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Special Aronszajn trees and Kurepa trees

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Abstract

My thesis is about special Aronszajn trees and Kurepa trees. First, I show that it follows from the existence of a supercompact cardinal and an inaccessible cardinal above that it is consistent that all $\aleph_2$-Aronszajn trees are special, there are such, and there is no $\aleph_1$-Kurepa tree and no $\aleph_2$-Kurepa tree.

Then I show that, assuming ω many supercompact cardinals, it is consistent that for all $0 < n < \omega$, all $\aleph_n$-Aronszajn trees are special and there exist such, and there are no $\aleph_n$-Kurepa trees.

Finally, I extend this result to a global version about all Aronszajn trees on successors of regular cardinals and all Kurepa trees on regular cardinals, using a proper class of supercompact cardinals.

Contents

Chapter 1

Introduction

Trees are very important and interesting combinatorial objects. They have been studied a lot in set theory over the last century. Aronszajn trees and Kurepa trees have been introduced in the 1930's and are of fundamental importance in combinatorial set theory. Two of the most interesting questions about them are the problem of their existence and the problem of the specialization of Aronszajn trees. Both questions are independent from ZFC for trees on uncountable successor cardinals.

In this thesis, I construct models in which all Aronszajn trees of some heights are special, there are such, and there are no Kurepa trees of certain heights. This is joint work with my advisor Sy-David Friedman.

Laver and Shelah showed in their paper [LS81] that it is consistent with CH that all $\aleph_2$-Aronszajn trees are special and there exists one. Golshani and Hayut extended this result in [GH20] with a similar but more involved technique to show that it is consistent that for all successors of regular cardinals all Aronszajn trees are special and there exists one. So the questions about Suslin trees and Aronszajn trees are settled in this model. What about Kurepa trees?

It was shown by Baumgartner in [Bau84] that PFA implies that there are no $\aleph_2$-Aronszajn trees and no $\aleph_1$-Kurepa trees. Cummings proved in [Cum18], assuming a weakly compact cardinal, that there is a generic extension in which there are no $\aleph_2$-Aronszajn trees and there is an $\aleph_1$-Kurepa tree.

Motivated by [GH20] and [Cum18], we have worked on models in which all Aronszajn trees on some cardinals are special and they exist, and there are no Kurepa trees of some heights. It turned out that in the forcing extension used in [GH20] $\aleph_n$-Kurepa trees actually exist (see Proposition 2.11), so we had to change the forcing iteration: Instead of using a product of Lévy collapses we use an iteration of these collapses in the beginning of the iteration. In fact, we have constructed a model in which for every $0 < n < \omega$, all $\aleph_n$-Aronszajn trees are special, there are such, and there exists no $\aleph_n$-Kurepa tree. To get this, we start with ω many supercompact cardinals which we collapse with an iteration of Lévy collapses to the $\aleph_n$'s. These collapses are followed by specializing forcings for the Aronszajn trees. We use supercompact embeddings to show that the specializing forcings have a suitable chain condition. Our proof of the chain condition is different than the one given in [LS81] and [GH20] and conceptually easier. Then we argue that there is no Kurepa tree in the final model. If there were such a tree, a small regular subforcing would capture it, which can be seen using supercompact embeddings. In the extension by the small subforcing, the tree cannot have many cofinal branches, and an analysis of the quotient shows that no cofinal branches are added and therefore the tree is not a Kurepa tree in the final model.

This result can also be generalized to a global version, using an Easton support iteration. This is done in Chapter 6.

Chapter 2

Preliminaries

In this chapter we provide the necessary definitions and results about trees and Lévy collapses, most of which are classical and well-known.

Definition 2.1. A *tree* is a set of nodes T, together with an order $<$, with the following properties:

1. There exists a *root*, i.e., an $r \in T$ with $r = s$ or $r < s$ for all $s \in T$.
2. $(\{t \in T \mid t < s\}, <)$ is a well-order for every $s \in T$.

For a tree T with the order $<$ we use the following notation:

- For $s \in T$ the *length of* s is the order type of $(\{t \in T \mid t < s\}, <)$; we denote the length of s by $|s|$.
- The *ξth level of* T, denoted T_ξ, is the set of nodes in T which have length ξ.
- The *height of* T is the smallest ordinal ξ such that $T_\xi = \emptyset$.
- A *cofinal branch* of T is a chain in $(T, <)$ whose order type is equal to the height of T and such that for each ξ smaller than the height of T it contains a $t \in T_\xi$. The set of all cofinal branches of a tree T is denoted by $[T]$.
- Let b be a cofinal branch of T and ξ smaller than the height of T. Then $b(\xi)$ denotes the unique node $t \in T_\xi \cap b$.
- For a cardinal κ, a *κ-tree* is a tree of height κ all whose levels are smaller than κ.

Now we can define Kurepa trees and Aronszajn trees:

Definition 2.2. Let κ be a cardinal.

1. A *κ-Aronszajn tree* is a κ-tree which has no cofinal branch.

2. A κ^+-Aronszajn tree T is *special* if there exists a function $f\colon T \to \kappa$ such that if $x < y$ then $f(x) \neq f(y)$.

3. A *κ-Kurepa tree* is a κ-tree which has more than κ many cofinal branches.

The following is easy to see:

Remark 2.3. If there exists a κ^+-(non-special-)Aronszajn tree or a κ^+-Kurepa tree, then there exists one with $T_\xi \subseteq \{\xi\} \times \kappa$ for each $\xi < \kappa^+$.

Therefore we will always assume that our trees of successor cardinal height κ^+ satisfy $T_\xi \subseteq \{\xi\} \times \kappa$ for all $\xi < \kappa^+$.

Kurepa trees and Aronszajn trees have been studied a lot. In the following we will present some of the classical results.

The existence of an $\aleph_1$-Kurepa tree is independent from ZFC. On the one hand, if $V = L$, then there exists an $\aleph_1$-Kurepa tree, on the other hand, it follows from the existence of an inaccessible that it is consistent with ZFC that there exists no $\aleph_1$-Kurepa tree. Since the proof of the second uses ingredients which we will use later, we will give the proof of it here.

First we define the Lévy collapse and prove its basic properties.

Definition 2.4. Let λ be an inaccessible cardinal and κ a regular cardinal with $\kappa < \lambda$. The *Lévy collapse of λ to κ^+*, written as $\mathrm{Col}(\kappa, {<}\lambda)$, is defined as follows: For each cardinal $\kappa < \alpha < \lambda$ let $\mathrm{Col}(\kappa, \alpha)$ be the set of partial functions of size $< \kappa$ from κ to α, ordered by $q \leq p$ if $q \supseteq p$. Now let $\mathrm{Col}(\kappa, {<}\lambda) := \prod_{\kappa<\alpha<\lambda} \mathrm{Col}(\kappa, \alpha)$ with ${<}\kappa$-support.

Lemma 2.5. *Let λ be an inaccessible cardinal and κ a regular cardinal with $\kappa < \lambda$. The Lévy collapse* $\mathrm{Col}(\kappa, {<}\lambda)$ *is ${<}\kappa$-closed.*

Proof. Let $\mu < \kappa$ and let $\langle p_i \mid i < \mu\rangle$ be a decreasing sequence in $\mathrm{Col}(\kappa, {<}\lambda)$. Define a condition p by letting $p(\alpha) := \bigcup_{i<\mu} p_i(\alpha)$ for every cardinal $\kappa < \alpha < \lambda$; it is easy to see that $p \in \mathrm{Col}(\kappa, {<}\lambda)$ by the following argument. Since $\mu < \kappa$ and for each $i < \mu$ and each $\kappa < \alpha < \lambda$ each $p_i(\alpha)$ is a partial function from κ to α with $|\mathrm{dom}(p_i(\alpha))| < \kappa$ and $p_j(\alpha) \subseteq p_i(\alpha)$ for $j < i$, also $p(\alpha)$ is such a function. Moreover $\mathrm{supp}(p) = \bigcup_{i<\mu} \mathrm{supp}(p_i)$, so it is a union of less than κ many sets of size $< \kappa$. Since κ is regular, this is of size $< \kappa$. So $p \in \mathrm{Col}(\kappa, {<}\lambda)$ and clearly $p \leq p_i$ for each $i < \mu$. □

Lemma 2.6. *Let λ be an inaccessible cardinal and κ a regular cardinal with $\kappa < \lambda$. Then the Lévy collapse* $\mathrm{Col}(\kappa, {<}\lambda)$ *has the λ-c.c..*

Proof. Let $A \subseteq \mathrm{Col}(\kappa, {<}\lambda)$ be a maximal antichain. Let M be a ${<}\kappa$-closed (i.e., ${}^{<\kappa}M \subseteq M$) elementary submodel of $H(\theta)$ for sufficiently large θ with $|M| < \lambda$, $A \in M$ and such that $\bar{\lambda} := M \cap \lambda$ is an ordinal. Since $|M| < \lambda$ and M is ${<}\kappa$-closed it follows that $\kappa \leq \bar{\lambda} < \lambda$.

Now consider $A \cap M$. Clearly $A \cap M$ is an antichain of size $< \lambda$; we will show that $A \cap M$ is maximal in $\mathrm{Col}(\kappa, {<}\lambda)$ in V from which it follows that $A = A \cap M$ and hence A is of size ${<}\lambda$. To show the maximality let $p \in \mathrm{Col}(\kappa, {<}\lambda)$. By definition of $\mathrm{Col}(\kappa, {<}\lambda)$, $p \restriction \bar{\lambda} \in \prod_{\kappa < \alpha < \bar{\lambda}} \mathrm{Col}(\kappa, \alpha)$ is a tuple where each coordinate is a partial function from κ to $\bar{\lambda}$, hence $p \restriction \bar{\lambda} \subseteq M$. Since M is ${<}\kappa$-closed and $|p| < \kappa$ it follows that $p \restriction \bar{\lambda} \in M$. By elementarity $A \cap M$ is a maximal antichain in M, hence there exists $q \in A \cap M$ which is compatible to $p \restriction \bar{\lambda}$. Since $|q| < \kappa$, $q \in M$ and M is ${<}\kappa$-closed, it follows that $q \subseteq M$, thus $q \in \prod_{\kappa < \alpha < \bar{\lambda}} \mathrm{Col}(\kappa, \alpha)$. So also $p \not\perp q$ because p is compatible with q on $\bar{\lambda}$ and $q \in \prod_{\kappa < \alpha < \bar{\lambda}} \mathrm{Col}(\kappa, \alpha)$. □

Lemma 2.7. *Let G be a generic filter over V for* $\mathrm{Col}(\kappa, {<}\lambda)$. *Then all cardinals α from V with $\alpha \leq \kappa$ or $\alpha \geq \lambda$ are cardinals in $V[G]$ and $V[G] \models \kappa^+ = \lambda$.*

Proof. The Lévy collapse $\mathrm{Col}(\kappa, {<}\lambda)$ is ${<}\kappa$-closed by Lemma 2.5, hence it does not change cardinals $\leq \kappa$. By Lemma 2.6, $\mathrm{Col}(\kappa, {<}\lambda)$ has the λ-c.c., hence it does not change cardinals $\geq \lambda$. For each $\kappa < \beta < \lambda$ the forcing $\mathrm{Col}(\kappa, \beta)$ adds a surjection from κ to β, so it forces that $\kappa = |\beta|$, hence this holds in $V[G]$. Since λ is a cardinal in $V[G]$, it follows that $\kappa^+ = \lambda$. □

The following lemma is essentially due to Silver [Sil71] and generalizes his lemma. For regular λ a further generalization can be found in Lemma 3.12.

