleph_1$ for all $\alpha \in I_0$,
- $\aleph_1 < \sigma$ and for all $\mu < \sigma$ we have $\mu^{<\aleph_1} = \mu^{\aleph_0} < \sigma$.

Choose $I_1 \in [I_0]^\sigma$ and R such that $\{D_\alpha \mid \alpha \in I_1\}$ is a Δ-system lemma with root R. For every $\theta \in \Theta$ let $R(\theta) := R \cap (\{\theta\} \times V)$. For every $\theta > \sigma$ we have $|\Psi(\theta)| > \sigma$, so we may assume that we extended every D_α for $\alpha \in I_1$ such that

(3) D_α is still countable and $\{D_\alpha \mid \alpha \in I_1\}$ is still a Δ-system with root R,
(4) For every $\alpha \in I_1$ and $\theta \in \Theta_R$ with $\theta > \sigma$ we have $|D_\alpha(\theta) \setminus R(\theta)| = \aleph_0$,
(5) For every $\alpha \in I_1$ and $\theta \in \Theta_\alpha \setminus \Theta_R$ we have $|D_\alpha(\theta)| = \aleph_0$.

Now, set $I_2 := \{\alpha \in I_1 \mid \text{For all } \theta \in \Theta_R \text{ with } \theta < \sigma \text{ we have } D_\alpha(\theta) \subseteq R(\theta)\}$. Then, $I_2 \in [I_1]^\sigma$ as for every $\theta \in \Theta_R$ with $\theta < \sigma$ there are only $<\sigma$-many $\alpha \in I_1$ with $D_\alpha(\theta) \setminus R(\theta) \neq \emptyset$, since $|\Psi(\theta)| = \theta$ and $\{D_\alpha \mid \alpha \in I_1\}$ is a Δ-system of size $\sigma > \theta$. Thus, we obtain

(6) For every $\alpha \in I_2$ and $\theta \in \Theta_R$ with $\theta < \sigma$ we have $D_\alpha(\theta) = R(\theta)$.

We extend our Δ-system by one more element as follows. Choose $\Theta_\lambda \subseteq \Theta$ countable such that $\Theta_R \subseteq \Theta_\lambda$, $|\Theta_\lambda \setminus \Theta_R| = \aleph_0$ and for all $\alpha < \lambda$ we have $\Theta_\lambda \cap \Theta_\alpha = \Theta_R$. This is possible since $|\Theta| = \max(\Theta) > \lambda$. Now, for $\theta \in \Theta$ we define $D_\lambda(\theta)$ as follows:

- If $\theta \in \Theta_R$ and $\theta < \sigma$ define $D_\lambda(\theta) := R(\theta)$,
- If $\theta \in \Theta_R$ and $\theta > \sigma$ we have $|\Psi(\theta)| = \theta > \lambda$, so choose $D_\lambda(\theta) \subseteq (\{\theta\} \times \Psi(\theta))$ countable with $R(\theta) \subseteq D_\lambda(\theta)$, $|D_\lambda(\theta) \setminus R(\theta)| = \aleph_0$ and for all $\alpha < \lambda$ we have $D_\lambda(\theta) \cap D_\alpha(\theta) = R(\theta)$,
- If $\theta \in \Theta_\lambda \setminus \Theta_R$ choose any countable subset $D_\lambda(\theta) \subseteq (\{\theta\} \times \Psi(\theta))$,
- If $\theta \in \Theta \setminus \Theta_\lambda$ set $D_\lambda(\theta) := \emptyset$.

Finally, we define $D_\lambda := \bigcup_{\theta \in \Theta} D_\lambda(\theta)$. By choice of Θ_λ we have that $\{\Theta_\alpha \mid \alpha \in I_2 \cup \{\lambda\}\}$ is a Δ-system with root Θ_R and similarly by choice of D_λ also $\{D_\alpha \mid \alpha \in I_2 \cup \{\lambda\}\}$ is a Δ-system with root R and properties (1) to (6) still hold for every $\alpha \in I_2 \cup \{\lambda\}$. Next, we define a Θ-subindexing function Φ_R of Ψ by $\operatorname{dom}(\Phi_R) := \Theta_R$ and for $\theta \in \Theta_R$ by

$$\Phi_R(\theta) := \{i \in \Psi(\theta) \mid (\theta, i) \in R(\theta)\}.$$

Analogously, for every $\alpha \in \lambda \cup \{\lambda\}$ define a Θ-subindexing function Φ_α of Ψ by $\operatorname{dom}(\Phi_\alpha) := \Theta_\alpha$ and for $\theta \in \Theta_\alpha$ by

$$\Phi_\alpha(\theta) := \{i \in \Psi(\theta) \mid (\theta, i) \in D_\alpha(\theta)\}.$$

As Θ_R and R are roots of their respective Δ-system we obtain $\Phi_R \subseteq \Phi_\alpha$ for every $\alpha \in I_2 \cup \{\lambda\}$. Since, $\operatorname{hsupp}(\dot{T}_\alpha) \subseteq D_\alpha$ we may pick a nice $\mathbb{P}^{\Phi_\alpha}_{\aleph_1}$-name $\dot{T}^*_\alpha$ with $\iota^{\Phi_\alpha, \Psi}_{\aleph_1}(\dot{T}^*_\alpha) = \dot{T}_\alpha$. Further, by (2) we may fix bijections $\langle g_\alpha : \Theta_\alpha \to \omega \mid \alpha \in I_2 \cup \{\lambda\}\rangle$ such that $g_\alpha \restriction \Theta_R = g_\beta \restriction \Theta_R$ for all $\alpha, \beta \in I_2 \cup \{\lambda\}$. Then, for $\alpha, \beta \in I_2 \cup \{\lambda\}$ we define $g_{\alpha,\beta} : \Theta_\alpha \to \Theta_\beta$ by

$$g_{\alpha,\beta}(\theta) := g_\beta^{-1}(g_\alpha(\theta)).$$

Note that $\Theta_\alpha \cap \Theta_\beta = \Theta_R$ and $g_\alpha \restriction \Theta_R = g_\beta \restriction \Theta_R$ implies that

$$g_{\alpha,\beta}(\theta) = g_\beta^{-1}(g_\alpha(\theta)) = \theta$$

for all $\theta \in \Theta_R$ and $\alpha, \beta \in I_2 \cup \{\lambda\}$. Hence, it is easy to verify that we obtain a system of bijections $\{g_{\alpha,\beta} : \Theta_\alpha \to \Theta_\beta \mid \alpha, \beta \in I_2 \cup \{\lambda\}\}$ with the following properties for all $\alpha, \beta, \gamma \in I_2 \cup \{\lambda\}$:

- (G1) $g_{\alpha,\alpha} = \operatorname{id}_{\Theta_\alpha}$ and $g^{-1}_{\alpha,\beta} = g_{\beta,\alpha}$,
- (G2) for all $\theta \in \Theta_R$ we have $g_{\alpha,\beta}(\theta) = \theta$,
- (G3) $g_{\alpha,\gamma} = g_{\beta,\gamma} \circ g_{\alpha,\beta}$.

Next, for every $\alpha \in I_2 \cup \{\lambda\}$ and $\theta \in \Theta_\alpha$ we may fix a bijection $h^\theta_\alpha : \Phi_\alpha(\theta) \to \omega$ such that for all $\alpha, \beta \in I_2 \cup \{\lambda\}$, $\theta \in \Theta_R$ and $i \in R(\theta)$ we have $h^\theta_\alpha(i) = h^\theta_\beta(i)$. This is possible, since by (4) and (6) we have $|D_\alpha(\theta) \setminus R(\theta)| = |D_\beta(\theta) \setminus R(\theta)|$ for every $\theta \in \Theta_R$. Then, for $\alpha, \beta \in I_2 \cup \{\lambda\}$ and $\theta \in \Theta_\alpha$ we define a map $h^\theta_{\alpha,\beta} : \Phi_\alpha(\theta) \to \Phi_\beta(g_{\alpha,\beta}(\theta))$ for $i \in \Phi_\alpha(\theta)$ by

$$h^\theta_{\alpha,\beta}(i) := ((h^{g_{\alpha,\beta}(\theta)}_\beta)^{-1} \circ h^\theta_\alpha)(i).$$

We verify the following properties for all $\alpha, \beta, \gamma \in I_2 \cup \{\lambda\}$ and $\theta \in \Theta_\alpha$:

- (H1) $h^\theta_{\alpha,\alpha} = \operatorname{id}_{\Phi_\alpha(\theta)}$ and the map $h^\theta_{\alpha,\beta} : \Phi_\alpha(\theta) \to \Phi_\beta(g_{\alpha,\beta}(\theta))$ is a bijection with inverse $h^{g_{\alpha,\beta}(\theta)}_{\beta,\alpha}$,
- (H2) for all $i \in R(\theta)$ we have $h^\theta_{\alpha,\beta}(i) = i$,
- (H3) $h^\theta_{\alpha,\gamma} = h^{g_{\alpha,\beta}(\theta)}_{\beta,\gamma} \circ h^\theta_{\alpha,\beta}$.

Proof.

(H1) Let $i \in \Phi_\alpha(\theta)$. Then, $g_{\alpha,\alpha}(\theta) = \theta$ by (G3), so that

$$h^\theta_{\alpha,\alpha}(i) = ((h^{g_{\alpha,\alpha}(\theta)}_\alpha)^{-1} \circ h^\theta_\alpha)(i) = ((h^\theta_\alpha)^{-1} \circ h^\theta_\alpha)(i) = i.$$

Next, by definition we have $h^{g_{\alpha,\beta}(\theta)}_{\beta,\alpha} : \Phi_\beta(g_{\alpha,\beta}(\theta)) \to \Phi_\alpha(g_{\beta,\alpha}(g_{\alpha,\beta}(\theta)))$. Further, by (G1) $g_{\beta,\alpha}(g_{\alpha,\beta}(\theta)) = \theta$, so that $h^{g_{\alpha,\beta}(\theta)}_{\beta,\alpha} : \Phi_\beta(g_{\alpha,\beta}(\theta)) \to \Phi_\alpha(\theta)$, i.e. the domains are correct.

Now, let $i \in \Phi_\alpha(\theta)$. Then, we compute

$$\begin{aligned}(h^{g_{\alpha,\beta}(\theta)}_{\beta,\alpha} \circ h^\theta_{\alpha,\beta})(i) &= ((h^{g_{\beta,\alpha}(g_{\alpha,\beta}(\theta))}_\alpha)^{-1} \circ h^{g_{\alpha,\beta}(\theta)}_\beta \circ (h^{g_{\alpha,\beta}(\theta)}_\beta)^{-1} \circ h^\theta_\alpha)(i) \\ &= ((h^\theta_\alpha)^{-1} \circ h^\theta_\alpha)(i) = i.\end{aligned}$$

Analogously, for $i \in \Phi_\beta(g_{\alpha,\beta}(\theta))$ we compute

$$\begin{aligned}(h^\theta_{\alpha,\beta} \circ h^{g_{\alpha,\beta}(\theta)}_{\beta,\alpha})(i) &= ((h^{g_{\alpha,\beta}(\theta)}_\beta)^{-1} \circ h^\theta_\alpha \circ (h^{g_{\beta,\alpha}(g_{\alpha,\beta}(\theta))}_\alpha)^{-1} \circ h^{g_{\alpha,\beta}(\theta)}_\beta)(i) \\ &= ((h^{g_{\alpha,\beta}(\theta)}_\beta)^{-1} \circ h^\theta_\alpha \circ (h^\theta_\alpha)^{-1} \circ h^{g_{\alpha,\beta}(\theta)}_\beta)(i) \\ &= ((h^{g_{\alpha,\beta}(\theta)}_\beta)^{-1} \circ h^{g_{\alpha,\beta}(\theta)}_\beta)(i) = i.\end{aligned}$$

(H2) Let $i \in R(\theta)$. Then, $\theta \in \Theta_R$ and $g_{\alpha,\beta}(\theta) = \theta$ by (G2). Hence, by choice of the bijections

$$\begin{aligned}h^\theta_{\alpha,\beta}(i) &= ((h^{g_{\alpha,\beta}(\theta)}_\beta)^{-1} \circ h^\theta_\alpha)(i) \\ &= ((h^\theta_\beta)^{-1} \circ h^\theta_\alpha)(i) \\ &= ((h^\theta_\beta)^{-1} \circ h^\theta_\beta)(i) \\ &= i.\end{aligned}$$

(H3) Finally, let $\theta \in \Theta_\alpha$ and $i \in \Phi_\alpha(\theta)$. Then, $g_{\alpha,\gamma} = g_{\beta,\gamma} \circ g_{\alpha,\beta}$ by (G3), so we compute

$$\begin{aligned}(h^{g_{\alpha,\beta}(\theta)}_{\beta,\gamma} \circ h^\theta_{\alpha,\beta})(i) &= ((h^{g_{\beta,\gamma}(g_{\alpha,\beta}(\theta))}_\gamma)^{-1} \circ h^{g_{\alpha,\beta}(\theta)}_\beta \circ (h^{g_{\alpha,\beta}(\theta)}_\beta)^{-1} \circ h^\theta_\alpha)(i) \\ &= ((h^{g_{\alpha,\gamma}(\theta)}_\gamma)^{-1} \circ h^\theta_\alpha)(i) \\ &= h^\theta_{\alpha,\gamma}(i).\end{aligned}$$

□

Now, for all $\alpha, \beta \in I_2 \cup \{\lambda\}$ define $\mathbf{x}_{\alpha,\beta} := (g_{\alpha,\beta}, \{h^\theta_{\alpha,\beta} \mid \theta \in \Theta_\alpha\})$. Then, the properties (G1) to (G3) and (H1) to (H3) may be rephrased as a system of isomorphisms of Θ-indexing functions $\langle \mathbf{x}_{\alpha,\beta} \mid \alpha, \beta \in I_2 \cup \{\lambda\}\rangle$, which for all $\alpha, \beta, \gamma \in I_2 \cup \{\lambda\}$ satisfies

(K1') $\mathbf{x}_{\alpha,\alpha} = \mathbf{1}_{\Phi_\alpha}$ and $\mathbf{x}^{-1}_{\alpha,\beta} = \mathbf{x}_{\beta,\alpha}$,
(K2') $\mathbf{x}_{\alpha,\alpha} \restriction \Phi_R = \mathbf{1}_{\Phi_R}$,
(K3') $\mathbf{x}_{\alpha,\gamma} = \mathbf{x}_{\beta,\gamma} \circ \mathbf{x}_{\alpha,\beta}$.

Applying the functor $\kappa^{\aleph_1}_\bullet$ from Corollary 8.14 to the system $\langle \mathbf{x}_{\alpha,\beta} \mid \alpha, \beta \in I_2 \cup \{\lambda\}\rangle$, we obtain a system of isomorphisms $\langle \kappa_{\mathbf{x}_{\alpha,\beta}} : \mathbb{P}^{\Phi_\alpha}_{\aleph_1} \to \mathbb{P}^{\Phi_\beta}_{\aleph_1} \mid \alpha, \beta \in I_2 \cup \{\lambda\}\rangle$ which satisfies

(K1) $\kappa_{\mathbf{x}_{\alpha,\alpha}} = \mathrm{id}_{\mathbb{P}^{\Phi_\alpha}_{\aleph_1}}$ and $\kappa^{-1}_{\mathbf{x}_{\alpha,\beta}} = \kappa_{\mathbf{x}_{\beta,\alpha}}$,
(K2) $\kappa_{\mathbf{x}_{\alpha,\beta}} \circ \iota^{\Phi_R,\Phi_\alpha}_{\aleph_1} = \iota^{\Phi_R,\Phi_\beta}_{\aleph_1}$,
(K3) $\kappa_{\mathbf{x}_{\alpha,\gamma}} = \kappa_{\mathbf{x}_{\beta,\gamma}} \circ \kappa_{\mathbf{x}_{\alpha,\beta}}$.

Fix $\beta_0 \in I_2$. For every $\alpha \in I_2$ we have that $\dot{T}^*_\alpha$ is a nice $\mathbb{P}^{\Phi_\alpha}_{\aleph_1}$-name for a tree. Thus, $\kappa_{\mathbf{x}_{\alpha,\beta_0}}(\dot{T}^*_\alpha)$ is a nice $\mathbb{P}^{\Phi_{\beta_0}}_{\aleph_1}$-name for a tree. However, Φ_{β_0} is countable, so by Lemma 6.9 there are only $\aleph_1$-many nice $\mathbb{P}^{\Phi_{\beta_0}}_{\aleph_1}$-names for such trees. Thus, choose $I_3 \in [I_2]^\sigma$ such that $\kappa_{\mathbf{x}_{\alpha,\beta_0}}(\dot{T}^*_\alpha) = \kappa_{\mathbf{x}_{\alpha',\beta_0}}(\dot{T}^*_{\alpha'})$ for all $\alpha, \alpha' \in I_3$.