Lemma 2.8. *Assume there exists some $\mu < \mathrm{cf}(\lambda)$ such that $2^\mu \geq \lambda$. Let $\mathbb{P}$ be a ${<}\lambda$-closed forcing and T a λ-tree. Then forcing with $\mathbb{P}$ does not add a new cofinal branch to T.*

Proof. Assume $\dot{b}$ is a name for a new cofinal branch. Let $p_{\langle\rangle} \in \mathbb{P}$, $x_{\langle\rangle} \in T$ and α_0 be such that $p_{\langle\rangle} \Vdash \dot{b}(\alpha_0) = x_{\langle\rangle}$.

Now continue inductively: Let $i < \mu$ and assume $\alpha_i < \lambda$ has been defined, and for each $v \in 2^i$, p_v and x_v have been defined such that $p_v \Vdash \dot{b}(\alpha_i) = x_v$. Since $\dot{b}$ is a new branch, there exists $\alpha_{i+1} > \alpha_i$ such that for every $v \in 2^i$ there exist two conditions $p_{v^\frown 0}, p_{v^\frown 1} \leq p_v$ and $x_{v^\frown 0} \neq x_{v^\frown 1}$ such that $p_{v^\frown 0} \Vdash \dot{b}(\alpha_{i+1}) = x_{v^\frown 0}$ and $p_{v^\frown 1} \Vdash \dot{b}(\alpha_{i+1}) = x_{v^\frown 1}$. For $w \in 2^{\leq \mu}$ of limit length, let $\alpha_{|w|} > \alpha_\delta$ for all $\delta < |w|$ and let $x_w \in T$ and p_w be such that p_w is a lower bound of $\langle p_{w \restriction \delta} \mid \delta < |w| \rangle$ such that $p_w \Vdash \dot{b}(\alpha_{|w|}) = x_w$; such a condition p_w exists because $\mathbb{P}$ is ${<}\lambda$-closed. It follows easily that $x_w \neq x_v$ for $w \neq v$ of the same length, hence $|\{x_w \mid w \in 2^\mu\}| = 2^\mu \geq \lambda$. But $\{x_w \mid w \in 2^\mu\} \subseteq T_{\alpha_\mu}$, which contradicts the fact that all levels of T are of size $< \lambda$. □

Let us fix the following notation: If $\mathbb{P}$ is a forcing, we denote with $G(\mathbb{P})$ a generic filter for $\mathbb{P}$, and $V[\mathbb{P}]$ is a shorthand for $V[G(\mathbb{P})]$.

Theorem 2.9 (Silver)**.** *Let $k \geq 1$, λ an inaccessible cardinal and $\mathbb{L}_\lambda = \mathrm{Col}(\aleph_k, {<}\lambda)$. Then there is no $\aleph_k$-Kurepa tree in $V[\mathbb{L}_\lambda]$.*

The following proposition is a generalization of Silver's theorem:

Proposition 2.10. *Let $k \geq 1$, λ an inaccessible cardinal and $\mathbb{L}_\lambda = \mathrm{Col}(\aleph_k, {<}\lambda)$. In $V[\mathbb{L}_\lambda]$ let $\mathbb{Q}$ be a forcing of size $\leq \aleph_k$ such that either $\mathbb{Q}$ is ${<}\aleph_k$-distributive or $\mathbb{Q}$ has the $\aleph_k$-c.c.. Then there is no $\aleph_k$-Kurepa tree in $V[\mathbb{L}_\lambda * \mathbb{Q}]$.*

Proof. Let $\dot{\mathbb{Q}}$ be an $\mathbb{L}_\lambda$-name for $\mathbb{Q}$. Since $\mathbb{Q}$ is a set of size $\leq \aleph_k$ and $\mathbb{L}_\lambda$ has the λ-c.c., we can assume that $\dot{\mathbb{Q}}$ has size less than λ. Conditions in $\mathbb{L}_\lambda$ have a support of size $< \aleph_k$, therefore there exists $\mu_{\mathbb{Q}} < \lambda$ such that the ${<}\lambda$ many conditions in $\dot{\mathbb{Q}}$ belong to $\prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha)$, hence $\mathbb{Q} \in V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha)]$. Therefore $\mathbb{L}_\lambda * \mathbb{Q}$ is equivalent to $\prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha<\lambda} \mathrm{Col}(\aleph_k, \alpha))$.

A similar argument works for an $\aleph_k$-tree: In $V[\mathbb{L}_\lambda * \mathbb{Q}]$ let T be an $\aleph_k$-tree. Since T with its order is an object of size $\aleph_k$ and $\mathbb{L}_\lambda * \mathbb{Q}$ has the λ-c.c., there exists a name $\dot{T}$ for it of size less than λ. Conditions in $\mathbb{L}_\lambda$ have a support of size $< \aleph_k$, therefore there exists $\mu_{\mathbb{Q}} < \mu_T < \lambda$ such that the ${<}\lambda$ many conditions in $\dot{T}$ belong to $\prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))$. Therefore $T \in V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))]$. In $V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))]$ still $2^{\aleph_k} < \lambda$, hence T has less than λ many cofinal branches there.

If $\mathbb{Q}$ is ${<}\aleph_k$-distributive, then $\prod_{\alpha>\mu_T} \mathrm{Col}(\aleph_k, \alpha)$ is still ${<}\aleph_k$-closed in $V[G \cap \prod_{\alpha\leq\mu_{\mathbb{Q}}} \mathrm{Col}(\aleph_k, \alpha) * (\mathbb{Q} \times \prod_{\mu_{\mathbb{Q}}<\alpha\leq\mu_T} \mathrm{Col}(\aleph_k, \alpha))]$, hence by Lemma 2.8 it does not add cofinal branches to T. So T is not a Kurepa tree in $V[\mathbb{L}_\lambda * \mathbb{Q}]$.

If $\mathbb{Q}$ has the $\aleph_k$-c.c., it follows by Lemma 3.12 below that $\prod_{\alpha>\mu_T} \mathrm{Col}(\aleph_k, \alpha)$ does not add cofinal branches to T. So T is not a Kurepa tree in $V[\mathbb{L}_\lambda * \mathbb{Q}]$. □

Proposition 2.11. *Let $\kappa_0 = \aleph_0$ and $\langle \kappa_n \mid 0 < n < \omega \rangle$ be inaccessible cardinals with $\kappa_n < \kappa_{n+1}$. Then for every $1 < m < \omega$ there exist $\aleph_m$-Kurepa trees in $V[\prod_{n\in\omega} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})]$.*

Proof. As $\prod_{m\leq n} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$ is ${<}\kappa_m$-closed and conditions in $\prod_{n<m} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$ are of size $< \kappa_m$, the product $\prod_{n\in\omega} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$ is equivalent to the iteration $\prod_{m\leq n} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1}) * \prod_{n<m} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$. In $V[\prod_{m\leq n} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})]$, let $T := 2^{<\kappa_m}$. Since κ_m is inaccessible in this model, all levels of T are of size $< \kappa_m$, and clearly $|[T]| > \kappa_m$. It follows that in $V[\prod_{m\leq n} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})][\prod_{n<m} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})]$, all levels of T are of size $< \aleph_m$ and $|[T]| > \aleph_m$, i.e., T is an $\aleph_m$-Kurepa tree. □

It follows easily from the above proposition that in the model of [GH20] there exist $\aleph_m$-Kurepa trees for each $1 < m < \omega$: Their forcing iteration starts with $\prod_{n\in\omega} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})$, and as the subsequent iteration of specializing forcings does not collapse cardinals, the $\aleph_m$-Kurepa trees from $V[\prod_{n\in\omega} \mathrm{Col}(\kappa_n, {<}\kappa_{n+1})]$ are still $\aleph_m$-Kurepa trees in the final model.

Lemma 2.12. *Let T be a κ^+-tree with order $<_T$ and with a specializing function, i.e., $f\colon T \to \kappa$ such that if $x <_T y$ then $f(x) \neq f(y)$. Then T is a κ^+-Aronszajn tree.*

Proof. Assume T has a cofinal branch, i.e., there exists $\{x_i \mid i < \kappa^+\} \subseteq T$ with $x_i <_T x_j$ for all $i < j$. It follows that $f(x_i) \neq f(x_j)$ for all $i < j < \kappa^+$, contradicting the fact that the values of f are elements of κ. □

Proposition 2.13. *If $2^{\aleph_n} = \aleph_{n+1}$, then there exists a special $\aleph_{n+2}$-Aronszajn tree.*

Proof. Let $(Q, <)$ be the set of those $x \in \aleph_{n+1}^{\aleph_n}$ with boundedly many non-zero elements, i.e., $|\{\alpha \in \aleph_n \mid x(\alpha) \neq 0\}| < \aleph_n$, with $<$ being the lexicographical ordering. First we show that for every $x < y \in Q$ and every $\gamma < \aleph_{n+2}$ there exists a strictly increasing sequence $\langle z_i \mid i < \gamma\rangle$ in Q with $x < z_i < y$ for every $i < \gamma$. The proof is by induction on ordinals $\gamma < \aleph_{n+2}$. For the first step let $\delta \leq \aleph_{n+1}$. Let α be large enough such that $x(\beta) = y(\beta) = 0$ for all $\beta \geq \alpha$. For each $i < \delta$, let $z_i(\alpha) = i$ and $z_i(\beta) = x(\beta)$ for $\beta \neq \alpha$. It is easy to see that the z_i form a strictly increasing sequence of length δ and $x < z_i < y$ for every $i < \delta$. Now let $\aleph_{n+1} < \gamma < \aleph_{n+2}$ and assume by induction that for each $x < y$ and for each $\delta < \gamma$ there exists a strictly increasing sequence of length δ in Q between x and y. Note that there exists a cofinal strictly increasing sequence $\{\gamma_j \mid j < \mathrm{cf}(\gamma)\} \subseteq \gamma$ such that the order type $\mathrm{ordtp}(\gamma_{j+1} \setminus \gamma_j) < \gamma$ for every $j < \mathrm{cf}(\gamma)$. Since $\mathrm{cf}(\gamma) \leq \aleph_{n+1}$, by step one we can take a strictly increasing sequence $\{z_j \mid j < \mathrm{cf}(\gamma)\}$ with $x < z_j < y$ for every j. Then take strictly increasing sequences $\{z_i^j \mid i < \mathrm{ordtp}(\gamma_{j+1} \setminus \gamma_j)\}$ for each j with $z_j < z_i^j < z_{j+1}$ for each i; such sequences exist by the induction hypothesis. Now $\{z_j \mid j < \mathrm{cf}(\gamma)\} \cup \bigcup_{j<\mathrm{cf}(\gamma)} \{z_i^j \mid i < \mathrm{ordtp}(\gamma_{j+1} \setminus \gamma_j)\}$ is as desired.

Further note that, using $2^{\aleph_n} = \aleph_{n+1}$, we have $|Q| = \aleph_{n+1}$.

We will construct a tree $T \subseteq Q^{<\aleph_{n+2}}$ with the order given by end-extension, and then show that it is a special $\aleph_{n+2}$-Aronszajn tree. The nodes of T will be bounded strictly increasing sequences. For such a sequence s and $x \in Q$, let $x > s$ denote the following assertion: there exists $x' \in Q$ with $x > x'$ and $x' \geq z$ for each $z \in \mathrm{range}(s)$; also, let $x \geq s$ denote $x \geq z$ for each $z \in \mathrm{range}(s)$. Note that for each $s \in T$, there will be an $x \in Q$ with $x > s$. We construct T by induction on the levels such that for each $\alpha < \aleph_{n+2}$ the following holds:

$$\text{For each } \beta < \alpha, \text{ for each } s \in T_\beta \text{ and each } x \in Q \text{ with } x > s \text{ there exists } t \in T_\alpha \text{ such that } s <_T t \text{ and } x \geq t. \tag{2.1}$$

Let $T_0 = \{\langle\rangle\}$. Assume now that T_α has been constructed and satisfies (2.1). Let $T_{\alpha+1} := \{s^\frown x \mid s \in T_\alpha \wedge x > s\}$. To see that (2.1) holds for $\alpha + 1$, first let $s \in T_\alpha$ and $x \in Q$ with $x > s$. So $s^\frown x \in T_{\alpha+1}$ is a witness for (2.1). Now let $\beta < \alpha$, $s \in T_\beta$ and $x \in Q$ with $x > s$. Let $x' \in Q$ be such that $s < x' < x$ (such an x' exists by a very easy version of the property shown in the beginning of the proof). By (2.1) for α (which holds by induction), there exists $t \in T_\alpha$ with $s <_T t$ and $x' \geq t$. So $t^\frown x \in T_{\alpha+1}$ is a witness for (2.1). Now let α be a limit and assume that T_β has been defined for every $\beta < \alpha$. Let $s \in \bigcup_{\beta<\alpha} T_\beta$ and $x > s$. By the above property there exists a strictly increasing sequence $\{z_i \mid |s| < i < \alpha\}$ with $s < z_i < x$ for every i. Using (2.1), inductively we get that there exists a sequence $\langle t_i \mid |s| < i < \alpha\rangle$ such that $t_i \in T_i$, $s <_T t_j <_T t_i$ and $z_i \geq t_i$ for all $|s| < j < i$. Therefore, taking the union of the t_i, we get that there is a strictly increasing sequence t of length α such that $s <_T t$ and $x \geq t$ and $t{\restriction}\beta \in T_\beta$ for every $\beta < \alpha$. For each $s \in \bigcup_{\beta<\alpha} T_\beta$ and each $x > s$, pick one such t, and let this set be $\tilde{T}_\alpha$. Then let $T_\alpha := \{s^\frown x \mid s \in \tilde{T}_\alpha \wedge x > s\}$. To see that (2.1) holds for α, let $\beta < \alpha$ and $s \in T_\beta$ and $x \in Q$ with $x > s$. Let $x > x' > s$. By construction there is $t \in \tilde{T}_\alpha$ with $x' \geq t$ and $t^\frown x \in T_\alpha$ is a witness for (2.1).

Let $T := \bigcup_{\alpha<\aleph_{n+2}} T_\alpha$. Note that every $s \in T$ is a strictly increasing sequence in Q of successor length.

Now we show that $|T_\alpha| < \aleph_{n+2}$ for every $\alpha < \aleph_{n+2}$ by induction: This is obvious for $\alpha = 0$. The successor step $\alpha+1$ follows by $|T_{\alpha+1}| = |T_\alpha|\cdot|Q| = |T_\alpha|\cdot\aleph_{n+1} = \aleph_{n+1}$ since $|T_\alpha| < \aleph_{n+2}$. Now let α be a limit: $|\tilde{T}_\alpha| \leq |\{(s, x) \mid s \in \bigcup_{\beta<\alpha} T_\beta \wedge x \in Q\}| = |\alpha| \cdot \aleph_{n+1} \cdot \aleph_{n+1} = \aleph_{n+1}$. As in the successor step, it follows that $|T_\alpha| = \aleph_{n+1}$.

Using Lemma 2.12 it only remains to show that T is special. To see this, let $\varphi\colon Q \to \aleph_{n+1}$ be a bijection. For $s \in T$ with x the last element of s, let $f(s) = \varphi(x)$. If $s <_T t$ then t extends s; since t is strictly increasing it follows that $f(s) \neq f(t)$. □

Chapter 3

General lemmata

In this chapter, we give some general lemmata, which will be needed in the later chapters.

Definition 3.1. $\mathbb{P}$ is a *regular subforcing* of $\mathbb{Q}$ if $\mathbb{P} \subseteq \mathbb{Q}$ and every maximal antichain of $\mathbb{P}$ is a maximal antichain in $\mathbb{Q}$.

Definition 3.2. For $\mathbb{P}$ a suborder of $\mathbb{Q}$, we say that $\pi\colon \mathbb{Q} \to \mathbb{P}$ is a *reduction map* if whenever $q \in \mathbb{Q}$ and $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} \pi(q)$ then p and q are compatible in $\mathbb{Q}$.

Lemma 3.3. *Let $\mathbb{P}$ be a suborder of $\mathbb{Q}$. Then $\mathbb{P}$ is a regular subforcing of $\mathbb{Q}$ if*

(1) *there exists a reduction map $\pi\colon \mathbb{Q} \to \mathbb{P}$, and*

(2) *if two conditions $p, q \in \mathbb{P}$ are compatible in $\mathbb{Q}$, then they are compatible in $\mathbb{P}$.*

Proof. See [Kun11, III.3.72]. □

Definition 3.4. For two forcing notions $\mathbb{P}$ and $\mathbb{Q}$, a function $\iota\colon \mathbb{P} \to \mathbb{Q}$ is a *regular embedding* if the following holds:

1. If $p \leq_{\mathbb{P}} p'$ then $\iota(p) \leq_{\mathbb{Q}} \iota(p')$.
2. $p \perp_{\mathbb{P}} p'$ iff $\iota(p) \perp_{\mathbb{Q}} \iota(p')$.
3. For every $q \in \mathbb{Q}$ there exists a *reduction* $p \in \mathbb{P}$, i.e., p is such that for each $p' \in \mathbb{P}$ if $p' \leq_{\mathbb{P}} p$ then $\iota(p') \not\perp_{\mathbb{Q}} q$.

Lemma 3.5. *Let $\mathbb{P}'$ be a regular subforcing of $\mathbb{P}$ and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name for a forcing. Then $\mathbb{P}'$ is a regular subforcing of $\mathbb{P} * \dot{\mathbb{Q}}$, and for a generic filter G' for $\mathbb{P}'$ the following equality holds: $(\mathbb{P} * \dot{\mathbb{Q}})/G' = \mathbb{P}/G' * \dot{\mathbb{Q}}$.*

Proof. Let $\iota\colon \mathbb{P}' \to \mathbb{P}$ be a regular embedding. Define $\iota'\colon \mathbb{P}' \to \mathbb{P} * \dot{\mathbb{Q}}$ by $\iota'(p) = (\iota(p), \mathbb{1}_{\dot{\mathbb{Q}}})$. Using ι', it is straightforward to check that the statements of the lemma hold. □

Lemma 3.6. *Let $\mathbb{P}$ be a forcing with a dense subset D and $\mathbb{P}^*$ a regular subforcing of $\mathbb{P}$ with a dense subset $D^* \subseteq D$. Then D^* is a regular subforcing of D.*

Proof. Let $A \subseteq D^*$ be a maximal antichain in D^*. Let $p \in \mathbb{P}^*$. There exists $p' \leq p$ with $p' \in D^*$. So there exists $q \in A$ with $q \not\perp p'$ and therefore $q \not\perp p$. On the other hand, let $q, q' \in A$ and assume $q \not\perp_{\mathbb{P}^*} q'$. So there exists $q'' \leq q, q'$ in $\mathbb{P}^*$. Since D^* is dense, there exists a condition extending q'' in D^*, thus q and q' are compatible in D^*, contradicting the fact that A is an antichain in D^*. This shows that A is a maximal antichain in $\mathbb{P}^*$.

Using that $\mathbb{P}^*$ is a regular subforcing of $\mathbb{P}$ we get that A is a maximal antichain in $\mathbb{P}$. Further, $A \subseteq D$ and hence A is an antichain in D. Since D is dense, A is maximal in D, finishing the proof. □

Lemma 3.7. *If $\mathbb{P}$ is a forcing and λ a cardinal in V, then $(2^\lambda)^{V[\mathbb{P}]} \leq (2^{|\mathbb{P}|\cdot\lambda})^V$.*

Proof. Every subset of λ in $V[\mathbb{P}]$ has a name of the form $\{(\check{\alpha}, p) \mid \alpha \in \lambda \wedge p \in A_\alpha\}$, where $A_\alpha \subseteq \mathbb{P}$ for each α. There are $(2^{|\mathbb{P}|\cdot\lambda})^V$ many such names, hence $(2^\lambda)^{V[\mathbb{P}]} \leq (2^{|\mathbb{P}|\cdot\lambda})^V$. □

Lemma 3.8. *If $\mathbb{P}$ is a forcing and λ a cardinal with $|\mathbb{P}| \leq \lambda$, then $(2^\lambda)^V = (2^\lambda)^{V[\mathbb{P}]}$.*

Proof. On the one hand, $(2^\lambda)^{V[\mathbb{P}]} \leq (2^\lambda)^V$ by Lemma 3.7. On the other hand, $\mathbb{P}$ does not collapse 2^λ, because $|\mathbb{P}| \leq \lambda$ and hence $\mathbb{P}$ has the λ^+-c.c.. □

The following lemma of Silver is useful to lift elementary embeddings to forcing extensions. A proof can be found in [Cum10].

Lemma 3.9 (Lifting Lemma)**.** *Let $j\colon V \to M$ be an elementary embedding and $\mathbb{P} \in V$ a forcing. Let $G(\mathbb{P})$ be generic for $\mathbb{P}$ and let $G(j(\mathbb{P}))$ be generic for $j(\mathbb{P})$. The following are equivalent:*

(1) $j[G(\mathbb{P})] \subseteq G(j(\mathbb{P}))$.

(2) *There exists an elementary embedding $j'\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ such that $j'(G(\mathbb{P})) = G(j(\mathbb{P}))$ and $j' \restriction V = j$.*

The following theorem is useful to represent some forcings as an easier forcing followed by a quotient which has a good closure. A good source for it is [Cum10].

Theorem 3.10 (Absorption)**.** *Let κ be regular and $\lambda > \kappa$ an inaccessible. Let $\mathbb{P}$ be separative, $<\kappa$-closed and $|\mathbb{P}| < \lambda$. Then there is a regular embedding $\iota\colon \mathbb{P} \to$ $\mathrm{Col}(\kappa, <\lambda)$ such that if G is $\mathbb{P}$-generic over V, then $\mathrm{Col}(\kappa, <\lambda)$ is forcing equivalent to $\mathrm{Col}(\kappa, <\lambda)/\iota[G]$.*

Lemma 3.11. *If $\mathbb{P} * \dot{\mathbb{Q}}$ is a two-step iteration which has the κ-c.c., then $\mathbb{P}$ has the κ-c.c. and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ has the κ-c.c.".*

Proof. Since it is easy to see that $\mathbb{P}$ has the κ-c.c., we only show that $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ has the κ-c.c.". Let $p_0 \in \mathbb{P}$ and assume $\dot{A}$ is a $\mathbb{P}$-name such that $p_0 \Vdash$ "$\dot{A} \subseteq \dot{\mathbb{Q}}$ and $\dot{A}$ has size κ". Furthermore, let $\dot{f}$ be a $\mathbb{P}$-name such that $p_0 \Vdash$ "$\dot{f}\colon \kappa \to \dot{A}$ is a bijection". For every $\alpha \in \kappa$ there exist $p_\alpha \leq p_0$ and $\dot{q}_\alpha$ such that $p_\alpha \Vdash \dot{f}(\alpha) = \dot{q}_\alpha$. Since $\mathbb{P} * \dot{\mathbb{Q}}$ has the κ-c.c., there exist distinct $\alpha, \beta \in \kappa$ such that $(p_\alpha, \dot{q}_\alpha)$ and $(p_\beta, \dot{q}_\beta)$ are compatible. Let $(p, \dot{q})$ be a witness for the compatibility. It follows that $p \Vdash$ "$\dot{f}(\alpha) = \dot{q}_\alpha \in \dot{A} \wedge \dot{f}(\beta) = \dot{q}_\beta \in \dot{A} \wedge \dot{q} \leq \dot{q}_\alpha, \dot{q}_\beta$". In particular $p \Vdash$ "$\dot{A}$ is not an antichain". □

The following lemma can be found in [Ung12] for the case where $\mathbb{P}$ has the μ^+-c.c. and $\mathbb{R}$ is $<\mu^+$-closed. It is a generalization of Lemma 2.8 and the proof is a refinement of the proof there.