Next, choose any $\alpha_0 \in I_3$ and define $\dot{T}^*_\lambda$ to be $\kappa_{\mathbf{x}_{\alpha_0,\lambda}}(\dot{T}^*_{\alpha_0})$. Then, $\dot{T}^*_\lambda$ is a nice $\mathbb{P}^{\Phi_\lambda}_{\aleph_1}$-name for a tree. We show that this definition is independent of the choice of $\alpha_0 \in I_3$, so let $\alpha \in I_3$. Then, we compute

$$\begin{aligned}
\dot{T}^*_\lambda &= \kappa_{\mathbf{x}_{\alpha_0,\lambda}}(\dot{T}^*_{\alpha_0}) && \text{(definition of } \dot{T}^*_\lambda) \\
&= \kappa_{\mathbf{x}_{\beta_0,\lambda}}(\kappa_{\mathbf{x}_{\alpha_0,\beta_0}}(\dot{T}^*_{\alpha_0})) && \text{(K3)} \\
&= \kappa_{\mathbf{x}_{\beta_0,\lambda}}(\kappa_{\mathbf{x}_{\alpha,\beta_0}}(\dot{T}^*_{\alpha})) && (\alpha \in I_3) \\
&= \kappa_{\mathbf{x}_{\alpha,\lambda}}(\dot{T}^*_{\alpha}) && \text{(K3)}.
\end{aligned}$$

Finally, let $\beta < \lambda$. Since $\{\Theta_\alpha \mid \alpha \in I_3\}$ is a Δ-system with root Θ_R and Θ_β is countable, there can only be countably many $\alpha \in I_3$ with $\Theta_\alpha \cap \Theta_\beta \not\subseteq \Theta_R$. Further, since $\{D_\alpha \mid \alpha \in I_3\}$ is a Δ-system with root R and for every $\theta \in \Theta_R$ the set $\Phi_\beta(\theta)$ is countable, there can only be countably many $\alpha \in I_3$ with $\Phi_\alpha(\theta) \cap \Phi_\beta(\theta) \not\subseteq R(\theta)$. Thus, we may choose $\alpha \in I_3 \setminus \{\beta\}$ such that $\Theta_\alpha \cap \Theta_\beta \subseteq \Theta_R$ and for all $\theta \in \Theta_R$ we have $\Phi_\alpha(\theta) \cap \Phi_\beta(\theta) \subseteq R(\theta)$. By definition of Θ_λ we also have $\Theta_\lambda \cap \Theta_\beta \subseteq \Theta_R$ and for all $\theta \in \Theta_R$ also $\Phi_\lambda(\theta) \cap \Phi_\beta(\theta) \subseteq R(\theta)$. For $\nu \in \{\alpha, \lambda\}$ we define a Θ-subindexing function Φ^*_ν of Ψ by $\mathrm{dom}(\Phi^*_\nu) := \Theta_\nu \cup \Theta_\beta$ and for $\theta \in \Theta^*_\nu$ by

$$\Phi^*_\nu(\theta) := \Phi_\nu(\theta) \cup \Phi_\beta(\theta),$$

where every undefined set is treated as the empty set. We define a bijection $g^*_{\alpha,\lambda} : \Theta^*_\alpha \to \Theta^*_\lambda$ for $\theta \in \Theta^*_\alpha$ by

$$g^*_{\alpha,\lambda}(\theta) := \begin{cases} g_{\alpha,\lambda}(\theta) & \text{if } \theta \in \Theta_\alpha, \\ \theta & \text{if } \theta \in \Theta_\beta. \end{cases}$$

This is well-defined by (G2) and $\Theta_\alpha \cap \Theta_\beta \subseteq \Theta_R$. Further, for every $\theta \in \Theta^*_\alpha$ define a bijection $h^{\theta,*}_{\alpha,\beta} : \Phi^*_\alpha(\theta) \to \Phi^*_\beta(g^*_{\alpha,\lambda}(\theta))$ as follows:

- If $\theta \in \Theta_R$ we have $\Phi^*_\alpha(\theta) \setminus \Phi_\alpha(\theta) = \Phi^*_\lambda(\theta) \setminus \Phi_\lambda(\theta)$ and $g^*_{\alpha,\lambda}(\theta) = \theta$, so we may extend the bijection $h^\theta_{\alpha,\lambda} : \Phi_\alpha(\theta) \to \Phi_\lambda(\theta)$ to $h^{\theta,*}_{\alpha,\beta} : \Phi^*_\alpha(\theta) \to \Phi^*_\beta(\theta)$ by

$$h^{\theta,*}_{\alpha,\lambda}(i) = \begin{cases} h^\theta_{\alpha,\lambda}(i) & \text{if } i \in \Phi_\alpha(\theta), \\ i & \text{if } i \in \Phi_\beta(\theta). \end{cases}$$

  This is well-defined by (H2) and $\Phi_\alpha(\theta) \cap \Phi_\beta(\theta) \subseteq R(\theta)$.
- If $\theta \in \Theta_\alpha \setminus \Theta_R$, then we have $\Phi^*_\alpha(\theta) = \Phi_\alpha(\theta)$, $\Phi^*_\lambda(\theta) = \Phi_\lambda(\theta)$, so we may define $h^{\theta,*}_{\alpha,\lambda} = h^\theta_{\alpha,\lambda}$.
- If $\theta \in \Theta_\beta \setminus \Theta_R$, then we have $\Phi^*_\alpha(\theta) = \Phi_\beta(\theta) = \Phi^*_\lambda(\theta)$ and $g^*_{\alpha,\lambda}(\theta) = \theta$, so we may define $h^{\theta,*}_{\alpha,\lambda} = \mathrm{id}_{\Phi^*_\alpha(\theta)}$.

Then, the tuple $\mathbf{x}^*_{\alpha,\lambda} = (g^*_{\alpha,\lambda}, \{h^{\theta,*}_{\alpha,\lambda} \mid \theta \in \Theta^*_\alpha\})$ is an isomorphism from Φ^*_α to Φ^*_λ. Further, we have $\Phi_\alpha, \Phi_\beta \subseteq \Phi^*_\alpha$ and $\Phi_\lambda, \Phi_\beta \subseteq \Phi^*_\lambda$ as well as

(L1') $\mathbf{x}^*_{\alpha,\lambda} \restriction \Phi_\alpha = \mathbf{x}_{\alpha,\lambda}$,
(L2') $\mathbf{x}^*_{\alpha,\lambda} \restriction \Phi_\beta = \mathbf{1}_{\Phi_\beta}$.

By Lemma 8.18 applying $\kappa^{\aleph_1}_\bullet$ from Corollary 8.14 yields an automorphism $\kappa_{\mathbf{x}^*_{\alpha,\lambda}} : \mathbb{P}^{\Phi^*_\alpha}_{\aleph_1} \to \mathbb{P}^{\Phi^*_\lambda}_{\aleph_1}$ with the following properties:

(L1) $\kappa_{\mathbf{x}^*_{\alpha,\lambda}} \circ \iota^{\Phi_\alpha,\Phi^*_\alpha}_{\aleph_1} = \iota^{\Phi_\lambda,\Phi^*_\lambda}_{\aleph_1} \circ \kappa_{\mathbf{x}_{\alpha,\lambda}}$,

(L2) $\kappa_{\mathbf{x}^*_{\alpha,\lambda}} \circ \iota^{\Phi_\beta,\Phi^*_\alpha}_{\aleph_1} = \iota^{\Phi_\beta,\Phi^*_\lambda}_{\aleph_1}$.

Since $\mathrm{hsupp}_\Theta(p) \subseteq \Theta_R$ and $\mathrm{hsupp}(p) \subseteq R$, choose $p^* \in \mathbb{P}^{\Phi_R}_{\aleph_1}$ with $\iota^{\Phi_R,\Psi}_{\aleph_1}(p^*) = p$. Then we have

$$\begin{aligned}
\kappa_{\mathbf{x}^*_{\alpha,\lambda}}(\iota^{\Phi_R,\Phi^*_\alpha}_{\aleph_1}(p^*)) &= \kappa_{\mathbf{x}^*_{\alpha,\lambda}}(\iota^{\Phi_\beta,\Phi^*_\alpha}_{\aleph_1}(\iota^{\Phi_R,\Phi_\beta}_{\aleph_1}(p^*))) && (\Phi_R \subseteq \Phi_\beta \subseteq \Phi^*_\alpha)\\
&= \iota^{\Phi_\beta,\Phi^*_\lambda}_{\aleph_1}(\iota^{\Phi_R,\Phi_\beta}_{\aleph_1}(p^*)) && \text{(L2)}\\
&= \iota^{\Phi_R,\Phi^*_\lambda}_{\aleph_1}(p^*) && (\Phi_R \subseteq \Phi_\beta \subseteq \Phi^*_\lambda).
\end{aligned}$$

Similarly, by (L2) we have $\kappa_{\mathbf{x}^*_{\alpha,\lambda}}(\iota^{\Phi_\beta,\Phi^*_\alpha}_{\aleph_1}(\dot{T}^*_\beta)) = \iota^{\Phi_\beta,\Phi^*_\lambda}_{\aleph_1}(\dot{T}^*_\beta)$. We also compute

$$\begin{aligned}
\kappa_{\mathbf{x}^*_{\alpha,\lambda}}(\iota^{\Phi_\alpha,\Phi^*_\alpha}_{\aleph_1}(\dot{T}^*_\alpha)) &= \iota^{\Phi_\lambda,\Phi^*_\lambda}_{\aleph_1}(\kappa_{\mathbf{x}_{\alpha,\lambda}}(\dot{T}^*_\alpha)) && \text{(L1)}\\
&= \iota^{\Phi_\lambda,\Phi^*_\lambda}_{\aleph_1}(\dot{T}^*_\lambda) && (\alpha \in I_3).
\end{aligned}$$

We may now finish the argument. Since

$$p \Vdash_{\mathbb{P}^\Psi_{\aleph_1}} [\dot{T}_\alpha] \cap [\dot{T}_\beta] = \emptyset,$$

we have

$$\iota^{\Phi^*_\alpha,\Psi}_{\aleph_1}(\iota^{\Phi_R,\Phi^*_\alpha}_{\aleph_1}(p^*)) \Vdash_{\mathbb{P}^\Psi_{\aleph_1}} [\iota^{\Phi^*_\alpha,\Psi}_{\aleph_1}(\iota^{\Phi_\alpha,\Phi^*_\alpha}_{\aleph_1}(\dot{T}^*_\alpha))] \cap [\iota^{\Phi^*_\alpha,\Psi}_{\aleph_1}(\iota^{\Phi_\beta,\Phi^*_\alpha}_{\aleph_1}(\dot{T}^*_\beta))] = \emptyset.$$

By Theorem 7.1 we may use downwards absoluteness to obtain

$$\iota^{\Phi_R,\Phi^*_\alpha}_{\aleph_1}(p^*) \Vdash_{\mathbb{P}^{\Phi^*_\alpha}_{\aleph_1}} [\iota^{\Phi_\alpha,\Phi^*_\alpha}_{\aleph_1}(\dot{T}^*_\alpha)] \cap [\iota^{\Phi_\beta,\Phi^*_\alpha}_{\aleph_1}(\dot{T}^*_\beta)] = \emptyset.$$

Applying the isomorphism $\kappa_{\mathbf{x}^*_{\alpha,\lambda}} : \mathbb{P}^{\Phi^*_\alpha}_{\aleph_1} \to \mathbb{P}^{\Phi^*_\lambda}_{\aleph_1}$ and the computation above yields

$$\iota^{\Phi_R,\Phi^*_\lambda}_{\aleph_1}(p^*) \Vdash_{\mathbb{P}^{\Phi^*_\lambda}_{\aleph_1}} [\iota^{\Phi_\lambda,\Phi^*_\lambda}_{\aleph_1}(\dot{T}^*_\lambda)] \cap [\iota^{\Phi_\beta,\Phi^*_\lambda}_{\aleph_1}(\dot{T}^*_\beta)] = \emptyset.$$

By Theorem 7.1 we may use Π^1_1-absoluteness to obtain

$$p \Vdash_{\mathbb{P}^\Psi_{\aleph_1}} [\dot{T}_\lambda] \cap [\dot{T}_\beta] = \emptyset,$$

so that

$$p \Vdash_{\mathbb{P}^\Psi_{\aleph_1}} \langle \dot{T}_\alpha \mid \alpha < \lambda \rangle \text{ is not maximal.}$$

□

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TIGHT COFINITARY GROUPS

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ABSTRACT. We introduce the notion of a tight cofinitary group, which captures forcing indestructibility of maximal cofinitary groups for a long list of partial orders, including Cohen, Sacks, Miller, Miller partition forcing and Shelah's poset for diagonalizing maximal ideals. Introducing a new robust coding technique, we establish the relative consistency of $\mathfrak{a}_g = \mathfrak{d} < \mathfrak{c} = \aleph_2$ alongside the existence of a Δ^1_3-well-order of the reals and a co-analytic witness for $\mathfrak{a}_g$.

1. INTRODUCTION

The study of the projective complexity of maximal cofinitary groups has already a comparatively long history. In 2008 [8] Gao and Zhang show that in the constructible universe L there is a maximal cofinitary group with a co-analytic set of generators. The result was improved a year later by Kastermans [11] who constructed a co-analytic maximal cofinitary group (in L). In both of those results the existence of a Σ^1_2 definable projective well-order of the reals, as well as the Continuum Hypothesis, play a crucial role, thus leaving aside the problem of providing projective maximal cofinitary groups of low projective complexity in models of large continuum.

The study of models of $\neg\,\mathrm{CH}$ with a projective well-order on the reals has a much longer history, initiated by the work of Leo Harrington [9], in which he obtained a model of $\mathfrak{c} = \aleph_2$ with a $\boldsymbol{\Delta^1_3}$-well-order of the reals. In [3], the first author of the paper and S. D. Friedman, introduced the method of coding with perfect trees to obtain models of $\mathfrak{c} = \aleph_2$, a light-face Δ^1_3-well-order of the reals and various cardinal characteristics constellations. Note that by a result of Mansfield if there is a Σ^1_2-well-order of the reals, then every real is constructible and so the existence of a Δ^1_3-well-order on the reals is optimal for models of $\mathfrak{c} > \aleph_1$. The well-order can be used to produce Σ^1_3-definable set of reals of interest of cardinality Σ^1_3, including maximal cofinitary groups. But the complexity of the maximal cofinitary group thus obtained is not lowest possible: Indeed, Horowitz and Shelah in [10] show that there is always a Borel maximal cofinitary group (which is necessarily of cardinality $\mathfrak{c}$). In [6], the first and third author of the current paper jointly with A. Törnquist construct a co-analytic maximal cofinitary group which is Cohen indestructible. Thus, they obtain a model in which $\mathfrak{a}_g = \mathfrak{b} < \mathfrak{d} = \mathfrak{c} = \kappa$ where κ can be an arbitrarily large cardinal of uncountable cofinality and $\mathfrak{a}_g$ is witnessed by a maximal cofinitary group of optimal projective complexity. Providing a model of $\mathfrak{d} = \mathfrak{a}_g = \aleph_1 < \mathfrak{c}$ with an optimal projective witness to $\mathfrak{a}_g$ remained open until the current paper and required significantly different ideas.

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Inspired by the construction of a tight almost disjoint family, in [7], Fischer and Switzer introduce the notion of a tight eventually different family of reals. Tight eventually different families are maximal in a strong sense, as they are indestructible by a long list of partial orders, including Cohen, Sacks (its products and countable support iterations), Miller, Miller partition forcing and others. They are never analytic, exist under CH and $\mathsf{MA}(\sigma\text{-centered})$, and are used in [7] to provide co-analytic witnesses to $\mathfrak{a}_e = \aleph_1$ in various forcing extensions. Moreover, in [1] the authors show that Sacks coding preserves (in a strong sense) co-analytic tight eventually different families leading to models of $\mathfrak{c} = \aleph_2$ with a Δ^1_3-well-order of the reals and co-analytic witnesses of $\mathfrak{a}_e = \aleph_1$. However, the consistency of the existence of a co-analytic maximal cofinitary group of size $\aleph_1$ in the presence of a Δ^1_3-well-order of the reals and $\mathfrak{c} = \aleph_2$ remained open until the current paper.

The paper is structured as follows: In section 2 we introduce the notion of a tight cofinitary group. Tight cofinitary groups are maximal in the strong sense that they are indestructible by a long list of partial orders. In section 3 we introduce a slight modification of Zhang's poset for adding a new generator to a cofinitary group and show that under $\mathsf{MA}(\sigma\text{-centered})$ tight cofinitary groups exist and moreover that every cofinitary group of size $< \mathfrak{c}$ is contained in a tight cofinitary group of size $\mathfrak{c}$. In Section 4, we argue that tight cofinitary groups are universally Sacks indestructible (see also [4]) and obtain our first main result, proving that Sacks coding strongly preserves tightness of cofinitary groups. In Section 5 we introduce a robust new coding technique allowing us to code reals into the lengths of orbits of every new word. Crucially, compared to other coding techniques for cofinitary groups (i.e. as in [5]) our new coding is parameter-less and hence may be applied to groups of uncountable size. Furthermore, as we code into orbits rather than actual function values, a more general generic hitting lemma (see Lemma 13) required for tightness holds. As an application of this new coding, we obtain our second main result, namely the existence of a co-analytic tight cofinitary group in L. Our results, together with earlier investigation into the existence of nicely definable, forcing indestructible combinatorial sets of reals of interest, imply that each of the constellations $\mathfrak{a}_g = \mathfrak{u} = \mathfrak{i} = \aleph_1 < \mathfrak{c} = \aleph_2$, $\mathfrak{a}_g = \mathfrak{u} = \aleph_1 < \mathfrak{i} = \mathfrak{c} = \aleph_2$, $\mathfrak{a}_g = \mathfrak{i} = \aleph_1 < \mathfrak{u} = \mathfrak{c} = \aleph_2$, $\mathfrak{a}_g = \aleph_1 < \mathfrak{i} = \mathfrak{u} = \aleph_2$ can hold alongside a Δ^1_3-well-order of the reals, a co-analytic tight cofinitary group of cardinality $\aleph_1$, as well as a co-analytic filter base of cardinality $\aleph_1$ in the first and second constellations, and a co-analytic selective independent family in the first and third constellations.