Lemma 3.12. *Let λ be a regular cardinal and $\mu < \lambda$ with $2^\mu \geq \lambda$. Let $\mathbb{P}$ be a forcing which has the λ-c.c. and $\mathbb{R}$ a forcing which is $<\lambda$-closed and $\dot{T}$ a $\mathbb{P}$-name for a λ-tree. Then forcing with $\mathbb{R}$ over $V[\mathbb{P}]$ does not add cofinal branches to T.*

Proof. Assume $\dot{b}$ is an $\mathbb{R}$-name for a new branch cofinal through T in $V[\mathbb{P}]$.

Claim. *For all $r_1, r_2 \in \mathbb{R}$ the set D_{r_1,r_2} of conditions $p \in \mathbb{P}$ with the following properties is dense.*

1. *$p \Vdash$ "there are $r_1' \leq r_1$ and $r_2' \leq r_2$ and $\gamma < \lambda$ such that r_1' and r_2' decide $\dot{b}(\gamma)$ in different ways".*
2. *p decides γ, r_1' and r_2'.*

Proof. Let $p^* \in \mathbb{P}$ and let $G_\mathbb{P}$ be generic for $\mathbb{P}$ with $p^* \in G_\mathbb{P}$. In $V[G_\mathbb{P}]$ the conditions r_1 and r_2 cannot decide all of $\dot{b}$, because it is a new branch. Hence there exists γ such that r_1 and r_2 do not decide $\dot{b}(\gamma)$. So there exist conditions $r_1' \leq r_1$ and $r_2' \leq r_2$ which decide $\dot{b}(\gamma)$ differently, and thus there exists a condition $p' \in G_\mathbb{P}$ which forces this and decides γ, r_1' and r_2'. Since both p^* and p' are in $G_\mathbb{P}$, they are compatible. Any witness of the compatibility is in D_{r_1,r_2} and stronger than p^*. □

Claim. *For every condition $r \in \mathbb{R}$ there exists a maximal antichain A in $\mathbb{P}$ and conditions $r_1, r_2 \leq r$ and $\gamma < \lambda$ such that for all $p \in A$, $p \Vdash$ "r_1 and r_2 decide $\dot{b}(\gamma)$ differently".*

Proof. By induction on α define increasing (w.r.t. $\subseteq$) antichains A_α and decreasing sequences r_1^α and r_2^α and an increasing sequence γ_α of ordinals $< \lambda$ such that for each $p \in A_\alpha$, $p \Vdash$ "r_1^α and r_2^α decide $\dot{b}(\gamma_\alpha)$ differently".

Let $r \in \mathbb{R}$ and $p_0 \in D_{r,r}$ and r_1^0, r_2^0 and $\gamma_0 < \lambda$ witnesses for this. Let $A_0 := \{p_0\}$.

For the successor step, assume A_α, r_1^α, r_2^α and γ_α have been defined. If A_α is a maximal antichain, we stop the construction here. If there exists $p \in \mathbb{P}$ which is incompatible to every condition in A_α, let $p' \leq p$ with $p' \in D_{r_1^\alpha, r_2^\alpha}$ and let $r_1^{\alpha+1}$, $r_2^{\alpha+1}$ and $\gamma_{\alpha+1}$ be witnesses for this. Since T is a tree, the set of γ's with $p' \Vdash$ "$r_1^{\alpha+1}$ and $r_2^{\alpha+1}$ decide $\dot{b}(\gamma)$ differently" is upwards closed, so we can assume that $\gamma_{\alpha+1} > \gamma_\alpha$. Let $A_{\alpha+1} := A_\alpha \cup \{p'\}$.

For the limit step, let α be a limit ordinal and let $A_\alpha := \bigcup_{\beta<\alpha} A_\beta$. Note that A_α is an antichain of size $|\alpha|$. Since $\mathbb{P}$ has the λ-c.c., it follows that $\alpha < \lambda$. Let r_1^α and r_2^α be lower bounds of the sequences $\langle r_1^\beta \mid \beta < \alpha \rangle$ and $\langle r_2^\beta \mid \beta < \alpha \rangle$ (such lower bounds exist because $\mathbb{R}$ is $<\lambda$-closed). Let $\gamma_\alpha := \sup\{\gamma_\beta \mid \beta < \alpha\}$. Since λ is regular, $\gamma_\alpha < \lambda$.

Since $\mathbb{P}$ has the λ-c.c., for some $\alpha < \lambda$ the antichain A_α will be maximal. Then we stop the induction and define $A := A_\alpha$, $r_1 := r_1^\alpha$, $r_2 := r_2^\alpha$ and $\gamma := \gamma_\alpha$.

To see that the claim is fulfilled, let $p \in A$. Hence $p \in A_{\beta+1}$ for some $\beta + 1 \leq \alpha$, so $p \Vdash$ "$r_1^{\beta+1}$ and $r_2^{\beta+1}$ decide $\dot{b}(\gamma_{\beta+1})$ differently". Since $r_1 \leq r_1^{\beta+1}$, $r_2 \leq r_2^{\beta+1}$, $\gamma \geq \gamma_{\beta+1}$ and T is a tree, it follows that $p \Vdash$ "r_1 and r_2 decide $\dot{b}(\gamma)$ differently". □

Let $\mu < \lambda$ with $2^\mu \geq \lambda$. For every $w \in 2^{\leq\mu}$ we construct r_w, x_w and α_i for every $i \leq \mu$ such that $r_w \Vdash \dot{b}(\alpha_{|w|}) = x_w$, for $w, w' \in 2^{\leq\mu}$ of the same length, $x_w \neq x'_w$, and $\alpha_j < \alpha_i < \lambda$ for $i > j$.

Let $r_{\langle\rangle} = \mathbb{1}_\mathbb{R}$, $\alpha_0 = 0$ and $x_{\langle\rangle} = \langle\rangle$, so $r_{\langle\rangle} \Vdash \dot{b}(\alpha_0) = x_{\langle\rangle}$. Now use the above claim inductively to get $r_{w^\frown 0} \in \mathbb{R}$ and $r_{w^\frown 1} \in \mathbb{R}$, together with $x_{w^\frown 0}$, $x_{w^\frown 1}$, $\alpha_{|w|} < \alpha_{|w|+1} < \lambda$ and a maximal antichain A_w such that each $p \in A_w$ forces that $r_{w^\frown 0}$ and $r_{w^\frown 1}$ decide $\dot{b}(\alpha_{|w|+1})$ differently. For $w \in 2^{\leq\mu}$ of limit length let $\alpha_\delta < \alpha_{|w|} < \lambda$ for all $\delta < |w|$. Since λ is regular and $\mu < \lambda$ such an $\alpha_{|w|} < \lambda$ exists. Let r_w be a lower bound of $\langle r_{w\restriction\delta} \mid \delta < |w| \rangle$. Such r_w exist because of the closure of $\mathbb{R}$.

Let $\alpha := \sup\{\alpha_\delta \mid \delta \leq \mu\}$. Since λ is regular and $\mu < \lambda$, it follows that $\alpha < \lambda$.

Let G be generic for $\mathbb{P}$. In $V[G]$ for all $v \neq w \in 2^\mu$, r_v and r_w decide $\dot{b}(\alpha)$ differently: Let $\delta < \mu$ be minimal such that $v(\delta) \neq w(\delta)$. Hence, every $p \in A_{w\restriction\delta}$ forces that $r_{v\restriction\delta+1}$ and $r_{w\restriction\delta+1}$ decide $\dot{b}(\alpha_{\delta+1})$ differently. $A_{w\restriction\delta}$ is a maximal antichain, so there exists $p \in G \cap A_{w\restriction\delta}$. So $V[G] \models$ "$r_{v\restriction\delta+1}$ and $r_{w\restriction\delta+1}$ decide $\dot{b}(\alpha_{\delta+1})$ differently". Since $r_v \leq r_{v\restriction\delta+1}$ and $r_w \leq r_{w\restriction\delta+1}$ and $\alpha > \alpha_{\delta+1}$ and T is a tree, it follows that $V[G] \models$ "r_v and r_w decide $\dot{b}(\alpha)$ differently".

Hence $V[G] \models |T_\alpha| \geq 2^\mu \geq \lambda$, contradicting that $\dot{T}$ is a name for a λ-tree. □

The following lemma is essentially due to Mitchell [Mit73]:

Lemma 3.13. *Let λ be a regular cardinal, $\mathbb{P}$ a forcing where $\mathbb{P} \times \mathbb{P}$ has the λ-c.c. and T a tree of height λ. Then forcing with $\mathbb{P}$ does not add a new cofinal branch to T.*

Proof. Assume $\mathbb{P}$ adds a new cofinal branch and let $\dot{b}$ be a $\mathbb{P}$-name for it. We inductively build a sequence of conditions $\{(p_i, q_i) \mid i < \lambda\} \subseteq \mathbb{P} \times \mathbb{P}$, a strictly increasing sequence $\{\gamma_i \mid i < \lambda\}$ of ordinals $< \lambda$ and sequences $\{x_i \mid i < \lambda\}$, $\{y_i \mid i < \lambda\}$ and $\{z_i \mid i < \lambda\}$ such that

1. $p_i, q_i \Vdash \dot{b}(\gamma_i) = x_i$,

2. $y_i \neq z_i$, $p_i \Vdash \dot{b}(\gamma_{i+1}) = y_i$ and $q_i \Vdash \dot{b}(\gamma_{i+1}) = z_i$.

Let $\gamma_0 = 0$ and $x_0 = \langle\rangle$. Clearly $\mathbb{1}_{\mathbb{P}} \Vdash \dot{b}(0) = \langle\rangle$. Since $\dot{b}$ is a name for a new branch, there exists $\alpha < \lambda$ such that $\mathbb{1}_{\mathbb{P}}$ does not decide $\dot{b}(\alpha)$. Find p_0 and q_0 and $y_0 \neq z_0$ with $p_0, q_0 \leq \mathbb{1}_{\mathbb{P}}$ and $p_0 \Vdash \dot{b}(\alpha) = y_0$ and $q_0 \Vdash \dot{b}(\alpha) = z_0$ and set $\gamma_1 := \alpha$.

Assume $p_i, q_i, \gamma_{i+1}, x_i, y_i$ and z_i have been defined. Let $x_{i+1} = y_i$ and $\gamma_{i+2} > \gamma_{i+1}$ such that p_i does not decide $\dot{b}(\gamma_{i+2})$. Find $p_{i+1}, q_{i+1} \leq p_i$ and $y_{i+1} \neq z_{i+1}$ such that $p_{i+1} \Vdash \dot{b}(\gamma_{i+2}) = y_{i+1}$ and $q_{i+1} \Vdash \dot{b}(\gamma_{i+2}) = z_{i+1}$.

Assume j is a limit and p_i, q_i, γ_i, x_i, y_i and z_i have been defined for every $i < j$. Let $\gamma_j = (\sup_{i<j} \gamma_i) + 1$. Let $p^* \in \mathbb{P}$ and x_j be such that $p^* \Vdash \dot{b}(\gamma_j) = x_j$ and let $\gamma_{j+1} > \gamma_j$ such that p^* does not decide $\dot{b}(\gamma_{j+1})$. Now let $p_j, q_j \leq p^*$, so $p_j, q_j \Vdash \dot{b}(\gamma_j) = x_j$, and let y_j and z_j be such that $y_j \neq z_j$, and $p_j \Vdash \dot{b}(\gamma_{j+1}) = y_j$ and $q_j \Vdash \dot{b}(\gamma_{j+1}) = z_j$. This finishes the construction.

Since $\mathbb{P} \times \mathbb{P}$ has the λ-c.c., there exist $i < j < \lambda$ such that (p_i, q_i) and (p_j, q_j) are compatible. Let $(p, q) \leq (p_i, q_i), (p_j, q_j)$. So $p \leq p_j$ and $q \leq q_j$, thus $p, q \Vdash \dot{b}(\gamma_j) = x_j$, and $p \leq p_i$ and $q \leq q_i$, thus $p \Vdash \dot{b}(\gamma_{i+1}) = y_i$ and $q \Vdash \dot{b}(\gamma_{i+1}) = z_i$. So in T there are two distinct nodes y_i and z_i on level $\gamma_{i+1} \leq \gamma_j$ with $y_i, z_i \leq x_j$, which contradicts the fact that T is a tree. We conclude that there is no new cofinal branch in $V[\mathbb{P}]$. □

3.1 Nicely closed forcings

The concept of nicely closed forcings is crucial for the main proofs of the thesis. In particular, we are interested in the closure of quotient forcings (see Lemma 3.16).

Definition 3.14. Let $\mathbb{P}$ be a partial order and λ be an ordinal. $\mathbb{P}$ is *nicely λ-closed* if for every decreasing sequence $\langle p_i \mid i < \lambda\rangle$ in $\mathbb{P}$ there exists a lower bound q with the property that if $p \in \mathbb{P}$ is compatible with every p_i then q is compatible with p. In this case we call q a *witnessing lower bound for the nice λ-closure*. Moreover, $\mathbb{P}$ is *nicely $<\kappa$-closed* if it is nicely λ-closed for every $\lambda < \kappa$.

We will later need the nice closure of the Lévy collapse:

Lemma 3.15. *Let λ be an inaccessible cardinal and κ a regular cardinal with $\kappa < \lambda$. Then the Lévy collapse* $\mathrm{Col}(\kappa, <\lambda)$ *is nicely $<\kappa$-closed.*

Proof. Let $\mu < \kappa$ and $\langle p_i \mid i < \mu \rangle$ a decreasing sequence in $\mathrm{Col}(\kappa, {<}\lambda)$. Let $q := \bigcup_{i<\mu} p_i$. Since $\mu < \kappa$, $q \in \mathrm{Col}(\kappa, {<}\lambda)$ is a lower bound of $\langle p_i \mid i < \mu \rangle$. Let p be compatible with every p_i. We have to show that q is compatible with p. It is easy to see that $p \cup q \in \mathrm{Col}(\kappa, {<}\lambda)$ and $p \cup q \leq p, q$. □

Lemma 3.16. *Let $\mathbb{P}$ be nicely ${<}\kappa$-closed and $\mathbb{Q}$ a regular subforcing of $\mathbb{P}$. Then $\mathbb{P}/G(\mathbb{Q})$ is ${<}\kappa$-closed.*

Proof. Let $G(\mathbb{Q})$ be a generic filter for $\mathbb{Q}$, $\lambda < \kappa$ and $\langle p_i \mid i < \lambda \rangle$ a decreasing sequence in $\mathbb{P}/G(\mathbb{Q})$. For every $q \in G(\mathbb{Q})$ every p_i is compatible with q. By the nice ${<}\kappa$-closure there exists a lower bound r of $\langle p_i \mid i < \lambda \rangle$ which is compatible with every condition which is compatible with every p_i. So in particular r is compatible with every $q \in G(\mathbb{Q})$. Hence $r \in \mathbb{P}/G(\mathbb{Q})$, and thus $\mathbb{P}/G(\mathbb{Q})$ is ${<}\kappa$-closed. □

Definition 3.17. A forcing $\mathbb{P}$ is *${<}\kappa$-closed with weakest lower bounds* if for each $\lambda < \kappa$ and for each decreasing sequence $\langle p_i \mid i < \lambda \rangle$ there exists a weakest lower bound q, i.e.,

1. $q \leq p_i$ for each $i < \lambda$, and
2. if $q' \leq p_i$ for each $i < \lambda$, then $q' \leq q$.

Definition 3.18. A forcing $\mathbb{P}$ is *well-met* if for all $p, p' \in \mathbb{P}$ with $p \not\perp p'$ there exists a weakest lower bound q, i.e.,

1. $q \leq p, p'$, and
2. if $q' \leq p, p'$, then $q' \leq q$.

Lemma 3.19. *Let $\mathbb{P}$ be a forcing which is ${<}\kappa$-closed with weakest lower bounds and well-met. Then $\mathbb{P}$ is nicely ${<}\kappa$-closed.*

Proof. Let $\lambda < \kappa$, $\langle p_i \mid i < \lambda \rangle$ be a decreasing sequence, and let p^* be the weakest lower bound of it. Now let $p \not\perp p_i$ for every $i < \lambda$. For every i let q_i be the weakest lower bound of p and p_i. Since the q_i are weakest lower bounds, it follows that they are decreasing, so there exists a lower bound q of the sequence $\langle q_i \mid i < \lambda \rangle$. In particular, $q \leq p$. Since $q_i \leq p_i$ for every i, q is also a lower bound of the sequence $\langle p_i \mid i < \lambda \rangle$. The weakest lower bound of this sequence is p^*, so $q \leq p^*$, hence q witnesses the compatibility of p and p^*. □

Definition 3.20. Let $\mathbb{P}_\mu$ be a forcing iteration of length μ. A condition $p \in \mathbb{P}_\mu$ is *decisive* if for each $\alpha < \mu$ there exists $q \in V$ such that $p{\restriction}\alpha \Vdash$ "$p(\alpha) = \check{q}$" and if $q' \in V$ such that $p{\restriction}\alpha \Vdash$ "$\check{q}' \in \dot{\mathbb{Q}}_\alpha$" and some extension of $p{\restriction}\alpha$ forces that $\check{q}$ is compatible to $\check{q}'$, then $p{\restriction}\alpha$ forces that $\check{q}$ is compatible to $\check{q}'$. In this case we say that $p{\restriction}\alpha$ *decides* $p(\alpha)$. We call $\mathbb{P}_\mu$ *decisive* if the set of decisive conditions is dense in $\mathbb{P}_\mu$.

Lemma 3.21. *Let $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \mu\}$ be a $<\kappa$-support forcing iteration such that for each $\alpha < \mu$*

(1) *$\mathbb{P}_\alpha$ is $<\kappa$-closed,*

(2) *$\mathbb{P}_\alpha \Vdash$ "$q \subseteq V$ and $|q| < \kappa$ for each $q \in \dot{\mathbb{Q}}_\alpha$",*

(3) *there exists a relation R definable in V such that for $p \in \mathbb{P}_\alpha$ and $q, q' \in V$ such that $p \Vdash$ "$\check{q}, \check{q}' \in \dot{\mathbb{Q}}_\alpha$", $p \Vdash$ "$\check{q}' \not\perp \check{q}$" if and only if $R(q', q)$, and*

(4) *for $p \in \mathbb{P}_\alpha$, $\lambda < \kappa$, and $q_i \in V$ for each $i < \lambda$ such that $p \Vdash$ "$\langle \check{q}_i \mid i < \lambda\rangle$ is decreasing in $\dot{\mathbb{Q}}_\alpha$" and p decides $\check{q}_i$ for every i, there exists $q^* \in V$ such that $p \Vdash$ "$\check{q}^*$ is a lower bound of $\langle \check{q}_i \mid i < \lambda\rangle$" and p decides $\check{q}^*$.*

Then the set of decisive conditions is dense in $\mathbb{P}_\mu$, i.e., $\mathbb{P}_\mu$ is decisive.

Proof. By induction on $\alpha \leq \mu$ we show that the set of decisive conditions is dense in $\mathbb{P}_\alpha$. For $\alpha = 0$, $\mathbb{P}_0$ is the trivial forcing consisting only of the weakest condition, which is decisive.

Next assume for $\mathbb{P}_\alpha$ that the set of decisive conditions is dense. Let $p \in \mathbb{P}_{\alpha+1}$. Since $\mathbb{P}_\alpha$ is $<\kappa$-distributive and $\mathbb{P}_\alpha \Vdash$ "$q \subseteq V$ and $|q| < \kappa$ for each $q \in \dot{\mathbb{Q}}_\alpha$", there exists $q \in V$ and $p' \leq p\restriction\alpha$ such that $p' \Vdash p(\alpha) = \check{q}$. Let $p'' \leq p'$ be in the set of decisive conditions in $\mathbb{P}_\alpha$ (which is dense by inductive hypothesis). Let $q' \in V$ be such that $p'' \Vdash \check{q}' \in \dot{\mathbb{Q}}_\alpha$, and assume there exists $p^* \leq p''$ with $p^* \Vdash \check{q}' \not\perp \check{q}$. By (3) in V we have $R(q', q)$ and hence by (3) $p'' \Vdash \check{q}' \not\perp \check{q}$. So $(p'', \check{q}) \leq p$ is decisive.

Now let α be a limit ordinal. If $\mathrm{cf}(\alpha) \geq \kappa$, then $\mathbb{P}_\alpha$ is a bounded support limit, so for every $p \in \mathbb{P}_\alpha$ there exists $\beta < \alpha$ such that $p \in \mathbb{P}_\beta$. The set of decisive conditions in $\mathbb{P}_\beta$ is dense by induction, and a decisive $p' \leq_{\mathbb{P}_\beta} p$ is still decisive in $\mathbb{P}_\alpha$ and $p' \leq_{\mathbb{P}_\alpha} p$.

If $\mathrm{cf}(\alpha) < \kappa$, let $\langle \beta_j \mid j < \mathrm{cf}(\alpha)\rangle$ be an increasing continuous cofinal sequence in α. Let $p \in \mathbb{P}_\alpha$. By induction, the set of decisive conditions in $\mathbb{P}_\beta$ is dense for each $\beta < \alpha$. Let $p_{\beta_0} \leq p\restriction\beta_0$ be decisive. By induction on j assume we have $p_{\beta_j} \leq p\restriction\beta_j$ decisive. Since $p_{\beta_j} \leq p\restriction\beta_j$ we know that $p_{\beta_j}{}^\frown p\restriction[\beta_j, \beta_{j+1}) \leq p\restriction\beta_{j+1}$. By the inductive assumption, there exists a decisive $p_{\beta_{j+1}} \leq p_{\beta_j}{}^\frown p\restriction[\beta_j, \beta_{j+1})$. It follows that $p_{\beta_{j+1}} \leq p_{\beta_j}$. If j is a limit, there exists (since $j < \kappa$) a lower bound p' of $\langle p_{\beta_k} \mid k < j\rangle$. Then let $p_{\beta_j} \leq p'$ be decisive. So we get a decreasing sequence $\langle p_{\beta_j} \mid j < \mathrm{cf}(\alpha)\rangle$ such that each p_{β_j} is decisive and $p_{\beta_j} \leq p\restriction\beta_j$.