In particular, we give a new proof of the existence of a Miller indestructible maximal cofinitary group (originally obtained in [12] using a diamond sequence) and answer Question 2 of [7].

2. Tight cofinitary groups

We will need the notion of a tight eventually different family of permutations as defined in [7]. A tree $T \subseteq \omega^{<\omega}$ is called injective iff every element of T is injective. Then, for a family $\mathcal{P} \subseteq \omega^\omega$ of eventually different permutations we define:

Definition 1. The injective tree ideal generated by $\mathcal{P}$, denoted $\mathcal{I}_i(\mathcal{P})$, is the set of all injective trees $T \subseteq \omega^{<\omega}$ such that there is a $s \in T$ and a finite set $\mathcal{P}_0 \subseteq \mathcal{P}$, so that for all $t \in T$ with $s \trianglelefteq t$

and $k \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ there is a $f \in \mathcal{P}_0$ with $t(k) = f(k)$. Dually, we denote with $\mathcal{I}_i(\mathcal{P})^+$ all injective trees $T \subseteq \omega^{<\omega}$ not in $\mathcal{I}_i(\mathcal{P})$.

Note that despite its name $\mathcal{I}_i(\mathcal{P})$ is not an ideal nor does it generate one. The purpose of this naming is its analogical role as the associated ideal for tight mad families.

Definition 2. Let $T \subseteq \omega^{<\omega}$ be a tree and $g \in \omega^\omega$. Then, we say g densely diagonalizes T iff for all $s \in T$ there is a $t \in T$ with $s \trianglelefteq t$ and $k \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ such that $t(k) = g(k)$.

Definition 3. An eventually different family of permutations $\mathcal{P}$ is called tight iff for all sequences $\langle T_n \in \mathcal{I}_i(\mathcal{P})^+ \mid n < \omega \rangle$ there is a $g \in \mathcal{P}$ such that g diagonalizes T_n for every $n < \omega$.

Since every cofinitary group is a family of eventually different permutations we may define tightness in terms of eventually different permutations.

Definition 4. Let G be a cofinitary group. Then we say G is tight iff G is tight as an eventually different family of permutations.

Remark 5. Since tight eventually different families of permutations are maximal eventually different families of permutations (cf. Proposition 5.3 in [7]), also tight cofinitary groups are maximal cofinitary groups as we have the following implication:

$$G \text{ group and m.e.d. family of permutations } \Rightarrow G \text{ maximal cofinitary group}$$

The reverse implication is not known even if the maximality is restricted to elements in $\mathrm{cofin}(S_\infty)$.

3. Existence of tight cofinitary groups under Martin's Axiom

Similarly to Theorem 5.4 in [7] for eventually different families of permutations, we verify that MA(σ-centered) implies the existence of tight cofinitary groups. To this end, we will need Zhang's forcing [14] adding a diagonalization real for a given cofinitary group. Following [4] we will use a version of Zhang's forcing which uses the notion of nice words.

Definition 6. Let G be a cofinitary group. Then, we denote with W_G the set of all words in the language $G \cup \{x, x^{-1}\}$, where x is treated as a new symbol. Given any (partial) injection s and $w \in W_A$ we denote with $w[s]$ the (partial) injection, where in w, x and x^{-1} are replaced by s and s^{-1} and then the corresponding (partial) functions are concatenated. Finally, for any $w \in W_G$ we denote with $w \restriction G$ the finite set of all $g \in G$ such that g or g^{-1} appears as a letter in w or $g = \mathrm{id}$. Similarly, for $E \subseteq W_G$ we set $E \restriction G := \bigcup_{w \in E} w \restriction G$.

Definition 7. Let G be a cofinitary group. We denote with W_G^* the following subset of W_G, which we call the set of nice words. A word $w \in W_A$ is nice iff $w = x^k$ for some $k > 0$ or there are $k_0 > 0$ and $k_1, \dots, k_l \in \mathbb{Z} \setminus \{0\}$ and $g_0, \dots, g_l \in G \setminus \{\mathrm{id}\}$ such that

$$w = g_l x^{k_l} g_{l-1} x^{k_{l-1}} \dots g_1 x^{k_1} g_0 x^{k_0}.$$

Further, we write W_G^1 for the set of all $w \in W_G^*$ with exactly one occurrence of x or x^{-1} and write $W_G^{>1} := W_G^* \setminus W_G^1$.

Remark 8. Note that our definition of nice words is slightly stronger than in [4] as we require the last block of x to be positive. However, as w and w^{-1} have the same fixpoints for every word w, without loss of generality we may work with this slightly more restrictive notion of nice words.

Definition 9. Let G be a cofinitary group. Then we define Zhang's forcing $\mathbb{Z}_G$ to be the set of all pairs (s, E) such that s is a finite partial function $s : \omega \overset{\text{partial}}{\longrightarrow} \omega$ and $E \in [W_G^*]^{<\omega}$. We order $(t, F) \leq (s, E)$ by

(1) $s \subseteq t$ and $E \subseteq F$,
(2) for all $w \in E$ we have $\operatorname{fix}(w[t]) = \operatorname{fix}(w[s])$.

Remark 10. For any cofinitary group G the forcing $\mathbb{Z}_G$ is σ-centered and if H is $\mathbb{Z}_G$-generic in $V[H]$ we have that

$$f_{\text{gen}} := \bigcup \{s \mid \exists E \subseteq W_G^* \ (s, E) \in H\} \in S_\infty.$$

Further, $G \cup \{f_{\text{gen}}\}$ generates a cofinitary group (e.g. see [14]).

To this end, Zhang's forcing satisfies the following well-known domain and range extension lemma, which forces the generic function to be a permutation (e.g. see [14] or [4]).

Lemma 11. Let G be a cofinitary group and $(s, E) \in \mathbb{Z}_G$. Then we have

(1) if $n \notin \operatorname{dom}(s)$, then for almost all $m < \omega$ we have $(s \cup \{(n, m)\}, E) \leq (s, E)$,
(2) if $m \notin \operatorname{ran}(s)$, then for almost all $n < \omega$ we have $(s \cup \{(n, m)\}, E) \leq (s, E)$.

Further, to show that Zhang's forcing yields tightness in the sense of Definition 4 we need the following additional technical lemmas, which yield a stronger version of the standard generic hitting lemma required for tightness.

Lemma 12. Let G be a cofinitary group. Let $T \in \mathcal{I}_i(G)^+$, $t \in T$, $(s, E_0) \in \mathbb{Z}_G$ and $N < \omega$ with $E_0 \subseteq W_G^1$. Then, there is a $t' \in T$ with $t \trianglelefteq t'$ satisfying

$$\left|\{k \in \operatorname{dom}(t') \setminus \operatorname{dom}(t) \mid (s \cup \{(k, t'(k))\}, E_0) \leq (s, E_0)\}\right| > N.$$

Proof. It suffices to verify the case $N = 0$. The general case then follows inductively. Since $T \in \mathcal{I}_i(G)^+$, choose $t' \in T$ with $t \trianglelefteq t'$ and $k \in \operatorname{dom}(t') \setminus \operatorname{dom}(t)$ with $k \notin \operatorname{dom}(s)$, $t'(k) \notin \operatorname{ran}(s)$ and $t'(k) \neq g(k)$ for all $g \in E_0 \restriction G$. But then, for $s' := s \cup \{(k, t'(k))\}$ we claim that $(s', E_0) \leq (s, E_0)$. But by choice of k we have that s' is still a partial injection. Further, for every $w \in E_0$ we may choose $g \in G$ such that $w = gx$. By choice of k we have $t'(k) \neq g^{-1}(k)$. Thus, we compute

$$w[s'](k) = g(s'(k)) = g(t'(k)) \neq k.$$

Finally, for every $l \in \operatorname{ran}(s) \cup \operatorname{dom}(s)$ we have $w[s](l) = w[s'](l)$, so that $\operatorname{fix}(w[s]) = \operatorname{fix}(w[s'])$. ☐

Lemma 13. Let G be a cofinitary group. Let $T \in \mathcal{I}_i(G)^+$, $t \in T$ and $(s, E) \in \mathbb{Z}_G$. Then, there is $(s', E) \in \mathbb{Z}_G$, $t' \in T$ and $k \in \operatorname{dom}(t') \setminus \operatorname{dom}(t)$ such that $(s', E) \leq (s, E)$, $t \trianglelefteq t'$ and $t'(k) = s'(k)$.

Proof. Let $E = E_0 \cup E_1$, where $E_0 \subseteq W_G^1$ and $E_1 \subseteq W_G^{>1}$. We may assume that for all $w \in E_1$ every subword of w in W_G^1 is in E_0. Choose $N < \omega$ large enough such that

(N1) $\operatorname{dom}(s) \cup \operatorname{ran}(s) \cup \operatorname{dom}(t) \subseteq N$,

(N2) for all $g \in E \restriction G$ we have $g[\mathrm{dom}(s) \cup \mathrm{ran}(s)] \subseteq N$,
(N3) for all $g \in E \restriction G \setminus \{\mathrm{id}\}$ we have $\mathrm{fix}(g) \subseteq N$.

By Lemma 12 choose $t' \in T$ with $t \trianglelefteq t'$ satisfying

$$\left|\{k \in \mathrm{dom}(t') \setminus \mathrm{dom}(t) \mid (s \cup \{(k, t'(k))\}, E_0) \leq (s, E_0)\}\right| > N.$$

By injectivity of t' choose such a $k_0 < \omega$ with $t'(k_0) \geq N$. We define $s' := s \cup \{(k_0, t'(k_0))\}$ and claim that (s', E) is as desired. First, since $N \leq k_0, s'(k_0)$ and by (N1) we have that s' is still a partial injection. Thus, by choice of k_0 it suffices to verify that for every $w \in E_1$ we have that $\mathrm{fix}(w[s]) = \mathrm{fix}(w[s'])$, so let $w \in E_1$. As w is nice and has at least two occurrences of x or x^{-1}, we may write $w = vx^{\pm 1}ux$ for some $u \in G$ and $v \in W_G$. First, notice that for any $k \in \mathrm{dom}(s) \cup \mathrm{ran}(s)$ we have that

$$w[s](k) = w[s'](k) \qquad \text{(in particular also if both sides are undefined)},$$

as by (N1) and (N2) the computation along $w[s']$ starting with k always stays below N. Thus, we may finish the proof by showing that $w[s'](k_0)$ is undefined. In case that $w = vxux$, we have $ux \in E_0$. Thus, by $(s', E_0) \leq (s, E_0)$ we have $(ux)[s'](k_0) \neq k_0$. But by (N2) we also have that $(ux)[s'](k_0) \notin \mathrm{dom}(s)$. Hence, $w[s'](k_0)$ is undefined.

Otherwise, $w = vx^{-1}ux$, so that $u \neq \mathrm{id}$. Then, by (M3) we get $(ux)[s'](k_0) = u(t'(k_0)) \neq t'(k_0)$. But by (N2) we also have that $(ux)[s'](k_0) \notin \mathrm{ran}(s)$. Hence, $w[s'](k_0)$ is undefined. □

Theorem 14. *Assume* MA(σ*-centered*). *Then, every cofinitary group of size* $<\mathfrak{c}$ *is contained in a tight cofinitary group of size* $\mathfrak{c}$.

Proof. As in Theorem 2.4 in [7] for any cofinitary group G of size $<\mathfrak{c}$ we use MA(σ-centered) to diagonalize against every ω-sequence of injective trees as in the definition of tightness in an iteration of length $\mathfrak{c}$. However, to this end we will instead use Zhang's forcing $\mathbb{Z}_G$ to obtain a cofinitary group, so we need to verify that the following sets are dense in $\mathbb{Z}_G$:

1. for every $n < \omega$ the set of all $(s, E) \in \mathbb{Z}_G$ such that $n \in \mathrm{dom}(s)$,
2. for every $m < \omega$ the set of all $(s, E) \in \mathbb{Z}_G$ such that $m \in \mathrm{ran}(s)$,
3. for every $T \in \mathcal{I}_i(G)^+$, $t \in T$ the set of all $(s, E) \in \mathbb{Z}_G$ such that there is a $t' \in T$ with $t \trianglelefteq t'$ and a $k \in \mathrm{dom}(t') \setminus \mathrm{dom}(t)$ with $s(k) = t'(k)$,
4. for every $w \in W_G^*$ the set of all $(s, E) \in \mathbb{Z}_G$ such that $w \in E$.

But (1) and (2) are dense by the domain and range extension Lemma 11, (3) is dense by the previous Lemma 13 and the density of (4) is trivial. Hence, a generic f hitting these $<\mathfrak{c}$-many dense sets will extend G to a cofinitary group $\langle G \cup \{f\} \rangle$ and diagonalize against a given witness of tightness. □

4. Indestructibility and strong preservation of tightness

In the next section, we will show how to obtain a model with a co-analytic tight cofinitary group and a Δ^1_3 well-order of the reals. To this end, we define Sacks coding and show that it preserves our notion of tightness in a strong sense. Also, note that in [7] it is essentially proved that $\mathbb{S}^{\aleph_0}$ strongly preserves tightness. Using the theory developed in [4] this implies that tight cofinitary groups are universally Sacks-indestructible, i.e. indestructible by any countable support iteration or product of Sacks forcing of any length.

Definition 15. Until the end of this section, let us assume $V = L[Y]$ for a fixed $Y \subseteq \omega_1$, and that $\omega_1 = (\omega_1)^L$. Define the sequence μ_i, for $i < \omega_1$ by induction as follows: Given $\langle \mu_j | j < i \rangle$, let μ_i be the least ordinal μ such that $\mu > \sup_{j<i} \mu_j$, and $L_\mu[Y \cap i]$ is Σ^1_5 elementary in $L_{\omega_1}[Y \cap i]$ and a model of ZF$^-$ and "ω is the largest cardinal". We shall say that $x \in 2^\omega$ codes Y below i, where $i < \omega_1$, to mean that for all $j < i$, $j \in Y$ if and only if $L_{\mu_i}[Y \cap i, r] \models \text{ZF}^-$. Moreover, let us use the short-hand

$$\mathcal{A}_i := L_{\mu_i}[Y \cap i].$$

Conditions of $C(Y)$ will be perfect trees $T \subseteq 2^{<\omega}$; given such T, we write $|T|$ for the least $i < \omega_1$ such that $T \in \mathcal{A}_i$. The forcing $C(Y)$, which we shall refer to as Sacks coding, consists of perfect trees $T \subseteq 2^{<\omega}$ such that each branch of T codes Y below $|T|$, ordered by reverse inclusion.

Definition 16. Let $\mathbb{P}$ be a forcing notion and G be a tight cofinitary group. We say that $\mathbb{P}$ strongly preserves tightness of G if for every sufficiently large regular cardinal θ, every $p \in \mathbb{P}$ and every countable $M \prec H_\theta$ with $\mathbb{P}, p, G \in M$, if $g \in G$ densely diagonalizes every element of $M \cap \mathcal{I}_i(G)^+$, then there is an $(M, \mathbb{P})$-generic $q \leq p$ in $\mathbb{P}$ so that q forces that g densely diagonalizes every element in $M[\dot{G}_{\mathbb{P}}] \cap \mathcal{I}_i(G)^+$; such q is called a $(M, \mathbb{P}, G, g)$-generic condition. Here and in what follows, we use $\dot{G}_{\mathbb{P}}$ for the canonical $\mathbb{P}$-name for the $\mathbb{P}$-generic filter (to avoid the unfortunate clash of notation caused by traditionally designating with G both a group and a forcing generic).

Theorem 17. *Sacks coding strongly preserves tightness.*

Proof. Suppose G is a tight cofinitary group. Suppose further we are given $p \in C(Y)$ and M such that $p, C(Y), G \in M \prec H_\theta$, along with $g \in G$ which densely diagonalizes every $T \in \mathcal{I}_i(G)^+ \cap M$. We must find a $(M, C(Y), G, g)$-generic condition in $C(Y)$ below p.