Now we define a decisive extension p^* of p. Let $p^*(0)$ be a lower bound of $\langle p_{\beta_j}(0) \mid j < \mathrm{cf}(\alpha)\rangle$. Now assume $p^*\restriction\beta$ has been defined such that it is decisive and stronger than $p_{\beta_j}\restriction\beta$ for each $\beta_j \geq \beta$. Since each p_{β_j} is decisive, it follows that there are $q_j \in V$ such that $p^*\restriction\beta \Vdash$ "$\check{q}_j = p_{\beta_j}(\beta)$ with $\beta_j > \beta$ is a decreasing sequence" and such that if $q'_j \in V$ with $p^*\restriction\beta \Vdash \check{q}'_j \in V$ and some extension of $p^*\restriction\beta$ forces that $\check{q}_j$ is compatible with $\check{q}'_j$, then $p^*\restriction\beta \Vdash \check{q}_j \not\perp \check{q}'_j$. Therefore, by assumption (4) of the lemma, there exists $q^* \in V$ such that $p^* \restriction \beta \Vdash$ "$\check{q}^*$ is a lower bound of $\langle p_{\beta_j}(\beta) \mid j < \mathrm{cf}(\alpha) \text{ with } \beta < \beta_j \rangle$" and $p^* \restriction \beta$ decides $\check{q}^*$. Let $q^* = \mathbb{1}$ if $q_j = \mathbb{1}$ for every j. Let $p^*(\beta) := \check{q}^*$. Continuing for length α we get p^*. Note that $\mathrm{supp}(p^*) \subseteq \bigcup_{j<\mathrm{cf}(\alpha)} \mathrm{supp}(p_{\beta_j})$, so it is smaller than κ and hence $p^* \in \mathbb{P}_\alpha$. Further, by definition $p^* \leq p$ and it is decisive. □

Lemma 3.22. *Let $\{\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \mu\}$ be a $<\kappa$-support forcing iteration such that*

(1) $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ *is $<\kappa$-closed with weakest lower bounds and well-met" for each α, and*

(2) $\mathbb{P}_\mu$ *is decisive.*

Further assume that for each $\alpha < \mu$ the following holds, which we will refer to as weakest lower bounds of decided conditions are decided:

- *for $p \in \mathbb{P}_\alpha$ and $q, q' \in V$ such that $(p, \check{q}) \not\perp (p, \check{q}')$ there exists $q^* \in V$ such that $p \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\check{q}$ and $\check{q}'$" and p decides $\check{q}^*$, and*
- *for $p \in \mathbb{P}_\alpha$, $\lambda < \kappa$ and $q_i \in V$ for each $i < \lambda$ such that $p \Vdash$ "$\langle \check{q}_i \mid i < \lambda \rangle$ is decreasing in $\dot{\mathbb{Q}}_\alpha$", there exists $q^* \in V$ such that $p \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\langle \check{q}_i \mid i < \lambda \rangle$" and p decides $\check{q}^*$.*

Then in $\mathbb{P}_\mu$ the dense subforcing of decisive conditions is $<\kappa$-closed with weakest lower bounds and well-met. In particular, this dense subforcing is nicely $<\kappa$-closed.

Proof. We show by induction on $\alpha \leq \mu$ that in $\mathbb{P}_\alpha$ the set of decisive conditions is dense and $<\kappa$-closed with weakest lower bounds and well-met. Applying Lemma 3.19, this in particular yields that in $\mathbb{P}_\alpha$ the set of decisive conditions is dense and nicely $<\kappa$-closed.

First note that $\mathbb{P}_\alpha$ is decisive for every $\alpha < \mu$, because $\mathbb{P}_\mu$ is decisive and each $p \in \mathbb{P}_\alpha$ is also a condition in $\mathbb{P}_\mu$. Let D_α be the dense set of the decisive conditions in $\mathbb{P}_\alpha$.

Let $\lambda < \kappa$. By induction on $\alpha \leq \mu$ we show that the dense set D_α in $\mathbb{P}_\alpha$ is λ-closed with weakest lower bounds and well-met, and that the weakest lower bounds for λ-sequences and for pairs are coherent. More precisely, if $\beta < \alpha$ and $p, p' \in D_\alpha$ are compatible, then there exists a weakest lower bound $p^* \in D_\alpha$ and $p^* \restriction \beta$ is the weakest lower bound for $p \restriction \beta$ and $p' \restriction \beta$ in D_β. If $\langle p_i \mid i < \lambda \rangle$ is a decreasing sequence in D_α, then there exists a weakest lower bound $p^* \in D_\alpha$ and $p^* \restriction \beta$ is the weakest lower bound of $\langle p_i \restriction \beta \mid i < \lambda \rangle$ in D_β.

For $\alpha = 0$, $\mathbb{P}_0$ is the trivial forcing consisting only of the weakest condition, so it is λ-closed with weakest lower bounds and well-met.

Now assume that D_α is λ-closed with weakest lower bounds and well-met. By assumption we know that $\mathbb{P}_\alpha \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is λ-closed with weakest lower bounds and well-met". Let $\langle (p_i, \dot{q}_i) \mid i < \lambda \rangle$ be a decreasing sequence in $D_{\alpha+1}$. So $\langle p_i \mid i < \lambda \rangle$ is a decreasing sequence in D_α, and by induction there exists a weakest lower bound $p \in D_\alpha$. Since p is a lower bound of the p_i, it follows that $p \Vdash$ "$\langle \dot{q}_i \mid i < \lambda \rangle$ is a decreasing sequence and $\dot{\mathbb{Q}}_\alpha$ is λ-closed with weakest lower bounds" and for each $i < \lambda$ there exists $q_i \in V$ such that $p \Vdash \dot{q}_i = \check{q}_i$. Therefore, there exists $q^* \in V$ such that $p \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\langle \dot{q}_i \mid i < \lambda \rangle$" and p decides $\check{q}^*$. Now it is easy to see that $(p, \check{q}^*)$ is the weakest lower bound of $\langle (p_i, \dot{q}_i) \mid i < \lambda \rangle$ and $(p, \check{q}^*) \in D_{\alpha+1}$. Note that the lower bounds are coherent, since $(p, \check{q}^*) \restriction \alpha = p$.

Now let $(p, \dot{q}) \not\perp (p', \dot{q}')$ in $D_{\alpha+1}$. Since D_α is well-met by induction there exists a weakest lower bound $p^* \leq p, p'$ in D_α. It follows that there exist $q, q' \in V$ such that $p^* \Vdash$ "$\dot{q} = \check{q}$ and $\dot{q}' = \check{q}'$", hence, since p^* is the weakest lower bound of p and p', $(p^*, \check{q}) \not\perp (p^*, \check{q}')$. So there exists $q^* \in V$ such that $p^* \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\dot{q}$ and $\dot{q}'$" and p^* decides $\check{q}^*$. It is easy to see that $(p^*, \check{q}^*)$ is a weakest lower bound of $(p, \dot{q})$ and $(p', \dot{q}')$ and $(p^*, \check{q}^*) \in D_{\alpha+1}$. Note that the lower bounds are coherent, since $(p^*, \check{q}^*) \restriction \alpha = p^*$. This finishes the proof of the successor step.

Next let α be a limit ordinal with $\mathrm{cf}(\alpha) \geq \kappa$, so $\mathbb{P}_\alpha$ is a bounded support limit. Let $\langle p_i \mid i < \lambda \rangle$ be a decreasing sequence in D_α. For each $i < \lambda$ let $\beta_i < \alpha$ be such that $p_i \in \mathbb{P}_{\beta_i}$. Let $\beta := \sup\{\beta_i \mid i < \lambda\}$; then $p_i \in D_\beta$ for every $i < \lambda$. Since $\mathrm{cf}(\alpha) \geq \kappa > \lambda$ we have $\beta < \alpha$. Using the induction hypothesis for β, there exists a weakest lower bound $p \in D_\beta$. It is easy to see that p is still a weakest lower bound in D_α and the coherence follows from the coherence up to β.

Similarly, for $p, p' \in D_\alpha$ with $p \not\perp p'$ there exists $\beta < \alpha$ with $p, p' \in D_\beta$ and $p \not\perp_{D_\beta} p'$. Using the induction hypothesis for β we find a weakest lower bound p^* of p and p' in D_β, and p^* is still a weakest lower bound in D_α and the coherence follows from the coherence up to β.

Finally let α be a limit ordinal with $\mathrm{cf}(\alpha) < \kappa$, so $\mathbb{P}_\alpha$ is an inverse limit, i.e., $\mathbb{P}_\alpha = \{p \mid p \restriction \beta \in \mathbb{P}_\beta \text{ for all } \beta < \alpha\}$. Let $\langle \beta_j \mid j < \mathrm{cf}(\alpha) \rangle$ be an increasing cofinal sequence in α.

Let $\langle p_i \mid i < \lambda \rangle$ be a decreasing sequence in D_α. By induction for each $j < \mathrm{cf}(\alpha)$ there exists a weakest lower bound $p_j^* \in D_{\beta_j}$ of $\langle p_i \restriction \beta_j \mid i < \lambda \rangle$. From the coherence of the weakest lower bounds it follows that $p_j^* \restriction \beta_k = p_k^*$ for $k < j$. Let $p^* := \bigcup \{p_j^* \mid j < \mathrm{cf}(\alpha)\}$. Clearly, this extends the sequence of the weakest lower bounds coherently and $p^* \in D_\alpha$. Since for each $i < \lambda$, $p^* \restriction \beta_j \leq p_i \restriction \beta_j$ for every j, it follows that $p^* \leq p_i$ for every $i < \lambda$. To see that this lower bound is the weakest lower bound, let p' be a lower bound. Then $p' \restriction \beta_j \leq p_i \restriction \beta_j$ for all $i < \lambda$ and all $j < \mathrm{cf}(\alpha)$, so $p' \restriction \beta_j \leq p^* \restriction \beta_j$ for every j and therefore $p' \leq p^*$.

Now let $p \not\perp p'$ in D_α. By induction, for each $j < \mathrm{cf}(\alpha)$ there exists a weakest lower bound $p_j^* \in D_{\beta_j}$ of $p \restriction \beta_j$ and $p' \restriction \beta_j$. From the coherence of the weakest lower bounds it follows that $p^*_j \upharpoonright \beta_k = p^*_k$ for $k < j$. Let $p^* := \bigcup\{p^*_j \mid j < \mathrm{cf}(\alpha)\}$. Clearly, this extends the sequence of the weakest lower bounds coherently and $p^* \in D_\alpha$. Since $p^* \upharpoonright \beta_j \le p \upharpoonright \beta_j, p' \upharpoonright \beta_j$ for every j, it follows that $p^* \le p, p'$. To see that this lower bound is the weakest lower bound, let $\tilde{p}$ be a lower bound. Then $\tilde{p} \upharpoonright \beta_j \le p \upharpoonright \beta_j, p' \upharpoonright \beta_j$ for all $j < \mathrm{cf}(\alpha)$, so $\tilde{p} \upharpoonright \beta_j \le p^* \upharpoonright \beta_j$ for every j and therefore $\tilde{p} \le p^*$. □

Lemma 3.23. *Let $\mathbb{P}$ be such that the set of decisive conditions is dense and nicely $<\kappa$-closed. Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name and R a relation definable in V such that $\mathbb{P} \Vdash$ "$q \subseteq V$ and $|q| < \kappa$ for each $q \in \dot{\mathbb{Q}}$, $\dot{\mathbb{Q}}$ is $<\kappa$-closed with weakest lower bounds and well-met and weakest lower bounds of decided conditions are decided" and for $q, q' \in V$ with $p \Vdash$ "$\check{q}, \check{q}' \in \dot{\mathbb{Q}}_\alpha$", $p \Vdash$ "$\check{q}' \not\perp \check{q}$" if and only if $R(q', q)$ holds. Then in $\mathbb{P} * \dot{\mathbb{Q}}$ the set of decisive conditions is dense and nicely $<\kappa$-closed.*

Proof. By Lemma 3.21, the set of decisive conditions is dense in $\mathbb{P} * \dot{\mathbb{Q}}$. Let $\lambda < \kappa$ and let $\langle (p_i, \dot{q}_i) \mid i < \lambda \rangle$ be a decreasing sequence of decisive conditions. Let p be decisive and a lower bound of $\langle p_i \mid i < \lambda \rangle$ witnessing the nice λ-closure. In particular, there exist $q_i \in V$ such that $p \Vdash$ "$\langle \dot{q}_i \mid i < \lambda \rangle$ is a decreasing sequence and $\dot{q}_i = \check{q}_i$ for every $i < \lambda$". Hence, there exists $q \in V$ such that $p \Vdash$ "$\check{q}$ is a weakest lower bound of $\langle \dot{q}_i \mid i < \lambda \rangle$" and p decides $\check{q}$. Clearly $(p, \check{q})$ is decisive and a lower bound of $\langle (p_i, \dot{q}_i) \mid i < \lambda \rangle$.