To this end, let $\pi\colon M \to \bar{M}$ be the transitive collapsing map and observe that $\bar{M} = L_\alpha[Y \cap \delta]$, where $\delta = (\omega_1)^{\bar{M}}$. Note $\pi(C(Y)) = C(Y) \cap M = C(Y) \cap \bar{M} = C(Y)^{\bar{M}}$. Since δ is uncountable in $\bar{M}$, by definition of μ_δ, $\bar{M} \in \mathcal{A}_\delta$. Let us fix a sequence $\langle \delta_n | n \in \omega \rangle \in \mathcal{A}_\delta$ which is cofinal in δ; this is possible as $\mathcal{A}_\delta \models$"$\bar{M}$ is countable". Similarly, we can fix an enumeration $\langle \bar{D}_n | n \in \omega \rangle \in \mathcal{A}_\delta$ of all dense subsets of $C(Y) \cap \bar{M}$ which are elements of $\bar{M}$, and an enumeration $\langle \dot{T}_n | n \in \omega \rangle \in \mathcal{A}_\delta$ of $\pi\left(\mathcal{I}_i(G)^+\right) = \bar{M} \cap \mathcal{I}_i(G \cap \bar{M})^+$. Moreover, fix a bijection $\varphi\colon \omega \to \omega^2$ such that $\varphi \in \mathcal{A}_\delta$ (e.g., a computable such map) with coordinate maps φ_0, φ_1, as a book-keeping device.

We now construct sequences $\langle q_n | n \in \omega \rangle, \langle \dot{t}_n | n \in \omega \rangle, \langle \dot{k}_n | n \in \omega \rangle \in \mathcal{A}_\delta$ with the following properties:

(1) $q_0 = p$,
(2) $q_{n+1} \in C(Y) \cap \bar{M}$ and $q_{n+1} \leq_{n+1} q_n$
(3) $|q_{n+1}| \geq \delta_n$,
(4) $q_{n+1} \Vdash \bar{D}_n \cap \dot{G} \neq \emptyset$,
(5) $\dot{t}_n$ is a $C(Y)^{\bar{M}}$-name in $\bar{M}$ for an element of $\dot{T}_{\varphi_0(n)}$ and $\dot{k}_n$ is a $C(Y)^{\bar{M}}$-name in $\bar{M}$ for an element of ω,
(6) $q_{n+1} \Vdash$"with $\dot{s}$ the $\varphi_1(n)$-th node in $\dot{T}_{\varphi_0(n)}$, $\dot{k}_n \in \text{dom}(\dot{t}_n) \setminus \text{dom}(\dot{s})$ and $\dot{t}_n(\dot{k}_n) = \check{g}(\dot{k}_n)$".

Suppose we have already constructed this sequence up to q_n. Let $q_{n+1} \leq_{n+1} q_n$ be the least condition (in the canonical well-ordering of $\mathcal{A}_\delta$) such that for each $(n+1)$-th splitting node t of q_n, $q := (q_{n+1})_t$ satisfies the following:

(i) $|q| \geq \delta_n$,

(ii) $q \in \bar{D}_n$ and moreover,

(iii) for some $s, t \in 2^{<\omega}$ with $s \subseteq t$ and some $k \in \omega$, it holds that $q \Vdash$"$\check{s}$ is the $\varphi_1(n)$-th node in $\dot{T}_{\varphi_0(n)}$, $\check{t} \in T_{\varphi_0(n)}$", and $k \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ and $t(k) = g(k)$.

It is clear that the set of q satisfying (i) and (ii) is dense in $C(Y) \cap \bar{M}$ by elementarity of M. To see a condition q as required exists, it therefore remains to show the following claim:

Claim 18. The set of conditions q satisfying (iii) is dense in $C(Y)$.

To see this claim, let us write $\dot{T} = \dot{T}_{\varphi_0(n)}$ and let $q^* \in C(Y)$ be arbitrary. Find $q' \leq q^*$ and s such that $q' \Vdash$"$\check{s}$ is the $\varphi_1(n)$-th node in $\dot{T}$". Notice that

$$T := \{t \in \omega^{<\omega} \colon q' \not\Vdash \check{t} \notin \dot{T}\}$$

is an element of $\mathcal{I}_i(G)^+$, as can be verified in a straightforward manner from the definition. Therefore, by assumption, g densely diagonalizes T and we can find k and $t \in T$ such that $s \subseteq t$, $k \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$, and $t(k) = g(k)$. Finally, as $t \in T$, we can find $q \leq q'$ such that $q \Vdash \check{t} \in \dot{T}$. This finishes the proof of the claim and hence the construction of q_{n+1}.

It is clear from Item (ii) that q_{n+1} satisfies (4); it is also clear that from Item (iii) that we can find names $\dot{k}_n, \dot{t}_n \in \bar{M}$ satisfying (5) and (6) (in fact, these names can be chosen to be finite). Finally, since $\mathcal{A}_\delta$ satisfies ZF^- and all the required data for the definition of these sequences is an element of $\mathcal{A}_\delta$, the above definition yields sequences which are also elements in this model, as required.

Define $q_\omega = \bigcap_{n\in\omega} q_n$. To see that $q_\omega \in C(Y)$, observe that $|q_\omega| = \delta$ and that q_ω codes Y up to δ since by (3), q_ω codes Y below δ and because $q_\omega \in \mathcal{A}_\delta$. It is now straightforward to verify from the definitions that q_ω is $(M, C(Y), G, g)$-generic. □

5. Co-analyticity and Zhang's forcing with coding into orbits

In order to obtain a co-analytic tight cofinitary group, we present a new parameter-less coding technique for maximal cofinitary groups. To this end, we will code a real using the parity of the length of the orbits of elements of our cofinitary group. First, we present a modification of Zhang's forcing, which codes a real into the lengths of orbits of the Zhang generic real. This, will yield a tight cofinitary group with a co-analytic set of generators. Secondly, we will expand our coding technique, so that the orbit function of every new word codes some real. Hence, we obtain that the entire tight cofinitary group is co-analytic.

5.1. Coding into orbits of the Zhang generic real.

Definition 19. Given $f \in \omega^\omega$ and $n < \omega$ let $O_f(n)$ be the orbit of f containing n, that is the smallest set containing n closed under applications of f and f^{-1}, and define $\mathcal{O}_f := \{O_f(n) \mid n < \omega\}$. There is a natural well-order on $\mathcal{O}_f$ defined for $O, P \in \mathcal{O}_f$ by $O < P$ iff $\min(O) < \min(P)$.

Assume f only has finite orbits; it follows that f has infinitely many orbits. Then, we may define a function $o_f : \omega \to 2$ by

$$o_f(n) := (|O_n| \mod 2),$$

where O_n is the n-th element in the well-order of $\mathcal{O}_f$.

Definition 20. For any finite partial injection $s : \omega \xrightarrow{\text{finite}} \omega$ and $n < \omega$ define $O_s(n)$ and $\mathcal{O}_s$ as above. We say an orbit $O \in \mathcal{O}_s$ is closed iff $O \subseteq \text{dom}(s) \cap \text{ran}(s)$ and denote with $\mathcal{O}_s^c$ the set of all closed orbits of s. Conversely, we denote with $\mathcal{O}_s^o := \mathcal{O}_s \setminus \mathcal{O}_s^c$ the set of all open orbits of s. We say s is nice iff for all $O \in \mathcal{O}_s^c$ we have $\min(O) < \min(\omega \setminus \bigcup \mathcal{O}_s^c)$. For any nice s define a function $o_s : |\mathcal{O}_s| \to 2$ by

$$o_s(n) := (|O_n| \mod 2),$$

where O_n is the n-th element in the well-order of $\mathcal{O}_s^c$. For a real $r \in 2^\omega$ and a nice s we say s codes r iff $o_s \subseteq r$.

Note that niceness makes sure that we do not prematurely close any orbit, in the sense that we do not know which function value should be coded, as the well-order of orbits is not decided up to that point yet.

Definition 21. Let $r \in 2^\omega$ be a real and G be a cofinitary group. Then, we define $\mathbb{Z}_G(r)$ as the set of all elements $(s, E) \in \mathbb{Z}_G$ such that s is nice and codes r, ordered by the restriction of the order on $\mathbb{Z}_G$.

Remark 22. We will show that $\dot{f}_{\text{gen}}$ generically codes as much information of r as desired. Hence, by definition of $\mathbb{Z}_G(r)$ in the generic extension we have $o_{\dot{f}_{\text{gen}}} = r$, and thus r can be decoded from the generic Zhang real. First, we verify range and domain extension:

Lemma 23. Let G be a cofinitary group and $(s, E) \in \mathbb{Z}_G(r)$. Then we have

(1) if $n \notin \text{dom}(s)$, then for almost all $m < \omega$ we have $(s \cup \{(n, m)\}, E) \leq (s, E)$ as well as $(s \cup \{(n, m)\}, E) \in \mathbb{Z}_G(r)$,

(2) if $m \notin \text{ran}(s)$, then for almost all $n < \omega$ we have $(s \cup \{(n, m)\}, E) \leq (s, E)$ as well as $(s \cup \{(n, m)\}, E) \in \mathbb{Z}_G(r)$.

Proof. This follows immediately from Lemma 11. Note that possibly only one choice of n (or m) may close an orbit of s, which immediately implies that $(s \cup \{(n, m)\}, E) \in \mathbb{Z}_G(r)$ for almost all n (or m). □

Secondly, we again verify the stronger generic hitting lemma required for tightness (as in Lemma 13). Hence, also $\mathbb{Z}_G(r)$ may be used to construct or force a tight cofinitary group.

Lemma 24. Let G be a cofinitary group. Let $T \in \mathcal{I}_i(G)^+$, $t \in T$ and $(s, E) \in \mathbb{Z}_G(r)$. Then, there is $(s', E) \in \mathbb{Z}_G(r)$, $t' \in T$ and $k \in \text{dom}(t') \setminus \text{dom}(t)$ such that $(s', E) \leq (s, E)$, $t \trianglelefteq t'$ and $t'(k) = s'(k)$.

Proof. The function pair (n, m) added to s in Lemma 12 satisfies $n, m \notin \text{ran}(s) \cup \text{dom}(s)$ and $n \neq m$. Hence, the extension $s \cup \{(n, m)\}$ does not close an orbit of s, which immediately implies that $s \cup \{(n, m)\} \in \mathbb{Z}_G(r)$. □

Finally, we need to verify that we may generically code as much information of r as desired. To this end, we prove that for any $(s, E) \in \mathbb{Z}_G$ and $O \in \mathcal{O}_s^o$ and almost all $k < \omega$ we may extend (s, E) to close O in length k.

Lemma 25. Let G be a cofinitary group, $(s, E) \in \mathbb{Z}_G$ and $n \in \omega \setminus \bigcup \mathcal{O}_s^c$. Then, there is a $K < \omega$ such that for all $k > K$ there is $(t, E) \in \mathbb{Z}_G$ with $(t, E) \leq (s, E)$, $O_t(n) \in \mathcal{O}_t^c$ and $|O_t(n)| = k$.

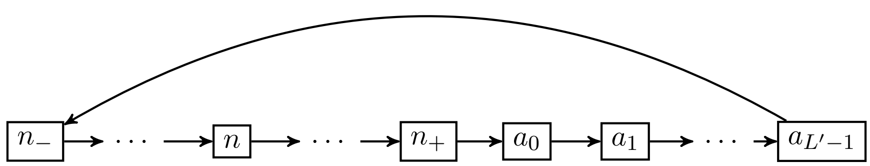
Proof. By Lemma 11 we may assume that $n \in \mathrm{dom}(s) \cup \mathrm{ran}(s)$. As $O_s(n) \in \mathcal{O}_s^o$ choose $n_- < \omega$ to be the unique element of $O_s(n) \setminus \mathrm{ran}(s)$ and n_+ be the unique element of $O_s(n) \setminus \mathrm{dom}(s)$. Let $L := \max\{|w| \mid w \in E\}$ and set $K := |O_s(n)| + L$. Now, let $k > K$. Choose $L' \geq L$ and pairwise different natural numbers $A := \{a_0, \ldots, a_{L'-1}\}$ such that

(1) $|O_s(n)| + L' = k$,
(2) for all $a \in A$ we have
 (a) $a \notin \mathrm{dom}(s) \cup \mathrm{ran}(s)$,
 (b) for all $g \in E \restriction G \setminus \{\mathrm{id}\}$ we have $g(a) \notin \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup A$.

Note that we can ensure (2b) as every $g \in E \restriction G \setminus \{\mathrm{id}\}$ only has finitely many fixpoints. We claim that for

$$t := s \cup \{(n_+, a_0)\} \cup \{(a_i, a_{i+1}) \mid i < L' - 1\} \cup \{(a_{L'-1}, n_-)\}$$

we have that $(t, E) \in \mathbb{Z}_G$. Clearly, then $(t, E) \leq (s, E)$, $O_t(n) \in \mathcal{O}_t^c$ and $|O_t(n)| = k$ by (1). Visualized, the orbit $O_t(n)$ then looks as follows:



So, let $w \in E$. If $w = x^{k_0}$ for some $k_0 > 0$, note that $\mathrm{fix}(w[s]) = \mathrm{fix}(w[t])$ as by choice of L we have $|O_t(n)| = k > L \geq |w| = k_0$. Hence, we may write $w = gv$ for some $g \in G \setminus \{\mathrm{id}\}$ and $v \in W_G$. Towards a contradiction, assume that $d \in \mathrm{fix}(w[t]) \setminus \mathrm{fix}(w[s])$. First, note that $v[t](d) \in \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup A$. Thus, by (2b) we have $d = w[t](d) = (gv)[t](d) \notin A$. Hence, $d \in \mathrm{dom}(s) \cup \mathrm{ran}(s)$.

If the entire computation of d along $w[t]$ stays in $\mathrm{dom}(s) \cup \mathrm{ran}(s)$, then $w[t](d) = w[s](d)$, contradicting $d \notin \mathrm{fix}(w[s])$. Thus, write $w = w_1 w_0$ with $w_0, w_1 \in W_G$ with w_0 minimal such that $w_0[t](d) \notin \mathrm{dom}(s) \cup \mathrm{ran}(s)$, i.e. $w_0[t](d) \in A$. By (2b) the left-most letter of w_0 has to be x or x^{-1}, so without loss of generality assume $w = w_1 x w_0'$ (the other case is symmetric). As we have $(w_0')[t](d) \in \mathrm{dom}(s) \cup \mathrm{ran}(s)$ and $(xw_0')[t](d) \in A$ we get $(w_0')[t](d) = n_+$. Finally, write $w = w_1' g x^l w_0'$ for some $w_1' \in W_G$, $g \in G \setminus \{\mathrm{id}\}$ and $l > 0$. Then, $l < |w| \leq L \leq L'$ implies that

$$(x^l w_0')[t](d) = (x^l)[t](n_+) \in A.$$

Hence, by (2b) we have $(gx^l w_0')[t](d) \notin \mathrm{dom}(s) \cup \mathrm{ran}(s) \cup A$. Thus, if w_1' is the empty word, this contradicts $d \in \mathrm{dom}(s) \cup \mathrm{ran}(s)$, and otherwise $(w_1' g x^l w_0')[t](d)$ is undefined, a contradiction. $\square$

Corollary 26. Let G be a cofinitary group and $(s, E) \in \mathbb{Z}_G(r)$. Then, there is $(t, E) \in \mathbb{Z}_G(r)$ such that $(t, E) \leq (s, E)$ and $|\mathcal{O}_s^c| < |\mathcal{O}_t^c|$.

Proof. Let $n := \min(\omega \setminus \bigcup \mathcal{O}_s^c)$. By Lemma 25 there is $(t, E) \in \mathbb{Z}_G$ such that $(t, E) \leq (s, E)$, $O_t(n) \in \mathcal{O}_t^c$ and

$$|O_t(n)| \equiv r(|\mathcal{O}_s^c|) \pmod 2.$$

Further, t can be chosen to not close any other orbits, so that by choice of n we have that t is nice and codes r. Hence, $(t, E) \in \mathbb{Z}_G(r)$. □

Corollary 27. Let G be a cofinitary group. Then we have

$$\mathbb{Z}_G \Vdash \dot{f}_{\text{gen}} \text{ only has finite orbits and } o_{\dot{f}_{\text{gen}}} \text{ is an unbounded real over } V.$$

Proof. Immediately follows from Lemma 25 as generically we may close any open orbit in arbitrarily long length. □

Theorem 28. *Consistently, there is a tight cofinitary group of size* $\aleph_1$ *(whence* $\mathfrak{a}_{\text{g}} = \aleph_1$*) with a co-analytic set of generators, a* Δ^1_3 *well-order of the reals, and* $\mathfrak{c} = \aleph_2$*.*

Proof. We omit the proof since we will prove a stronger result below, see Theorem 43. □

5.2. **Coding into orbits of every new word.** In the last section we have seen how to code a real into the orbit function of the Zhang generic. Now, we will consider a slightly different orbit function, which is stable under cyclic permutations and inverses thereof. Hence, in order to code a real into every new element of our cofinitary group, we will we able to restrict to nice words.

Definition 29. Assume $f \in \omega^\omega$ only has finite orbits and for every $n < \omega$ only finitely many orbits of length n. Let $\{p_n \mid n < \omega\}$ enumerate all primes. Then, define a function $o_f^\dagger : \omega \to 2$ by

$$o_f^\dagger(n) := (|\{O \in \mathcal{O}_f \mid |O| = p_n\}| \mod 2).$$

Similarly, for any finite partial injection $s : \omega \xrightarrow{\text{finite}} \omega$ define a function $o_s^\dagger : \omega \to 2$ by

$$o_s^\dagger(n) := (|\{O \in \mathcal{O}_s^c \mid |O| = p_n\}| \mod 2).$$

Hence, for every $n < \omega$ we are counting how many orbits of size p_n there are. For a real r and $n < \omega$ we say s codes r up to n iff $r \restriction (n+1) = o_s^\dagger \restriction (n+1)$. Finally, we say f codes r iff $o_f^\dagger = r$.

Remark 30. In the previous section, we have seen that the Zhang generic only has closed orbits (see Lemma 25). Further, note that for every $n < \omega$ closing a new orbit in length n implies that x^n has a new fixpoint. Hence, every $(s, E) \in \mathbb{Z}_G$ with $x^n \in E$ forces that the number of orbits of length of $\dot{f}_{\text{gen}}$ is decided, i.e. $\dot{f}_{\text{gen}}$ only has finitely many orbits of length n. Hence, $\dot{f}_{\text{gen}}$ satisfies the assumption of Definition 29.

Remark 31. As f has the same orbits as f^{-1} we get that if f only has finite orbits and for every $n < \omega$ only finitely many orbits of length n, then the same holds for f^{-1} and $o_f^\dagger = o_{f^{-1}}^\dagger$.

Lemma 32. Let $f, g \in \omega^\omega$ be bijections. Then, the map $\pi : \mathcal{O}_{fg} \to \mathcal{O}_{gf}$ defined for $O \in \mathcal{O}_{fg}$ by

$$\pi(O) = g[O],$$

defines a bijection. Further, π maps every orbit of fg to an orbit of gf of the same length.

Proof. The second part follows immediately as g is a bijection. First, we verify that π maps to $\mathcal{O}_{gf}$. So, let $O \in \mathcal{O}_{fg}$, i.e. O is a minimal non-empty set closed under applications of fg and $(fg)^{-1}$. Clearly, $\pi[O]$ is non-empty and for $m \in g[O]$ choose $n < \omega$ such that $g(n) = m$. Then, we compute

$$(gf)(m) = (gfg)(n) = g(fg(n)) \in g[O],$$

as $fg(n) \in O$. Similarly, we have

$$(gf)^{-1}(m) = (f^{-1}g^{-1}g)(n) = (gg^{-1}f^{-1})(n) = g((fg)^{-1}(n)) \in g[O],$$

as $(fg)^{-1}(n) \in O$. Hence, $g[O]$ is closed under applications of gf and $(gf)^{-1}$. Now, let $P \subseteq g[O]$ be a non-empty subset closed under applications of gf and $(gf)^{-1}$. By the same argument as above we have that $g^{-1}[P] \subseteq O$ is a non-empty set closed under applications of fg and $(fg)^{-1}$. But $O \in \mathcal{O}_{fg}$, so that $g^{-1}[P] = O$. Hence, $P = g[O]$, which shows that $g[O] \in \mathcal{O}_{gf}$. Finally, note that by the same argumentation $\psi : \mathcal{O}_{gf} \to \mathcal{O}_{fg}$ defined for $O \in \mathcal{O}_{fg}$ by

$$\psi(O) = g^{-1}[O],$$

is well-defined and clearly the inverse of π. Hence, π is bijective. □

Corollary 33. Let $f, g \in \omega^\omega$ be bijections and assume fg only has finite orbits and for every $n < \omega$ only finitely many orbits of length n. Then, gf has only finite orbits, for every $n < \omega$ only finitely many orbits of length n and $o^\dagger_{fg} = o^\dagger_{gf}$.

Next, we will show that our orbit function is also stable under finite powers, so that we may restrict to even nicer words, in whose orbit functions we will code a real in the end. This is, where we use the sequence of primes in Definition 29.

Definition 34. Let G be a cofinitary group. We say a nice word $w \in W^*_G$ is very nice iff there is no $v \in W^*_G$ and $k > 1$ such that $v^k = \underbrace{v \ldots v}_{k\text{-times}} = w$. We denote with $W^\dagger_G$ the set of all very nice words.

Lemma 35. Let $f \in \omega^\omega$ be a bijection with only finite orbits and for every $n < \omega$ only finitely many orbits of length n and let $k < \omega$. Then, f^k only has finite orbits, for every $n < \omega$ only finitely many orbits of length n and $o^\dagger_f(n) = o^\dagger_{f^k}(n)$ for almost all $n < \omega$.

Proof. Every orbit of f^k of size n is contained in an orbit of f of length at most kn, so that f^k only has finite orbits and for every $n < \omega$ only finitely many orbits of length $n < \omega$.

In particular, f^k only has finitely many orbits of length 1, i.e. only finitely many fixpoints. We show for almost all $n < \omega$ that $o^\dagger_f(n) = o^\dagger_{f^k}(n)$. So assume $p_n > \left|\mathrm{fix}(f^k)\right|$, where p_n is the n-th prime number. Fix an orbit $O \in \mathcal{O}_f$ of size p_n and let $\mathcal{P}$ be the set of all $P \in \mathcal{O}_{f^k}$ with $P \subseteq O$. Applying f induces bijections between members of $\mathcal{P}$, i.e. all of them have the same size. As $\mathcal{P}$

partitions O we obtain that $|P|$ divides $|O| = p_n$ for all $P \in \mathcal{P}$. But p_n is prime, so either $|P| = 1$ for all $P \in \mathcal{P}$, which implies that f^k has at least $|O| = p_n$-many fixpoints, contradicting the choice of n. Thus, $|\mathcal{P}| = 1$, i.e. f and f^k have the same number of orbits of size p_n. Hence, we proved $o^\dagger_f(n) = o^\dagger_{f^k}(n)$. □

Corollary 36. Let G be a cofinitary group and let $f \in \omega^\omega \setminus G$ such that $\langle G \cup \{f\}\rangle$ is cofinitary. Further, assume that there is $r \in 2^\omega$ such that for every $w \in W^\dagger_G$ we have $w[f]$ codes r. Then, for every $g \in \langle G \cup \{f\}\rangle \setminus G$ we have g almost codes r, i.e. $r(n) = o^\dagger_g(n)$ for almost all $n < \omega$.

Proof. Let $g \in \langle G \cup \{f\}\rangle \setminus G$. By our slight modification of nice words (see Definition 7) and the properties of nice words in [4], we may choose $w_0, w_1 \in W_G$ such that $w := w_0w_1 \in W^*_G$ and $g = (w_1w_0)[f]$ or $g^{-1} = (w_1w_0)[f]$. By Remark 31 and Lemma 32 it suffices to verify that $r(n) = o^\dagger_{w[f]}(n)$ for almost all $n < \omega$. If $w \in W^\dagger_G$ we are done by assumption, so let $v \in W^\dagger_G$ and $k > 1$ such that $w = v^k$. Then, $v[f]$ codes r, so by Lemma 35 we obtain

$$o^\dagger_{w[f]}(n) = o^\dagger_{v^k[f]}(n) = o^\dagger_{v[f]}(n) = r(n)$$

for almost all $n < \omega$. □

Hence, we will only have to make sure that every very nice word codes r. Next, we introduce the variation of Zhang's forcing, which ensures exactly this property.

Definition 37. Let $r \in 2^\omega$ be a real and G be a cofinitary group. Then, we define $\mathbb{Z}^\dagger_G(r)$ as the set of all elements $(s, E) \in \mathbb{Z}_G$ such that E is closed under cyclic permutations and inverses thereof in W^*_G, and for all $w \in E$ if $w = v^k$ for $v \in W^\dagger_G$ and $k < \omega$, then $v^l \in E$ for all $0 < l \leq k$ and for all $n < \omega$ with $p_n \leq k$ also $o^\dagger_{v[s]}$ codes r up to n. We let the order on $\mathbb{Z}^\dagger_G(r)$ be the restriction of the order on $\mathbb{Z}_G$.

Proposition 38. Let G be a cofinitary group and $(s, E) \in \mathbb{Z}^\dagger_G(r)$. Then, for every $(t, E) \in \mathbb{Z}_G$ with $(t, E) \leq (s, E)$ we have $(t, E) \in \mathbb{Z}^\dagger_G(r)$.

Proof. If not, there are $v \in W^\dagger_G$, $n < \omega$ and $O \in \mathcal{O}^c_t \setminus \mathcal{O}^c_s$ such that $v^{p_n} \in E$ and $|O| = p_n$. But then, for every $k \in O$ we have $k \in \mathrm{fix}(v^{p_n}[t]) \setminus \mathrm{fix}(v^{p_n}[s])$, contradicting $(t, E) \leq (s, E)$. □

Corollary 39. Let G be a cofinitary group and $(s, E) \in \mathbb{Z}^\dagger_G(r)$. Then we have

(1) if $n \notin \mathrm{dom}(s)$, then for almost all $m < \omega$ we have $(s \cup \{(n, m)\}, E) \leq (s, E)$ as well as $(s \cup \{(n, m)\}, E) \in \mathbb{Z}^\dagger_G(r)$,
(2) if $m \notin \mathrm{ran}(s)$, then for almost all $n < \omega$ we have $(s \cup \{(n, m)\}, E) \leq (s, E)$ as well as $(s \cup \{(n, m)\}, E) \in \mathbb{Z}^\dagger_G(r)$,

Corollary 40. Let G be a cofinitary group. Let $T \in \mathcal{I}_i(G)^+$, $t \in T$ and $(s, E) \in \mathbb{Z}^\dagger_G(r)$. Then, there is $(s', E) \in \mathbb{Z}^\dagger_G(r)$, $t' \in T$ and $k \in \mathrm{dom}(t') \setminus \mathrm{dom}(t)$ such that $(s', E) \leq (s, E)$, $t \trianglelefteq t'$ and $t'(k) = s'(k)$.

Proof. Follows immediately from Proposition 38 and Lemma 11 or Lemma 13, respectively. □

Hence, we only have to verify, that densely we may add any nice word to the second component of every condition in $\mathbb{Z}^{\dagger}_{G}(r)$. To this end, we verify that we can find extensions which add exactly one orbit to some very nice word.

Lemma 41. Let G be a cofinitary group. Let $(s,E) \in \mathbb{Z}^{\dagger}_{G}(r)$, $v \in W^{\dagger}_{G}$ and $k < \omega$ such that $v^k \notin E$. Then there is $(t,E) \in \mathbb{Z}^{\dagger}_{G}(r)$ such that $(t,E) \leq (s,E)$ and $\mathcal{O}^c_{v[t]} = \mathcal{O}^c_{v[s]} \cup \{O\}$ for some $O \in \mathcal{O}^c_{v[t]} \setminus \mathcal{O}^c_{v[s]}$ with $|O| = k$.

Proof. Let $F := E \cup \{v\}$. Choose $N < \omega$ such that

(1) for all $g \in F \restriction G$ we have $g[\mathrm{dom}(s) \cup \mathrm{ran}(s)] \subseteq N$,
(2) for all $g_0 \neq g_1 \in F \restriction G$ and $n > N$ we have $g_0(n) \neq g_1(n)$.

In particular, as $\mathrm{id} \in F \restriction G$, we have

(1) $\mathrm{dom}(s) \cup \mathrm{ran}(s) \subseteq N$,
(2) for all $g \in F \restriction G \setminus \{\mathrm{id}\}$ we have $\mathrm{fix}(g) \subseteq N$.

Inductively, we define a sequence of pairwise different natural numbers $A := \{a_i \mid i < k \cdot |v|\}$ and a sequence of finite partial injections $\langle t_i \mid 0 < i \leq k \cdot |v|\rangle$ as follows. Write $v^k = v_{k\cdot|v|-1} \dots v_0$, choose $a_1 > N$ arbitrarily and set $t_1 := s$. Now, assume a_i and t_i are defined for some $0 < i < k \cdot |v| - 1$. If $v_i \in G$ set $a_{i+1} := v_i(a_i)$ and $t_{i+1} := t_i$. Otherwise, $v_i = x^{\pm 1}$. Choose, $a_{i+1} > N$ such that for all $g \in F \restriction G$ we have $g(a_{i+1}) \notin \{a_1, \dots, a_i\}$. Further, if $v_i = x$ set $t_{i+1} := t_i \cup \{(a_i, a_{i+1})\}$ and if $v_i = x^{-1}$ set $t_{i+1} := t_i \cup \{(a_{i+1}, a_i)\}$.

Finally, assume $a_{k\cdot|v|-1}$ and $t_{k\cdot|v|-1}$ are defined. In case $v_{k\cdot|v|-1} \in G$, set $a_0 := v_{k\cdot|v|-1}(a_{k\cdot|v|-1})$ and define $t_{k\cdot|v|} := t_{k\cdot|v|-1} \cup \{(a_0, a_1)\}$. Otherwise, $v_{k\cdot|v|-1} = x$ (so that $v = x$, because $v \in W^{\dagger}_{G}$). Then, we may choose $a_0 > N$ such that for all $g \in F \restriction G$ we have $g(a_0) \notin \{a_1, \dots, a_{k\cdot|v|-1}\}$ and we set $t_{k\cdot|v|} := t_{k\cdot|v|-1} \cup \{(a_{k\cdot|v|-1}, a_0), (a_0, a_1)\}$.

By construction, we have that all a_i are pairwise distinct, $O := \{a_{i\cdot|v|} \mid i < k\} \in \mathcal{O}^c_{v[t]} \setminus \mathcal{O}^c_{v[s]}$ and $|O| = k$. Let $t := t_{k\cdot|v|}$. Note that for every $i < k \cdot |v|$ exactly one of the following cases is satisfied:

(1) $a_i \in \mathrm{dom}(t) \cap \mathrm{ran}(t)$ and for every $g \in F \restriction G \setminus \{\mathrm{id}\}$ we have that $g(a_i) \notin \mathrm{dom}(t) \cup \mathrm{ran}(t)$ and $\{t(a_i), t^{-1}(a_i)\} = \{a_{i-1}, a_{i+1}\}$,
(2) $a_i \in \mathrm{dom}(t) \setminus \mathrm{ran}(t)$ and there is a unique $g \in F \restriction G \setminus \{\mathrm{id}\}$ such that $g(a_i) \in \mathrm{dom}(t) \cup \mathrm{ran}(t)$ and $\{t(a_i), g(a_i)\} = \{a_{i-1}, a_{i+1}\}$,
(3) $a_i \in \mathrm{ran}(t) \setminus \mathrm{dom}(t)$ and there is a unique $g \in F \restriction G \setminus \{\mathrm{id}\}$ such that $g(a_i) \in \mathrm{dom}(t) \cup \mathrm{ran}(t)$ and $\{t^{-1}(a_i), g(a_i)\} = \{a_{i-1}, a_{i+1}\}$,

where $a_{-1} := a_{k\cdot|v|-1}$ and $a_{k\cdot|v|} := a_0$. Finally, it remains to show that $(t,E) \leq (s,E)$ and $\mathcal{O}^c_{v[t]} = \mathcal{O}^c_{v[s]} \cup \{O\}$.

So, let $w \in E$ and assume $d \in \mathrm{fix}(w[t]) \setminus \mathrm{fix}(w[s])$. By choice of N we have $d \in A$ and the entire computation of d along $v[t]$ is in A. So choose $i < k \cdot |v|$ with $d = a_i$ and let w_0 be the rightmost letter of w. By the three properties above and the fact that the entire computation of d along $w[t]$ is defined, we obtain $w_0[t](a_i) = a_{i+1}$ or $w_0[t](a_i) = a_{i-1}$. By considering the word w^{-1} we may restrict to the first case. Write $v = v_1 v_0$ for $v_0, v_1 \in W_G \setminus \{\mathrm{id}\}$ such that $v_0(a_0) = a_i$. Inductively, using the three properties above, there is only one computation using $F \restriction G \setminus \{\mathrm{id}\}$ and

t, t^{-1} starting at a_i and proceeding to a_{i+1}. As d is a fixpoint of $w[t]$, the length of w is a multiple of $|A| = k \cdot |v|$. But then, by uniqueness of the computation we have $w = ((v_0v_1)^k)^l$ for some $l > 0$. But $v^k \notin E$, so by the closure of E (recall Definition 37) we get $w \notin E$, a contradiction.

Finally, we have to show that we only added exactly one orbit to $\mathcal{O}^c_{v[s]}$. So let $a_i \in A \setminus O$ and assume $a_i \in P \in \mathcal{O}^c_{v[t]}$. Again, there are only two possible computations using $F \restriction G \setminus \{\mathrm{id}\}$ and t, t^{-1} starting with a_i; one proceeding with a_{i+1} and the other with a_{i-1}. As before, by uniqueness we obtain the following two cases. Either, there are $v_0, v_1 \in W_G \setminus \{\mathrm{id}\}$ with $v = v_1v_0$ and $v = v_0v_1$, or there are $v_0, v_1 \in W_G \setminus \{\mathrm{id}\}$ with $v = v_1v_0$ and $v = v_0^{-1}v_1^{-1}$. It is easy to verify that the first case contradicts that v is very nice and the second case contradicts that v is reduced. □

Corollary 42. Let G be a cofinitary group and $w \in W^*_G$. Then the set of all (s, E) with $w \in E$ is dense in $\mathbb{Z}^\dagger_G(r)$.