Let $(p', \dot{q}')$ be decisive such that $(p', \dot{q}') \not\perp (p_i, \dot{q}_i)$ for every i. Since p is a witness for the nice closure, $p \not\perp p'$. Let $p^* \le p, p'$ be decisive. Hence there exist $q_i, q' \in V$ for every i such that $p^* \Vdash$ "$\check{q}' = \dot{q}'$ and $\check{q}_i = \dot{q}_i$ for every i" and p^* decides $\check{q}'$ and $\check{q}_i$. By the assumption, weakest lower bounds of decided conditions are decided, i.e., there exist $q^*_i \in V$ such that $p^* \Vdash$ "$\check{q}^*_i$ is a weakest lower bound of $\check{q}_i$ and $\check{q}'$" and p^* decides $\check{q}^*_i$. It follows that it is forced by p^* that $\langle \check{q}^*_i \mid i < \lambda \rangle$ is a decreasing sequence. Therefore there exists $q^* \in V$ such that $p^* \Vdash$ "$\check{q}^* \le \check{q}^*_i$ for every $i < \lambda$" and p^* decides $\check{q}^*$. So $(p^*, \check{q}^*) \le (p', \dot{q}'), (p, \check{q})$ and $(p^*, \check{q}^*)$ is decisive. □

Chapter 4

$\aleph_2$-trees

From the existence of a supercompact cardinal and an inaccessible above, we prove that it is consistent that all $\aleph_2$-Aronszajn trees are special, there are such, and there are no $\aleph_1$- or $\aleph_2$-Kurepa trees.

4.1 Definition of the forcing

Let $\kappa_2 < \kappa_3$ be cardinals with κ_2 supercompact and κ_3 inaccessible.

Definition 4.1. For a κ_2-Aronszajn tree T let $\mathbb{S}(T)$ be the forcing to specialize T, defined as follows: $\mathbb{S}(T)$ consists of partial functions f from T to $[\omega_1]^{\leq\omega}$ such that $|\mathrm{dom}(f)| \leq \omega$, and $f(s) \cap f(t) = \emptyset$ whenever $s \neq t \in \mathrm{dom}(f)$ are comparable in T. The order is given by $g \leq f$ if $g \supseteq f$.

In the forcing iteration we use the following variant to specialize names for trees, instead of specializing the tree in the extension. The reason we need to do this is Lemma 4.6.

Definition 4.2 (Specializing names)**.** Assume that $\mathbb{P}$ is a forcing with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{T}$ is a κ_2-Aronszajn tree with $\dot{T}_\xi = \{\xi\} \times \aleph_1$". Let $\mathbb{S}_{\mathbb{P}}(\dot{T})$ be the following forcing: Conditions are countable partial functions f from $\dot{T}$ to $[\aleph_1]^{\leq\omega}$ such that, for $s \neq t \in \mathrm{dom}(f)$, if $f(s) \cap f(t) \neq \emptyset$, then $\mathbb{1}_{\mathbb{P}} \Vdash$ "s is incomparable to t in $\dot{T}$". The order is given by $g \leq f$ if $g \supseteq f$.

Remark 4.3. Note that $\not\perp_{\mathbb{S}_{\mathbb{P}}(\dot{T})}$ is definable in V, more precisely, for $p \in \mathbb{P}$ and $f, g \in V$, $p \Vdash \check{f} \not\perp_{\mathbb{S}_{\mathbb{P}}(\dot{T})} \check{g}$ if and only if

- for all $s \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, $f(s) = g(s)$, and
- for all $s \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)$ and $t \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)$, if $f(s) \cap g(t) \neq \emptyset$ then $\mathbb{1}_{\mathbb{P}} \Vdash$ "s is incomparable to t in $\dot{T}$".

We use multi-valued functions, i.e., functions with values in $[\aleph_1]^{\leq\omega}$ instead of values in $\aleph_1$, to make some of the proofs easier.

To get then a specializing function in the usual sense, a multi-valued function $F\colon T \to [\aleph_1]^{\leq\omega}$ can be translated as follows: Let F' be defined by $F'(s) = \min(F(s))$. Then F' is a specializing function since images under F of comparable nodes are disjoint, hence the minima are different.

Note that if $\dot{T}$ is a $\mathbb{P}$-name for a κ_2-Aronszajn tree, then $|\mathbb{S}_{\mathbb{P}}(\dot{T})| = \kappa_2$.

Lemma 4.4. *Let $\dot{T}$ be a $\mathbb{P}$-name for an $\aleph_2$-Aronszajn tree. For every $(\xi,\beta) \in \aleph_2 \times \aleph_1$ the set $\{g \in \mathbb{S}_{\mathbb{P}}(\dot{T}) \mid (\xi,\beta) \in dom(g)\}$ is dense in $\mathbb{S}_{\mathbb{P}}(\dot{T})$.*

Proof. Let $f \in \mathbb{S}_{\mathbb{P}}(\dot{T})$, $\xi \in \aleph_2$ and $\beta \in \aleph_1$. Since $|\mathrm{dom}(f)| \leq \omega$, and $|f(s)| \leq \omega$ for every $s \in \mathrm{dom}(f)$, there exists $i \in \aleph_1 \setminus \bigcup \mathrm{rng}(f)$. If $(\xi,\beta) \notin \mathrm{dom}(f)$, let $g := f \cup \{((\xi,\beta),\{i\})\}$. So $g \in \mathbb{S}_{\mathbb{P}}(\dot{T})$, $g \leq f$ and $(\xi,\beta) \in \mathrm{dom}(g)$. □

Lemma 4.5. *Let $\mathbb{P}$ be a forcing with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{T}$ is an $\aleph_2$-Aronszajn tree". Then $\mathbb{P} * \mathbb{S}_{\mathbb{P}}(\dot{T}) \Vdash$ "there is a specializing function from $\dot{T}$ to $[\aleph_1]^{\leq\omega}$".*

Proof. In $V[\mathbb{P}]$ let G be a generic filter for $\mathbb{S}_{\mathbb{P}}(\dot{T})$. Let $F := \bigcup\{f \in \mathbb{S}_{\mathbb{P}}(\dot{T}) \mid f \in G\}$. It follows from the above lemma that $\mathrm{dom}(F) = \aleph_2\times\aleph_1$. For distinct $s, t \in \aleph_2\times\aleph_1$ with $F(s) = F(t)$ we have that $\mathbb{1}_{\mathbb{P}} \Vdash$ "s and t are incomparable in $\dot{T}$", hence $F(s) \neq F(t)$ if $s <_T t$. So F is a specializing function of T to $[\aleph_1]^{\leq\omega}$. □

Now we define the forcing iteration to specialize all $\aleph_2$-Aronszajn trees. Let $\mathbb{L}_2 = \mathrm{Col}(\omega_1, {<}\kappa_2)$ and $\mathbb{L}_3 = \mathrm{Col}(\kappa_2, {<}\kappa_3)^{V[\mathbb{L}_2]}$. Let $\dot{T}_0$ be an $\mathbb{L}_2 * \mathbb{L}_3$-name for a κ_2-Aronszajn tree and $\mathbb{S}_{\mathbb{L}_2*\mathbb{L}_3}(\dot{T}_0)$ the forcing to specialize the name $\dot{T}_0$ as in Definition 4.2. Let $\mathbb{P}_1 := \mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\mathbb{L}_2*\mathbb{L}_3}(\dot{T}_0)$ and $\mathbb{S}_1 := \mathbb{S}_{\mathbb{L}_2*\mathbb{L}_3}(\dot{T}_0)$.

Assume $\mathbb{P}_i$ has been defined. Continue the iteration in the same way, i.e., let $\dot{T}_i$ be a $\mathbb{P}_i$-name for a κ_2-Aronszajn tree and $\mathbb{S}_{\mathbb{P}_i}(\dot{T}_i)$ the forcing to specialize $\dot{T}_i$ as in the case of $\dot{T}_0$. Let $\mathbb{P}_{i+1} := \mathbb{P}_i * \mathbb{S}_{\mathbb{P}_i}(\dot{T}_i)$ and $\mathbb{S}_{i+1} := \mathbb{S}_i * \mathbb{S}_{\mathbb{P}_i}(\dot{T}_i)$. Continue this as a countable support iteration for κ_3 many steps, using a bookkeeping function for the nice names of κ_2-Aronszajn trees. Let $\mathbb{P}^{\aleph_2}_{\kappa_3}$ be this forcing iteration. We will show that in $V[\mathbb{L}_2 * \mathbb{L}_3]$ the forcing iteration $\mathbb{S}_{\kappa_3}$ to specialize κ_2-Aronszajn trees has the κ_2-c.c. (see Lemma 4.13). Recall that by Lemma 2.7, $\kappa_2 = \aleph_2$ in $V[\mathbb{L}_2]$. By Lemma 2.5, $\mathbb{L}_3$ is ${<}\kappa_2$-closed, so it does not collapse κ_2 and, since the forcing iteration to specialize κ_2-Aronszajn trees has the κ_2-c.c., also the rest of the iteration does not collapse κ_2. Thus $V[\mathbb{P}^{\aleph_2}_{\kappa_3}] \models \kappa_2 = \aleph_2$.

Since, using Lemma 2.6, $\mathbb{L}_2 * \mathbb{L}_3$ has the κ_3-c.c., it follows that $\mathbb{P}^{\aleph_2}_{\kappa_3}$ has the κ_3-c.c., therefore κ_3 is preserved and every κ_2-Aronszajn tree in $V[\mathbb{P}^{\aleph_2}_{\kappa_3}]$ has a nice $\mathbb{P}^{\aleph_2}_{\kappa_3}$-name of size smaller than κ_3 and so there are only κ_3 many nice names for κ_2-Aronszajn trees. Hence the bookkeeping function can make sure that all κ_2-Aronszajn trees have been specialized in $V[\mathbb{P}^{\aleph_2}_{\kappa_3}]$ with a specializing function to $[\aleph_1]^{\leq\omega}$.