Proof. Let $(s, E) \in \mathbb{Z}^\dagger_G(r)$ and $w = v^k$ for some $k < \omega$ and $v \in W^\dagger_G$ with $w \notin E$. Let F be the closure of $E \cup \{w\}$ under cyclic permutations and inverses thereof in W^*_G. If $k = 1$, then $(s, F) \in \mathbb{Z}^\dagger_G(r)$ and $(s, F) \leq (s, E)$. If $k > 1$ by induction we may assume $v^{k-1} \in E$. If k is not prime, we have $(s, F) \in \mathbb{Z}^\dagger_G(r)$ and $(s, F) \leq (s, E)$. So assume $k = p_n$ for some $n < \omega$. If

$$o^\dagger_{v[s]}(n) \not\equiv r(n) \qquad \text{mod } 2,$$

use the previous Lemma 41 to find $(t, E) \in \mathbb{Z}_G(r)$ with $(t, E) \leq (s, E)$ and

$$o^\dagger_{v[t]}(n) \equiv r(n) \qquad \text{mod } 2.$$

Hence, $v[t]$ codes r up to n. By Proposition 31 and Corollary 33 the same is true for every cyclic permutation of v and inverses thereof in $W^\dagger_G$. Thus, $(t, F) \in \mathbb{Z}^\dagger_G(r)$, $(t, F) \leq (s, E)$ and $w \in F$. □

Theorem 43. *Consistently, there is a co-analytic tight cofinitary group of size $\aleph_1$ (thus $\mathfrak{a}_{\mathrm{g}} = \aleph_1$), a Δ^1_3 well-order of the reals, and $\mathfrak{c} = \aleph_2$.*

Sketch of proof. The first part of the proof is similar to that of [6, Theorem 4.1], so we only give a sketch, point out the necessary changes, and otherwise refer the reader to said paper for any omitted details.

We start by working in L. Construct a sequence $\langle \delta_\xi, z_\xi, G_\xi, \sigma_\xi | \xi < \omega_1 \rangle$ satisfying the following:

(i) δ_ξ is a countable ordinal such that L_{δ_ξ} projects to ω and $\delta_\xi > \delta_\nu + \omega \cdot 2$ for each $\nu < \xi$,
(ii) $z_\xi \in 2^\omega$ such that the canonical surjection from ω to L_{δ_ξ} is computable from z_ξ,
(iii) G_ξ is the group generated by $\{\sigma_\nu \colon \nu < \xi\}$,
(iv) σ_ξ is the generic permutation over L_{δ_ξ} for Zhang's forcing with coding z_ξ into orbits, over the group G_ξ.

Such a sequence is easily constructed by induction: Supposing we already have constructed $\langle \delta_\xi, z_\xi, G_\xi, \sigma_\xi | \xi < \nu \rangle$ let $(\delta_\nu, z_\nu, G_\nu, \sigma_\nu)$ be the $\leq_L$-least triple such that the above items are satisfied. Let

$$G = \bigcup_{\xi < \omega_1} G_\xi.$$

We have seen that G is tight. We verify that G is co-analytic: Fix a formula $\Psi(g)$ such that

$$\Psi(g) \Leftrightarrow (\exists \nu < \omega_1)\ g = \langle \delta_\xi, z_\xi, G_\xi, \sigma_\xi | \xi < \nu \rangle$$

and such that Φ is absolute for initial segments of L, i.e.,

$$(\forall \alpha \in \mathrm{ORD})\ \big[g \in L_\alpha \Rightarrow (\Psi(\vec{g}) \Leftrightarrow L_\alpha \vDash \Psi(\vec{g})\big]$$

A standard argument shows that

$$\begin{aligned} g \in G \Leftrightarrow (\exists y \in 2^\omega)\ & y \text{ codes a well-founded model} \\ & \text{whose transitive collapse } M \text{ satisfies} \\ & g \in M \wedge M \vDash (\exists \vec{g})\ \Psi(\vec{g}) \wedge \vec{g} = \langle \delta_\xi, z_\xi, \sigma_\xi, G_\xi \mid \xi < \nu + 1 \rangle \wedge g \in G_\nu \end{aligned}$$

where, crucially, the right-hand can be written $(\exists y \in 2^\omega)\ \Phi(y, g)$ with Φ a Π^1_1 formula. We now show that

$$g \in G \Leftrightarrow (\exists y \leq_h g)\ \Phi(y, g). \tag{1}$$

It suffices to show $\Rightarrow$. But if $g \in G$, $g = w[\sigma_\xi]$ for some $\xi < \omega_1$, with w a word in G_ξ, and so z_ξ is computable in g. But y as required is computable from z_ξ. As in [6, Theorem 4.1], it can be shown that the right-hand side in (1) is equivalent to a Π^1_1 formula.

Finally, force over L with an iteration of shooting clubs and Sacks coding as in [1, Theorem 6.1]. Since shooting clubs is S-proper and as we have seen that Sacks coding strongly preserves tightness (see Theorem 17), an argument as in [1] shows that the entire iteration also preserves tightness of cofinitary groups. In particular, the tightness of G is preserved. □

6. Further applications

Our construction provides not only a new proof of the existence of a Miller indestructible maximal cofinitary group (originally proved by Kastermans and Zhang with the use of a diamond sequence, see [12]) and answers Question 2 of [1], but gives a uniform proof of the existence of a co-analytic witness to $\mathfrak{a}_g$ in various forcing extensions:

Corollary 44. Each of the following cardinal characteristics constellations is consistent with the existence of a co-analytic tight witness to $\mathfrak{a}_g$ and a Δ^1_3-well-order of the reals:

(1) $\mathfrak{a}_g = \mathfrak{u} = \mathfrak{i} = \aleph_1 < \mathfrak{c} = \aleph_2$,
(2) $\mathfrak{a}_g = \mathfrak{u} = \aleph_1 < \mathfrak{i} = \mathfrak{c} = \aleph_2$,
(3) $\mathfrak{a}_g = \mathfrak{i} = \aleph_1 < \mathfrak{u} = \mathfrak{c} = \aleph_2$,
(4) $\mathfrak{a}_g = \aleph_1 < \mathfrak{i} = \mathfrak{u} = \mathfrak{c} = \aleph_2$.

In addition, in each of the above constellations the characteristics $\mathfrak{a}$, $\mathfrak{a}_e$, $\mathfrak{a}_p$ can have tight co-analytic witnesses of cardinality $\aleph_1$; in items (1) and (2), the ultrafilter number $\mathfrak{u}$ can be witnessed by a co-analytic ultrafilter base for a p-point; in items (1) and (3) the independence number can be witnessed by a co-analytic selective independent family.

Proof. Work over L and proceed with a countable support iteration as in [1, Theorem 6.1] for item (1) and as in [1, Theorem 6.2] for items (2) – (4). □

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$\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ HAS A COFINITARY REPRESENTATION

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Abstract. In [3] Kastermans proved that consistently $\bigoplus_{\aleph_1} \mathbb{Z}_2$ has a cofinitary representation. We present a short proof that $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ always has a cofinitary representation.

1. Introduction

A cofinitary group is a subgroup $G \subseteq S_\omega$ such that every $g \in G \setminus \{\mathrm{id}\}$ only has finitely many fixpoints. It is maximal iff it is maximal with respect to inclusion. We are interested in the possible isomorphism types of (maximal) cofinitary groups. A full classification of all possible isomorphism types of (maximal) cofinitary groups is still open, but there are some results that realize certain groups as (maximal) cofinitary groups and conversely that maximal cofinitary groups cannot possibly have certain isomorphism types. Equivalently, in terms of group actions we may think about which groups may possess a (maximal) cofinitary representation when acting on ω. We may summarize the known restrictions on the possible isomorphism types of maximal cofinitary groups as follows:

Theorem (Truss, [5]; Adeleke, [1])**.** *Every countable cofinitary group is not maximal.*

Theorem (Kastermans, [4])**.** *Every cofinitary group with infinitely many orbits is not maximal.*

As a consequence of this theorem, Blass noticed the following:

Corollary (Blass, [3])**.** *Every abelian cofinitary group is not maximal.*

In terms of definability, Kastermans also proved the following restriction. Recall that a set is K_σ iff it is a countable union of compact sets.

Theorem (Kastermans, [3])**.** *Every K_σ cofinitary group is not maximal.*

On the positive side, Zhang's forcing [7] may be used to force the existence of a maximal cofinitary representation of the free group in κ generators for any κ of uncountable cofinality. For uncountable κ of countable cofinality one may use a product version of Zhang's forcing as in [2]. Further, as a converse to the restriction above, Kastermans [4] proved that consistently for every $n \in \mathbb{N}$ and $m \in \mathbb{N} \setminus \{0\}$ one may force the existence of a cofinitary group with exactly n finite and m infinite orbits. He also proved that consistently there is a locally finite maximal cofinitary group [4]. Finally, a modification of Zhang's forcing also yields:

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Theorem (Kastermans, [3])**.** *There exists a c.c.c. forcing which forces the existence of a cofinitary representation of* $\bigoplus_{\aleph_1} \mathbb{Z}_2$*.*

2. A cofinitary representation of $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$

We will prove that the existence of such a cofinitary representation is not just consistent, but directly follows from ZFC. We also prove the statement for $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ and not just $\bigoplus_{\aleph_1} \mathbb{Z}_2$. For the following proof recall that the main idea for Cayley's theorem is that every group acts freely on itself. Furthermore, $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ has the following representation:

Remark 1. The group $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ can be represented as the free group in $\mathfrak{c}$-many generators modulo the relations a^2 and $ab = ba$ for all generators a, b. Hence, for any set I we may define an action of $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ on I by defining it on the set of generators and verifying the two types of relations above.

Theorem 2. *There is a cofinitary representation of* $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$*.*

Proof. Let $H := \bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ be generated by $\{h_f \mid f \in {}^{\omega}2\}$. Inductively, we will define an interval partition $\langle I_n \mid n < \omega \rangle$ of ω and for every real $f \in 2^{\omega}$ an action of h_f on I_n.

Given $n < \omega$ let $H_n := \bigoplus_{i=1}^{2^n} \mathbb{Z}_2$ be generated by $\{h_s \mid s \in 2^n\}$. Further, let I_n be the interval above $\bigcup_{m<n} I_m$ of size $|H_n|$. H_n acts freely on itself, so also on I_n. Thus, for further computations we may identify I_n with H_n. For every $f \in {}^{\omega}2$ let the action of h_f on H_n be defined as the action of $h_{f \restriction n}$ on H_n. First, we show that this generates a well-defined group action of H on H_n, for if $f \in {}^{\omega}2$ and $h \in H_n$, then we compute

$$h_f.(h_f.h) = h_{f\restriction n}.(h_{f\restriction n}.h) = (h_{f\restriction n} \circ h_{f\restriction n}).h = e.h = h.$$

Now, let $f, g \in {}^{\omega}2$ and $h \in H_n$. Then we compute

$$h_f.(h_g.h) = h_{f\restriction n}.(h_{g\restriction n}.h) = (h_{f\restriction n} \circ h_{g\restriction n}).h = (h_{g\restriction n} \circ h_{f\restriction n}).h = h_{g\restriction n}.(h_{f\restriction n}.h) = h_g.(h_f.h).$$

By the previous remark this suffices. Now, we define a group action of H on ω for $f \in {}^{\omega}2$ by

$$e_f.k := e_{f\restriction n}.k, \quad \text{where } k \in I_n.$$

We already verified that all group actions of H on I_n are well-defined, so we obtain a well-defined group action of H on $\omega = \bigcup_{n<\omega} I_n$. It remains to verify that the action is cofinitary. So let $h \in H$. Choose $F_0 \subseteq {}^{\omega}2$ finite such that $h = \sum_{f \in F_0} h_f \neq e$. Choose $N < \omega$ such that for all $n > N$ and $f \neq g \in F_0$ we have $f \restriction n \neq g \restriction n$. We finish the proof by showing that for $n > N$ we have that h acting on H_n has no fixpoints.

But on one hand, by choice of N we have $\sum_{f \in F_0} e_{f\restriction n} \neq e$. On the other hand, H_n acts freely on itself which implies that the action of $\sum_{f \in F_0} e_{f\restriction n}$ on H_n has no fixpoints. But $h = \sum_{f \in F_0} e_f$ acts on H_n the same way $\sum_{f \in F_0} e_{f\restriction N}$ does, so that also h acting on H_n has no fixpoints. □

Remark 3. Note that since $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2$ is abelian, by the Blass' Corollary above it cannot have a maximal cofinitary representation. One may also directly observe that the representation in the proof above is not maximal as the action of H has infinitely many orbits, namely the I_n's.

3. Questions

By our theorem there are always groups of size $\mathfrak{c}$ which have a cofinitary representation, but no maximal cofinitary representation. However, it is not known, whether possibly every subgroup of S_ω (consistently) has a cofinitary representation:

Question 4. Does S_ω (consistently) have a (maximal) cofinitary representation?

Let CGA (cofinitary group axiom) be the statement that every group G with $|G| < \mathfrak{c}$ has a cofinitary representation. Note that every countable group has a cofinitary representation [1], i.e. CH implies CGA. Similar to Martin's axiom we may ask:

Question 5. Is CGA consistent with large continuum?

Question 6. Is there always a group of size $\mathfrak{c}$ without a cofinitary representation?

Note that these questions are very closely related to the problem of finding minimal permutation representations of groups. For finite groups this has been intensively studied for many different families of groups, see [6] for a nice summary. We let GA (group axiom) be the statement that every group G with $|G| < \mathfrak{c}$ may be embedded into S_ω. Clearly, CGA implies GA. Note that Cayley's theorem states that any group G can be embedded into $S_{|G|}$. In particular, Cayley's theorem shows that CH implies GA.

Question 7. Is GA consistent with large continuum?

Question 8. Is there always a group of size $\mathfrak{c}$ which cannot be embedded into S_ω?

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VAN DOUWEN AND MANY NON VAN DOUWEN FAMILIES

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ABSTRACT. We prove that the spectrum of Van Douwen families is closed under singular limits. For any maximal eventually different family in [5] Raghavan defined an associated ideal which measures how far the family is from being Van Douwen. Under CH we prove that every non-principal ideal is realized as the associated ideal of some maximal eventually different family, i.e. there are many different non Van Douwen families.

1. Introduction

Let $\mathrm{spec}(\mathfrak{a}_\mathrm{e})$ be the set of all sizes of maximally eventually different families and $\mathfrak{a}_\mathrm{e}$ its minimum (see Definition 2.1). Of particular interest for us are the following two well-known open questions. Since $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_\mathrm{e}$ [1] and $\mathfrak{a} < \mathrm{non}(\mathcal{M})$ is consistent (for example in the random model), also $\mathfrak{a} < \mathfrak{a}_\mathrm{e}$ is consistent. However, the consistency of the other direction is an open problem:

Question 1.1. Does ZFC prove $\mathfrak{a} \leq \mathfrak{a}_\mathrm{e}$?

Secondly, it seems to be a shared property among spectra of combinatorial families to be closed under singular limits. Hechler first proved this property for mad families in [3]. Recently, Brian also verified that the spectrum of partitions of Baire space into compact sets is closed under singular limits [2]. The analogous question for maximal eventually different families is still open:

Question 1.2. Is $\mathrm{spec}(\mathfrak{a}_\mathrm{e})$ closed under singular limits?

Instead of answering these questions for maximal eventually different families, we will instead consider them for the stronger notion of Van Douwen families. A maximal eventually different family is Van Douwen iff it is also maximal with respect to infinite partial functions (see Definition 2.3). Van Douwen asked whether families with this strong kind of maximality always exists (see problem 4.2 in Miller's problem list [4]). In [6] Zhang proved that Van Douwen families of desired sizes may be forced by a c.c.c. forcing, so that MA implies the existence of a Van Douwen family of size $\mathfrak{c}$. Later, Raghavan [5] proved that there always is a Van Douwen family of size $\mathfrak{c}$. So, let $\mathrm{spec}(\mathfrak{a}_\mathrm{v})$ be the set of sizes of Van Douwen families and $\mathfrak{a}_\mathrm{v}$ its minimum. It is not hard to prove and well-known that $\mathfrak{a} \leq \mathfrak{a}_\mathrm{v}$ holds (see Corollary 2.7), so Question 1.1 has a positive answer for Van Douwen families. In Theorem 3.3 we provide a short argument that the standard forcing adding a maximal eventually different family of desired size $\mathbb{E}_{\mathcal{F}}(I)$ also adds a maximal almost disjoint family of the same size:

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Theorem. *Let $\mathcal{F}$ be an e.d. family and I an uncountable index set. Then*

$$\mathbb{E}_{\mathcal{F}}(I) \Vdash \max(|\mathcal{F}|, |I|) \in \operatorname{spec}(\mathfrak{a}).$$

Hence, the standard forcing for realizing a desired spectrum of $\mathfrak{a}_{\mathrm{e}}$ also forces $\mathfrak{a}$ to have the same spectrum. Further, as it is the case for $\mathfrak{a}$ and $\mathfrak{a}_{\mathrm{T}}$, in Theorem 4.1 we show that Question 1.2 also has a positive answer for Van Douwen families:

Theorem. $\operatorname{spec}(\mathfrak{a}_{\mathrm{v}})$ *is closed under singular limits.*

Clearly, $\operatorname{spec}(\mathfrak{a}_{\mathrm{v}}) \subseteq \operatorname{spec}(\mathfrak{a}_{\mathrm{e}})$. One of the central open questions regarding Van Douwen families is if we always have equality:

Question 1.3. Does $\operatorname{spec}(\mathfrak{a}_{\mathrm{v}}) = \operatorname{spec}(\mathfrak{a}_{\mathrm{e}})$ hold?

Notice that a positive answer together with our Theorem 4.1 would yield a positive answer for the well-known open Question 1.2. Moreover, in order to answer Question 1.1 the following weaker version would suffice:

Question 1.4. Does $\mathfrak{a}_{\mathrm{v}} = \mathfrak{a}_{\mathrm{e}}$ hold?

Towards this question, it is interesting to study non Van Douwen families. For any maximal eventually different family $\mathcal{F}$ in [5] Raghavan defined an associated ideal $\mathcal{I}_0(\mathcal{F})$ which measures how far the family is from being Van Douwen (see Definition 5.1). We prove that under CH any non-principal ideal may be realized as the associated ideal of some maximal eventually different family (see Theorem 5.6), i.e. there are many different non Van Douwen families.

Theorem. *Assume* CH *and let $\mathcal{I}$ be a non-principal ideal. Then there is a maximal eventually different family such that $\mathcal{I} = \mathcal{I}_0(\mathcal{F})$.*

2. Preliminaries

Definition 2.1. We say $f, g \in {}^{\omega}\omega$ are eventually different iff $\{n < \omega \mid f(n) = g(n)\}$ is finite. A family $\mathcal{F} \subseteq {}^{\omega}\omega$ is called eventually different (e.d.) iff all $f \neq g \in \mathcal{F}$ are eventually different. It is called maximal (m.e.d.) iff it is maximal with respect to inclusion. Finally, we define the associated spectrum and cardinal characteristic

$$\operatorname{spec}(\mathfrak{a}_{\mathrm{e}}) := \{|\mathcal{F}| \mid \mathcal{F} \text{ is a m.e.d. family}\},$$
$$\mathfrak{a}_{\mathrm{e}} := \min(\operatorname{spec}(\mathfrak{a}_{\mathrm{e}})).$$

Remark 2.2. For any countably infinite A, B we may equivalently consider m.e.d. families $\mathcal{F} \subseteq {}^{A}B$ by using bijections with ω. In most cases $A, B = \omega$, however we will also consider the cases $A \in [\omega]^{\omega}$ and $B = \omega \times \omega$.

Definition 2.3. Let $\mathcal{F} \subseteq {}^{\omega}\omega$ and $A \in [\omega]^{\omega}$. Then we define $\mathcal{F} \restriction A := \{f \restriction A \mid f \in \mathcal{F}\}$. We call $\mathcal{F}$ Van Douwen iff $\mathcal{F} \restriction A$ is a m.e.d. family for all $A \in [\omega]^{\omega}$. Analogously, we define the associated spectrum and cardinal characteristic

$$\operatorname{spec}(\mathfrak{a}_{\mathrm{v}}) := \{|\mathcal{F}| \mid \mathcal{F} \text{ is Van Douwen}\},$$
$$\mathfrak{a}_{\mathrm{v}} := \min(\operatorname{spec}(\mathfrak{a}_{\mathrm{v}})).$$

Clearly, we have $\operatorname{spec}(\mathfrak{a}_{\mathrm{v}}) \subseteq \operatorname{spec}(\mathfrak{a}_{\mathrm{e}})$. Unlike as for other notions of strong maximality, such as ω-maximality or tightness, Raghavan [5] proved that there always exists a Van Douwen family of size $\mathfrak{c}$, i.e. the cardinal characteristic $\mathfrak{a}_{\mathrm{v}}$ is well-defined. Next, we present a short argument why $\mathfrak{a} \leq \mathfrak{a}_{\mathrm{v}}$ holds.

Definition 2.4. Let $\mathcal{F}$ be an e.d. family. Then we define

$$\operatorname{cov}(\mathcal{F}) := \{g \in {}^{\omega}\omega \mid \exists \mathcal{F}_0 \in [\mathcal{F}]^{<\omega}\ \exists N < \omega\ \forall n \geq N\ \exists f \in \mathcal{F}_0\ f(n) = g(n)\},$$

i.e. $g \in \operatorname{cov}(\mathcal{F})$ iff its graph can almost be covered by finitely many elements of $\mathcal{F}$. Further, we set $\operatorname{cov}^+(\mathcal{F}) := {}^{\omega}\omega \setminus \operatorname{cov}(\mathcal{F})$.

Proposition 2.5. *Let $\mathcal{F}$ be an e.d. family, $g \in {}^{\omega}\omega$, $\mathcal{F}_1 \in [\mathcal{F}]^{\omega}$ and assume $g =^{\infty} f$ for every $f \in \mathcal{F}_1$. Then $g \in \operatorname{cov}^+(\mathcal{F})$.*

Proof. Let $\mathcal{F}_0 \in [\mathcal{F}]^{<\omega}$ and $N_0 < \omega$. Choose $f \in \mathcal{F}_1 \setminus \mathcal{F}_0$. Since $\mathcal{F}$ is e.d. choose $N_1 \geq N_0$ such that $f(n) \neq f_0(n)$ for all $f_0 \in \mathcal{F}_0$ and $n \geq N_1$. Since $g =^{\infty} f$ choose $n \geq N_1$ such that $f(n) = g(n)$. Hence, $g(n) \neq f_0(n)$ for all $f_0 \in \mathcal{F}_0$. Thus, $g \in \operatorname{cov}^+(\mathcal{F})$. □

Proposition 2.6. *Let $\mathcal{F}$ be a Van Douwen family and $g \in {}^{\omega}\omega$. For every $f \in \mathcal{F}$ define*

$$E_g^f := \{n < \omega \mid f(n) = g(n)\}.$$

Then, $\mathcal{A}_g^{\mathcal{F}} := \{E_g^f \mid f \in \mathcal{F}$ such that $E_g^f \in [\omega]^{\omega}\}$ is a (possibly finite) mad family. Further, $\mathcal{A}_g^{\mathcal{F}}$ is infinite iff $g \in \operatorname{cov}^+(\mathcal{F})$.

Proof. Note that $\mathcal{A}_g^{\mathcal{F}}$ is almost disjoint as for all $f \neq f' \in \mathcal{F}$ we have that f and f' are eventually different. Hence, $E_g^f \cap E_g^{f'}$ is finite. Towards maximality, let $A \in [\omega]^{\omega}$. Since $\mathcal{F}$ is Van Douwen choose $f \in \mathcal{F}$ and $B \in [A]^{\omega}$ such that $f \restriction B = g \restriction B$. Thus, $B \subseteq E_g^f$, which shows that $E_g^f \in \mathcal{A}_g^{\mathcal{F}}$ and $A \cap E_g^f$ is infinite.

Now, assume $g \in \operatorname{cov}(\mathcal{F})$. Choose $\mathcal{F}_0 \in [\mathcal{F}]^{<\omega}$ as in the definition of $\operatorname{cov}(\mathcal{F})$ and let $f \in \mathcal{F} \setminus \mathcal{F}_0$. But if $E_g^f \in [\omega]^{\omega}$ there would be a $f_0 \in \mathcal{F}_0$ such that $E_g^f \cap E_g^{f_0}$ is infinite, i.e. f and f_0 are not eventually different, a contradiction. Thus, $\mathcal{A}_g^{\mathcal{F}}$ is finite.

Now, assume $\mathcal{A}_g^{\mathcal{F}}$ is finite, so choose $\mathcal{F}_0 \in [\mathcal{F}]^{<\omega}$ with $\mathcal{A}_g^{\mathcal{F}} = \{E_g^f \mid f \in \mathcal{F}_0 \text{ and } E_g^f \in [\omega]^{\omega}\}$. Since $\mathcal{A}_g^{\mathcal{F}}$ is maximal we have $\omega \subseteq^* \bigcup_{f \in \mathcal{F}_0} E_g^f$. Hence, $g \in \operatorname{cov}(\mathcal{F})$ is witnessed by $\mathcal{F}_0$. □

Corollary 2.7. $\mathfrak{a} \leq \mathfrak{a}_{\mathrm{v}}$.

Proof. Let $\mathcal{F}$ be a witness for $\mathfrak{a}_{\mathrm{v}}$. By the previous proposition it suffices to find a $g \in \operatorname{cov}^+(\mathcal{F})$. Choose a disjoint partition $\omega = \bigcup_{k<\omega} A_k$ into infinite sets and a subset $\{f_k \mid k < \omega\}$ from $\mathcal{F}$. Then, we define

$$g(n) := f_k(n), \text{ where } n \in A_k.$$

By Proposition 2.5 we have $g \in \operatorname{cov}^+(\mathcal{F})$. □

Remark 2.8. As in Proposition 2.6 for any eventually different family $\mathcal{F}$ we may define its trace

$$\operatorname{tr}(\mathcal{F}) := \{g \in {}^{\omega}\omega \mid \mathcal{A}_g^{\mathcal{F}} \text{ is a mad family}\}.$$

Proposition 2.6 then implies that $\operatorname{cov}(\mathcal{F}) \subseteq \operatorname{tr}(\mathcal{F})$ and Van Douwen families satisfy $\operatorname{tr}(\mathcal{F}) = {}^{\omega}\omega$. Conversely, $\operatorname{tr}(\mathcal{F}) = {}^{\omega}\omega$ also implies that $\mathcal{F}$ is Van Douwen, for if $A \in [\omega]^{\omega}$ and $g : A \to \omega$, let g^* be any extension of g to ω. By assumption $\mathcal{A}^{\mathcal{F}}_{g^*}$ is mad, so choose $f \in \mathcal{F}$ with $E^f_{g^*} \cap A$ infinite. But then $f \restriction (E^f_{g^*} \cap A) = g \restriction (E^f_{g^*} \cap A)$, i.e. $\mathcal{F}$ is Van Douwen. Note that the trace is one of the crucial ingredients of the construction of a Van Douwen family in [5].

3. $\mathbb{E}_{\mathcal{F}}(I)$ adds a mad family of size $\max(|\mathcal{F}|, |I|)$

We recall the standard c.c.c. forcing $\mathbb{E}_{\mathcal{F}}(I)$ for extending an eventually different family $\mathcal{F}$ by I-many new eventually different reals:

Definition 3.1. Let $\mathcal{F}$ be an e.d. family and I an index set. Let $\mathbb{E}_{\mathcal{F}}(I)$ be the partial order of pairs (s, E), where $s : I \times \omega \overset{\text{part}}{\to} \omega$ is a finite partial function and $E \in [\mathcal{F}]^{<\omega}$. For $(s, E) \in \mathbb{E}_{\mathcal{F}}(I)$ and $i \in I$ we define the finite partial function $s_i : \omega \overset{\text{part}}{\to} \omega$ by $s_i := \{(n, m) \mid (i, n, m) \in s\}$ and set $\operatorname{supp}(s) := \{i \in I \mid s_i \neq \emptyset\}$. For $(s, E), (t, F) \in \mathbb{E}_{\mathcal{F}}(I)$ we define $(t, F) \leq (s, E)$ iff

(1) $s \subseteq t$ and $E \subseteq F$,
(2) for all $i \neq j \in \operatorname{supp}(s)$ and $n \in \operatorname{dom}(t_i) \setminus \operatorname{dom}(s_i)$ we have $n \notin \operatorname{dom}(t_j)$ or $t_i(n) \neq t_j(n)$,
(3) for all $i \in \operatorname{supp}(s)$, $f \in E$ and $n \in \operatorname{dom}(t_i) \setminus \operatorname{dom}(s_i)$ we have $t_i(n) \neq f(n)$.

For $|I| = 1$ Zhang [6] proved that iterating this forcing of uncountable cofinality yields a Van Douwen family. Similar density arguments give the same result for the product version:

Lemma 3.2. *Let $\mathcal{F}$ be an e.d. family and I an uncountable index set. Then*

$$\mathbb{E}_{\mathcal{F}}(I) \Vdash \mathcal{F} \cup \dot{\mathcal{F}}_{gen} \text{ is a Van Douwen family,}$$

where $\dot{\mathcal{F}}_{gen}$ is the family I-many eventually different reals added by $\mathbb{E}_{\mathcal{F}}(I)$.

Note that in contrast to the iteration-version Zhang considered in [6], with the product-version it is possible to add Van Douwen families of uncountable size with countable cofinality. We use Lemma 3.2 and a similar argument as in the proof of $\mathfrak{a} \leq \mathfrak{a}_{\mathrm{v}}$ (Corollary 2.7) to prove that the standard forcing to realize a desired spectrum of $\mathfrak{a}_{\mathrm{e}}$ also forces $\mathfrak{a}$ to have the same spectrum.

Theorem 3.3. *Let $\mathcal{F}$ be an e.d. family and I an uncountable index set. Then*

$$\mathbb{E}_{\mathcal{F}}(I) \Vdash \max(|\mathcal{F}|, |I|) \in \operatorname{spec}(\mathfrak{a}).$$

Proof. Choose a disjoint partition $\omega = \bigcup_{i<\omega} A_i$ into infinite sets and a subset $I_0 = \{i_k \mid k < \omega\}$ from I. Let G be $\mathbb{E}_{\mathcal{F}}(I)$-generic. In $V[G]$ we define

$$g(n) := f^{i_k}_{\text{gen}}(n), \text{ where } n \in A_k.$$

By construction, we have $E_g^{f^{i_k}_{\text{gen}}} \in [\omega]^{\omega}$ for all $k < \omega$. We show that also $E^f_g \in [\omega]^{\omega}$ for all $f \in \mathcal{F}$. To this end, in V let $N < \omega$ and $(s, E) \in \mathbb{E}_{\mathcal{F}}(I)$. Choose $k < \omega$ such that $i_k \notin \operatorname{supp}(s)$ and $n \geq N$ with $n \in A_k$. Then $(s \cup \{(i_k, n, f(n))\}, E) \in \mathbb{E}_{\mathcal{F}}(I_0)$, $(s \cup \{(i_k, n, f(n))\}, E) \leq (s, E)$ and

$$(s \cup \{(i_k, n, f(n))\}, E) \Vdash \dot{g}(n) = \dot{f}^{i_k}_{\text{gen}}(n) = f(n).$$

Finally, we show that $E_g^{f^i_{\rm gen}} \in [\omega]^\omega$ for all $i \in I \backslash I_0$. To this end, in V let $N < \omega$ and $(s, E) \in \mathbb{E}_{\mathcal{F}}(I)$. We may assume that $i \in \operatorname{dom}(s)$. Choose $k < \omega$ such that $i_k \notin \operatorname{supp}(s)$ and $n \geq N$ with $n \in A_k$ and $n \notin \operatorname{dom}(s_j)$ for all $j \in \operatorname{dom}(s)$. Finally, choose $m \in \omega \setminus \{f(n) \mid f \in E\}$. Then we have that $(s \cup \{(i_k, n, m), (i, n, m)\}, E) \in \mathbb{E}_{\mathcal{F}}(I_0)$, $(s \cup \{(i_k, n, m), (i, n, m)\}, E) \leq (s, E)$ and

$$(s \cup \{(i_k, n, m), (i, n, m)\}, E) \Vdash \dot{g}(n) = \dot{f}^{i_k}_{\rm gen}(n) = m = \dot{f}^{i}_{\rm gen}(n).$$

Hence, by Lemma 3.2 and Proposition 2.6 we obtain

$$\mathbb{E}_{\mathcal{F}}(I) \Vdash \mathcal{A}_{\dot{g}}^{\mathcal{F} \cup \dot{\mathcal{F}}_{\rm gen}} \text{ is a mad family of size } \max(|\mathcal{F}|, |I|),$$

completing the proof. □

4. Spectrum of Van Douwen families

In this section, similar to $\mathfrak{a}$ [3] and $\mathfrak{a}_{\rm T}$ [2] we prove that the spectrum of Van Douwen families is closed under singular limits. The main idea is that we may glue a sequence of Van Douwen families together in order to obtain a bigger Van Douwen family. A similar argument fails for maximal eventually different families as the gluing argument we provide might not preserve maximality. Hence, the corresponding Question 1.2 for $\mathfrak{a}_{\rm e}$ is still open.

Theorem 4.1. $\operatorname{spec}(\mathfrak{a}_{\rm v})$ *is closed under singular limits.*

Proof. Let $\kappa = \operatorname{cof}(\lambda) < \lambda$, $\langle \kappa_\alpha \mid \alpha < \lambda \rangle$ be an increasing sequence of cardinals cofinal in λ with $\kappa < \kappa_0$ and $\langle \mathcal{F}_\alpha \mid \alpha < \kappa \rangle$ a sequence of Van Douwen families with $|\mathcal{F}_\alpha| = \kappa_\alpha$. Choose pairwise different elements $\mathcal{G} = \langle g_\alpha \in \mathcal{F}_0 \mid \alpha < \kappa \rangle$. We construct an eventually different family of functions from $\omega \to \omega \times \omega$ of size λ as follows:

- Given $\alpha < \kappa$ and $f \in \mathcal{F}_\alpha$ define $(g_\alpha \times f) : \omega \to (\omega \times \omega)$ for $k < \omega$ by
$$(g_\alpha \times f)(k) := (g_\alpha(k), f(k)).$$
- Given $f_0 \in \mathcal{F}_0 \setminus \mathcal{G}$ and $f_1 \in \mathcal{F}_0$ define $(f_0 \times f_1) : \omega \to (\omega \times \omega)$ for $k < \omega$ by
$$(f_0 \times f_1)(k) := (f_0(k), f_1(k)).$$

Finally, we define the family $\mathcal{F}$ to be family of all functions from $\omega \to (\omega \times \omega)$ of one of the two forms above. Then $\mathcal{F}$ is of size λ and we claim that $\mathcal{F}$ is Van Douwen. First, we prove that $\mathcal{F}$ is e.d., so we have to consider the following cases:

- Let $\alpha < \kappa$ and $f \neq f' \in \mathcal{F}_\alpha$. Since f and f' are e.d. choose $K < \omega$ such that $f(k) \neq f'(k)$ for all $k \geq K$. But then for every $k \geq K$ we have
$$(g_\alpha \times f)(k) = (g_\alpha(k), f(k)) \neq (g_\alpha(k), f'(k)) = (g_\alpha \times f')(k).$$
- Let $\alpha \neq \beta < \kappa$ and $f \in \mathcal{F}_\alpha, f' \in \mathcal{F}_\beta$. Since g_α and g_β are e.d. choose $K < \omega$ such that $g_\alpha(k) \neq g_\beta(k)$ for all $k \geq K$. But then for every $k \geq K$ we have
$$(g_\alpha \times f)(k) = (g_\alpha(k), f(k)) \neq (g_\beta(k), f'(k)) = (g_\beta \times f')(k).$$

- Let $\alpha < \kappa$, $f \in \mathcal{F}_\alpha$, $f_0 \in \mathcal{F}_0 \setminus \mathcal{G}$ and $f_1 \in \mathcal{F}_0$. Since g_α and f_0 are e.d. choose $K < \omega$ such that $g_\alpha(k) \neq f_0(k)$ for all $k \geq K$. But then for every $k \geq K$ we have
$$(g_\alpha \times f)(k) = (g_\alpha(k), f(k)) \neq (f_0(k), f_1(k)) = (f_0 \times f_1)(k).$$
- Let $f_0, f_0' \in \mathcal{F}_0 \setminus \mathcal{G}$ and $f_1, f_1' \in \mathcal{F}_0$ with $(f_0', f_1') \neq (f_0, f_1)$. W.l.o.g. assume $f_0 \neq f_0'$. Then f_0 and f_0' are e.d., so choose $K < \omega$ such that $f_0(k) \neq f_0'(k)$ for all $k \geq K$. But then for every $k \geq K$ we have
$$(f_0 \times f_1)(k) = (f_0(k), f_1(k)) \neq (f_0'(k), f_1'(k)) = (f_0' \times f_1')(k).$$

Hence, it remains to prove that $\mathcal{F}$ is Van Douwen. So let $A \in [\omega]^\omega$ and $h : A \to (\omega \times \omega)$. For $i \in 2$ let $p_i(h) : A \to \omega$ be the projection of h to the i-th component. As $\mathcal{F}_0$ is Van Douwen choose $B \in [A]^\omega$ and $f_0 \in \mathcal{F}_0$ such that $f_0 \restriction B = p_0(h) \restriction B$. We consider the following two cases:

$f_0 \in \mathcal{G}$. Choose $\alpha < \kappa$ such that $f_0 = g_\alpha$. As $\mathcal{F}_\alpha$ is Van Douwen choose $C \in [B]^\omega$ and $f \in \mathcal{F}_\alpha$ such that $p_1(h) \restriction C = f \restriction C$. But then for every $k \in C$ we have
$$(g_\alpha \times f)(k) = (g_\alpha(k), f(k)) = (p_0(h)(k), p_1(h)(k)) = h(k).$$

Otherwise, $f_0 \in \mathcal{F}_0 \setminus \mathcal{G}$. As $\mathcal{F}_0$ is Van Douwen choose $C \in [B]^\omega$ and $f_1 \in \mathcal{F}_0$ with $p_1(h) \restriction C = f_1$. But then for every $k \in C$ we have
$$(f_0 \times f_1)(k) = (f_0(k), f_1(k)) = (p_0(h)(k), p_1(h)(k)) = h(k).$$

Hence, in both cases h is not eventually different from some element in $\mathcal{F} \restriction A$. □

5. Many non Van Douwen families

Given a maximal eventually different family $\mathcal{F}$ in [5] Raghavan introduced the following ideal $\mathcal{I}_0(\mathcal{F})$ measuring how far $\mathcal{F}$ is from being Van Douwen.

Definition 5.1. Let $\mathcal{F}$ be m.e.d. family. Then we define
$$\mathcal{I}_0(\mathcal{F}) := \{A \in [\omega]^\omega \mid \mathcal{F} \restriction A \text{ is not a m.e.d. family}\} \cup \mathrm{Fin}\,.$$

Proposition 5.2. *$\mathcal{I}_0(\mathcal{F})$ is a non-principal ideal.*

Proof. Since $\mathcal{F}$ is maximal we have $\omega \notin \mathcal{I}_0(\mathcal{F})$. If $A \in \mathcal{I}_0(\mathcal{F})$ and $B \in [A]^\omega$ choose $g : A \to \omega$ eventually different from $\mathcal{F} \restriction A$. But then $g \restriction B : B \to \omega$ is eventually different from $\mathcal{F} \restriction B$. Hence, $B \in \mathcal{I}_0(\mathcal{F})$.

Finally, let $A, B \in \mathcal{I}_0(\mathcal{F})$. We may assume that $A \in [\omega]^\omega$, so choose $g : A \to \omega$ eventually different from $\mathcal{F} \restriction A$. If B is finite, then any extension of $f : A \to \omega$ to $f^* : (A \cup B) \to \omega$ is eventually different from $\mathcal{F} \restriction (A \cup B)$, so assume that $B \in [\omega]^\omega$ and choose $h : B \to \omega$ eventually different from $\mathcal{F} \restriction B$. We claim that $g \cup h \restriction (A \setminus B) : (A \cup B) \to \omega$ is eventually different from $\mathcal{F} \restriction (A \cup B)$, so let $f \in \mathcal{F}$. By choice of g and h there are $K_0, K_1 < \omega$ such that $g(k) \neq f(k)$ for every $k \in A \setminus K_0$ and $h(k) \neq f(k)$ for every $k \in B \setminus K_1$. Hence, $(g \cup h \restriction (B \setminus A))(k) \neq f(k)$ for all $k \in (A \cup B) \setminus (K_0 \cup K_1)$. □

Corollary 5.3. *$\mathcal{F}$ is Van Douwen iff $\mathcal{I}_0(\mathcal{F}) = \mathrm{Fin}$.*

We prove that under CH any non-principal ideal may be realized as the associated $\mathcal{I}_0$-ideal of some maximal eventually different family. Towards this goal, we need the following two diagonalization lemmata:

Lemma 5.4. *Let $\mathcal{F} = \langle f_n \mid n < \omega \rangle$ be e.d. and $A \in [\omega]^\omega$. Then there is $g : A \to \omega$ such that $\mathcal{F} \restriction A \cup \{g\}$ is e.d.*

Proof. Enumerate A by $\{a_n \mid n < \omega\}$. Inductively, choose $g(a_n)$ different from $\{f_m(a_n) \mid m < n\}$. By construction $\mathcal{F} \restriction A \cup \{g\}$ is e.d. □

Lemma 5.5. *Let $\mathcal{I}$ be a non-principal ideal, $\mathcal{F} = \langle f_n \mid n < \omega \rangle$ be e.d. and $\langle g_n : A_n \to \omega \mid n < \omega \rangle$ be such that $A_n \in \mathcal{I}$ and $\mathcal{F} \restriction A_n \cup \{g_n\}$ is e.d. for all $n < \omega$. Further, let $h : B \to \omega$ such that $B \notin \mathcal{I}$ and $\mathcal{F} \restriction B \cup \{h\}$ is e.d. Then there is $f : \omega \to \omega$ such that*

1. *$\mathcal{F} \cup \{f\}$ is e.d.,*
2. *$(\mathcal{F} \cup \{f\}) \restriction A_n \cup \{g_n\}$ is e.d. for all $n < \omega$,*
3. *$f \restriction C = h \restriction C$ for some $C \in [B]^\omega$.*

Proof. We define an increasing sequence of finite partial functions $\langle s_n \mid n < \omega \rangle$ as follows. Set $s_0 := \emptyset$. Now, let $n < \omega$ and assume s_n is defined. By assumption, choose $K < \omega$ such that $\mathrm{dom}(s_n) \subseteq K$ and for all $k \in B \setminus K$ and $m < n$ we have $f_m(k) \neq h(k)$. Since $\mathcal{I}$ is a non-principal ideal and $B \notin \mathcal{I}$, the set $B \setminus \bigcup_{m<n} A_m$ is infinite, so choose $k \in B \setminus K$ with $k \notin A_m$ for all $m < n$. Now, set $s'_{n+1} := s_n \cup \{(k, h(k))\}$. Finally, if $n \in \mathrm{dom}(s'_{n+1})$ set $s_{n+1} := s'_{n+1}$, otherwise choose $l < \omega$ such that $l \neq f_m(n)$ for all $m < n$ and $l \neq g_m(n)$ for all $m < n$ with $n \in A_m$ and set $s_{n+1} := s'_{n+1} \cup \{(n, l)\}$.

Set $f := \bigcup_{n<\omega} s_n$. Then, $f : \omega \to \omega$ as $n \in \mathrm{dom}(s_{n+1})$ for all $n < \omega$. Further, we verify (1-3):

1. Let $m < \omega$, then for every $n > m$ and $k \in \mathrm{dom}(s_{n+1}) \setminus \mathrm{dom}(s_n)$ we compute that $f(k) = s_{n+1}(k) \neq f_m(k)$ by choice of K or l, i.e. f and f_m are e.d.,
2. Let $m < \omega$, then for every $n > m$ and $k \in (A_m \cap \mathrm{dom}(s_{n+1})) \setminus \mathrm{dom}(s_n)$ we compute that $f(k) = s_{n+1}(k) \neq g_m(k)$ by choice of k or l, i.e. $f \restriction A_m$ and g_m are e.d.,
3. For every $n < \omega$ there is a $k \in \mathrm{dom}(s_{n+1}) \setminus \mathrm{dom}(s_n)$ such that $f(k) = s_{n+1}(k) = h(k)$, i.e. $f \restriction C = h \restriction C$ for some $C \in [B]^\omega$.

Hence, f is as desired. □

Theorem 5.6. *Assume* CH *and let $\mathcal{I}$ be a non-principal ideal. Then there is a m.e.d. family $\mathcal{F}$ such that $\mathcal{I} = \mathcal{I}_0(\mathcal{F})$.*

Proof. Enumerate all functions $\{h_\alpha : B_\alpha \to \omega \mid \alpha < \aleph_1\}$, where $B_\alpha \notin \mathcal{I}$, and $\mathcal{I}$ by $\langle A_\alpha \mid \alpha < \aleph_1 \rangle$. We inductively construct a m.e.d. family $\langle f_\alpha \mid \alpha < \aleph_1 \rangle$ and functions $\langle g_\alpha : A_\alpha \to \omega \mid \alpha < \aleph_1 \rangle$ while preserving the following properties for every $\alpha < \aleph_1$:

1. $\mathcal{F}_{<\alpha} := \{f_\beta \mid \beta < \alpha\}$ is e.d.,
2. $\mathcal{F}_{<\alpha} \restriction A_\beta \cup \{g_\beta\}$ is e.d. for all $\beta < \alpha$,
3. For all $\beta < \alpha$ if $\mathcal{F}_{<\beta} \restriction B_\beta \cup \{h_\beta\}$ is e.d., then $f_\beta \restriction C = h_\beta \restriction C$ for some $C \in [B_\beta]^\omega$.

Let $\alpha < \aleph_1$. By induction assumption we may apply Lemma 5.5 to $\mathcal{F}_{<\alpha}$, $\langle g_\beta : A_\alpha \to \omega \mid \beta < \alpha \rangle$ and $h_\alpha : B_\alpha \to \omega$ to obtain $f_\alpha : \omega \to \omega$ such that

(1) $\mathcal{F}_{<\alpha} \cup \{f_\alpha\}$ is e.d.,
(2) $(\mathcal{F}_{<\alpha} \cup \{f_\alpha\}) \restriction A_\beta \cup \{g_\beta\}$ is e.d. for all $\beta < \alpha$,
(3) If $\mathcal{F}_{<\alpha} \restriction B_\alpha \cup \{h_\alpha\}$ is e.d., then $f_\alpha \restriction C = h_\alpha \restriction C$ for some $C \in [B_\alpha]^\omega$.

Finally, by Lemma 5.4 choose $g_\alpha : A_\alpha \to \omega$ such that $(\mathcal{F}_{<\alpha} \cup \{f_\alpha\}) \restriction A_\alpha \cup \{g_\alpha\}$ is e.d.

Then, by (1) $\mathcal{F} := \langle f_\alpha \mid \alpha < \aleph_1 \rangle$ is e.d. and we claim that $\mathcal{I}_0(\mathcal{F}) = \mathcal{I}$. First, let $A \in \mathcal{I}$. Choose $\alpha < \aleph_1$ such that $A = A_\alpha$. By (2) we have that $\mathcal{F} \restriction A \cup \{g_\alpha\}$ is e.d., which witnesses $A \in \mathcal{I}_0(\mathcal{F})$.

Secondly, let $B \notin \mathcal{I}$ and assume $B \in \mathcal{I}_0(\mathcal{F})$. Choose $h : B \to \omega$ such that $\mathcal{F} \restriction B \cup \{h\}$ is e.d. Choose $\alpha < \aleph_1$ such that $B = B_\alpha$ and $h = h_\alpha$. Then also $\mathcal{F}_{<\alpha} \restriction B \cup \{h\}$ is e.d. Thus, by (3) we have $f_\alpha \restriction C = h \restriction C$ for some $C \in [B]^\omega$, which contradicts that $f_\alpha \restriction B$ and h are e.d. □

6. Questions

Notice that with Theorem 5.6 under CH we may construct Van Douwen families $\mathcal{F}$, so that $\mathcal{I}_0(\mathcal{F})$ is a maximal ideal. These kind of maximal eventually families are in some sense as far as possible away from being Van Douwen while still being maximal. Hence, we may define

Definition 6.1. A maximal eventually family is called very non Van Douwen iff $\mathcal{I}_0(\mathcal{F})$ is a maximal ideal. Equivalently, for any $A \in [\omega]^\omega \setminus \mathrm{cofin}(\omega)$ exactly one of the following holds:
(1) Either there is $g : A \to \omega$ such that $\mathcal{F} \restriction A \cup \{g\}$ is e.d.,
(2) or there is $g : A^c \to \omega$ such that $\mathcal{F} \restriction A^c \cup \{g\}$ is e.d.

Similar to Van Douwen's question we may ask:

Question 6.2. Does there always exist a very non Van Douwen family?

Further, under CH these very non Van Douwen families may have a selective ideal as their associated ideal, whose selectivity is preserved under various forcings. Note, this does not imply that the maximality of $\mathcal{F}$ is preserved, but we may ask if our constructions can be adapted to also make sure that the ideal $\mathcal{I}_0(\mathcal{F})$ interpreted in the forcing extension is equal to the generated ideal of $\mathcal{I}_0(\mathcal{F})$ interpreted in the ground model. Then in particular, the maximality of $\mathcal{F}$ is preserved.

Question 6.3. Can we construct non Douwen families, so that their associated ideals are preserved under various forcings?

These considerations are particularly interesting towards a possible proof of the consistency of $\mathfrak{a}_\mathrm{e} < \mathfrak{a}_\mathrm{v}$, as in this case a non Van Douwen family which remains maximal under some forcing which destroys the Van Douwenness of some other family is desirable.

Finally, all our families were constructed under the assumption of CH, but one may also consider similar constructions under large continuum. Remember, that Lemma 3.2 implies that $\mathbb{E}_\mathcal{F}(I)$ only adds Van Douwen families, but by adapting the combinatorics of the forcing $\mathbb{E}_\mathcal{F}(I)$ according to the proofs in this section, one may consider forcing notions which add maximal eventually different families with certain prescribed associated ideals.

Question 6.4. Which ideals may be realized as associated ideals of maximal eventually different families under large continuum? Analogously to Theorem 5.6 can we have that all ideals are realized as associated ideals of maximal eventually different families under large continuum?

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