Lemma 4.6. *Let $\mathbb{P}$ be a forcing and $\dot{T}$ a $\mathbb{P}$-name for a κ_2-Aronszajn tree. Then $\mathbb{S}_{\mathbb{P}}(\dot{T})$ is σ-closed with weakest lower bounds and well-met. Moreover, weakest lower bounds of decided conditions in $\mathbb{P} * \mathbb{S}_{\mathbb{P}}(\dot{T})$ are decided (see Lemma 3.22).*

Proof. Let $\langle q_i \mid i < \omega \rangle$ be a decreasing sequence in $\mathbb{S}_{\mathbb{P}}(\dot{T})$. Let $q^* := \bigcup_{i<\omega} q_i$. It is easy to see that $q^* \in \mathbb{S}_{\mathbb{P}}(\dot{T})$, $q^* \leq q_i$ for every $i \in \omega$ and q^* is a weakest lower bound. Now let $q \not\perp q'$ in $\mathbb{S}_{\mathbb{P}}(\dot{T})$. Then clearly $q \cup q'$ is a weakest lower bound of q and q'.

It remains to show that weakest lower bounds of decided conditions are decided: Let $p \in \mathbb{P}$ and $q, q' \in V$ such that $p \Vdash \check{q}, \check{q}' \in \mathbb{S}_{\mathbb{P}}(\dot{T})$ and $(p, \check{q}) \not\perp (p, \check{q}')$. So there exists $p^* \leq p$ such that $p^* \Vdash \check{q} \not\perp \check{q}'$. Let $q^* := q \cup q'$. It follows that $q^* \in V$ and $p^* \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\check{q}$ and $\check{q}'$" and p^* decides $\check{q}^*$, since $\not\perp_{\mathbb{S}_{\mathbb{P}}(\dot{T})}$ is definable in V. So $|q^*| \leq \omega$ and for s, t with $q^*(s) \cap q^*(t) \neq \emptyset$ we know that $\mathbb{1}_{\mathbb{P}} \Vdash$ "s is incomparable to t in $\dot{T}$". Since this does not depend on p^* it follows that $p \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\check{q}$ and $\check{q}'$" and p decides $\check{q}^*$, since $\not\perp_{\mathbb{S}_{\mathbb{P}}(\dot{T})}$ is definable in V.

Similarly, for $p \in \mathbb{P}$ and $q_i \in V$ for each $i < \omega$ such that $p \Vdash$ "$\langle \check{q}_i \mid i < \omega \rangle$ is decreasing in $\mathbb{S}_{\mathbb{P}}(\dot{T})$" let $q^* := \bigcup_{i \in \omega} q_i$. Note that $|q^*| \leq \omega$. For $s, t \in \mathrm{dom}(q^*)$ with $q^*(s) \cap q^*(t) \neq \emptyset$ we know that there exists $i < \omega$ such that $s, t \in \mathrm{dom}(q_i)$, therefore $\mathbb{1}_{\mathbb{P}} \Vdash$ "s is incomparable to t". So $q^* \in \mathbb{S}_{\mathbb{P}}(\dot{T})$ and $p \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\langle \check{q}_i \mid i < \omega \rangle$" and p decides $\check{q}^*$, since $\not\perp_{\mathbb{S}_{\mathbb{P}}(\dot{T})}$ is definable in V. □

We prove the following lemma for all iterations of length $< \kappa_3^+$ since we will need it for iterations longer than κ_3 in Lemma 4.13.

Lemma 4.7. *Let $\alpha < \kappa_3^+$ be a limit ordinal and let $\mathbb{P}$ be a forcing with $V[G(\mathbb{P})] \models \kappa_2 = \aleph_2$. Let $\mathbb{S}_\alpha$ be an iteration of limit length α of forcings to specialize names for $\aleph_2$-Aronszajn trees with countable support in $V[G(\mathbb{P})]$. Then in $V[G(\mathbb{P})]$ the set of decisive conditions in $\mathbb{S}_\alpha$ is dense and nicely σ-closed.*

Proof. We want to use Lemma 3.21 and Lemma 3.22. First note that by Lemma 4.6 $\mathbb{S}_{\mathbb{P}}(\dot{T})$ is σ-closed and well-met, and weakest lower bounds of decided conditions are decided for sequences of length ω and for pairs of conditions, hence the first and the last requirement of Lemma 3.22, and the last requirement of Lemma 3.21 hold.

Now we argue that also the other requirements of Lemma 3.21 hold. Each iterand is σ-closed, therefore also the iteration with countable support is σ-closed, which shows the first requirement. The second requirement holds by the definition of the forcing $\mathbb{S}_{\mathbb{P}}(\dot{T})$. For the third requirement let $R(q', q)$ as explained in Remark 4.3, i.e., for $p \in \mathbb{P}$ and $q, q' \in V$ such that $p \Vdash \check{q}, \check{q}' \in \mathbb{S}_{\mathbb{P}}(\dot{T})$, $p \Vdash \check{q}' \not\perp \check{q}$ if and only if $R(q', q)$ holds in V. So we can apply Lemma 3.21 and get that the set of decisive conditions is dense.

Thus, all the requirements of Lemma 3.22 are fulfilled, and we get the nice σ-closure. □

4.2 Chain condition and regular subforcings

From now on let $j\colon V \to M$ be a supercompact embedding for κ_2 such that $j(\kappa_2) > \kappa_3$ and ${}^{\leq\kappa_3}M \subseteq M$.

Lemma 4.8. *There exists an $\mathbb{L}_2$-name $\mathbb{L}_3^*$ such that in $V[\mathbb{L}_2]$, $\mathbb{L}_3^*$ is a regular subforcing of $j(\mathbb{L}_2)/G(\mathbb{L}_2)$, $(j(\mathbb{L}_2)/G(\mathbb{L}_2))/G(\mathbb{L}_3^*)$ is σ-closed, $\mathbb{L}_3^*$ is forcing equivalent to $\mathbb{L}_3$ and $|\mathbb{L}_3^*| < j(\kappa_2)$.*

Proof. Let $G(\mathbb{L}_2)$ be generic for $\mathbb{L}_2$. We work in $V[G(\mathbb{L}_2)]$ and apply Theorem 3.10: $\mathbb{L}_3$ is σ-closed and $|\mathbb{L}_3| < j(\kappa_2)$, thus in $V[G(\mathbb{L}_2)]$ there exists a regular embedding $\iota\colon \mathbb{L}_3 \to \mathrm{Col}(\omega_1, {<}j(\kappa_2)) = j(\mathbb{L}_2)$ such that $j(\mathbb{L}_2)$ is equivalent to $j(\mathbb{L}_2)/\iota[G(\mathbb{L}_3)]$. It follows that in V there exists a regular embedding $\iota\colon \mathbb{L}_3 \to j(\mathbb{L}_2)/G(\mathbb{L}_2)$ such that $j(\mathbb{L}_2)/G(\mathbb{L}_2)$ is equivalent to $(j(\mathbb{L}_2)/G(\mathbb{L}_2))/\iota[G(\mathbb{L}_3)]$. Again in $V[G(\mathbb{L}_2)]$, by Lemma 3.15 $j(\mathbb{L}_2) = \mathrm{Col}(\omega_1, {<}j(\kappa_2))$ is nicely σ-closed, thus in V, $j(\mathbb{L}_2)/G(\mathbb{L}_2)$ is nicely σ-closed. Therefore by Lemma 3.16 it follows that $(j(\mathbb{L}_2)/G(\mathbb{L}_2))/\iota[G(\mathbb{L}_3)]$ is σ-closed. $\mathbb{L}_3^* := \iota[\mathbb{L}_3]$ is the forcing we are looking for. □

Corollary 4.9. *There exists a reduction map $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}_2 * \mathbb{L}_3^*$.*

Proof. Clearly there exists a reduction map $\pi_1\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to j(\mathbb{L}_2)$. Note that $j(\mathbb{L}_2)$ is forcing equivalent to $\mathbb{L}_2 * (j(\mathbb{L}_2)/\mathbb{L}_2)$. By Lemma 4.8, $j(\mathbb{L}_2)/\mathbb{L}_2$ has $\mathbb{L}_3^*$ as a regular subforcing, hence there exists a reduction map $\pi_2\colon \mathbb{L}_2 * (j(\mathbb{L}_2)/\mathbb{L}_2) \to \mathbb{L}_2 * \mathbb{L}_3^*$. This shows that there exists a reduction map $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}_2 * \mathbb{L}_3^*$. □

To be able to use the supercompact embedding, we have to lift it to the forcing extensions. To lift a supercompact embedding for κ to the extension by a Lévy collapse for some larger cardinal κ' we use absorption, i.e., the fact that the Lévy collapse for κ' contains the collapse for κ as a regular subforcing:

Lemma 4.10. *Let $G(\mathbb{L}_2)$ be generic for $\mathbb{L}_2$ and $G(\mathbb{L}_3)$ generic for $\mathbb{L}_3$ over $V[G(\mathbb{L}_2)]$. The supercompact embedding j can be lifted to*

$$j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3)] \to M[G(j(\mathbb{L}_2) * j(\mathbb{L}_3))].$$

Proof. Let $\iota\colon \mathbb{L}_3 \to j(\mathbb{L}_2)/G(\mathbb{L}_2)$ be a regular embedding as in Lemma 4.8. We can choose $G(j(\mathbb{L}_2))$ such that $G(j(\mathbb{L}_2)) \cap \mathrm{range}(\iota) = \iota[G(\mathbb{L}_3)]$, thus $\iota[G(\mathbb{L}_3)] \in V[G(j(\mathbb{L}_2))]$ and $G(\mathbb{L}_2) \subseteq G(j(\mathbb{L}_2))$; that is possible because $\mathbb{L}_2 * \iota[\mathbb{L}_3]$ is a regular subforcing of $j(\mathbb{L}_2)$. Thus it follows that $\iota[G(\mathbb{L}_3)] \in V[G(j(\mathbb{L}_2))]$ and since

ι and $j \restriction \mathbb{L}_3 \in V[G(j(\mathbb{L}_2))]$ it follows that $j[G(\mathbb{L}_3)] \in V[G(j(\mathbb{L}_2))]$. Since M is closed under subsets of size $\leq \kappa_3$ the same holds for $M[G(j(\mathbb{L}_2))]$ and therefore $j[G(\mathbb{L}_3)] \in M[G(j(\mathbb{L}_2))]$.

Now $j[G(\mathbb{L}_3)] \subseteq j[\mathbb{L}_3] \subseteq j(\mathbb{L}_3)$, $j[G(\mathbb{L}_3)]$ is a directed set of size $< j(\kappa_2)$ and $j(\mathbb{L}_3)$ is $<j(\kappa_2)$-directed closed, therefore there exists a master condition $p \in j(\mathbb{L}_3)$ for $j[G(\mathbb{L}_3)]$. Let $G(j(\mathbb{L}_3))$ be generic for $j(\mathbb{L}_3)$ with $p \in G(j(\mathbb{L}_3))$. It follows that $j[G(\mathbb{L}_3)] \subseteq G(j(\mathbb{L}_3))$.

Finally, we can use the Lifting Lemma (Lemma 3.9) to lift j to an embedding $j\colon V[G(\mathbb{L}_2)][G(\mathbb{L}_3)] \to M[G(j(\mathbb{L}_2))][G(j(\mathbb{L}_3))]$. □

As a summary of the previous lemmata we get the following:

Corollary 4.11. *There exists a regular subforcing $\mathbb{L}^*_3$ of $j(\mathbb{L}_2 * \mathbb{L}_3)$ with the following properties:*

(1) *$\mathbb{L}^*_3$ is also a regular subforcing of $j(\mathbb{L}_2) \times \{\mathbb{1}_{j(\mathbb{L}_3)}\}$.*

(2) *$\mathbb{L}^*_3$ is forcing equivalent to $\mathbb{L}_2 * \mathbb{L}_3$.*

(3) *There exists a reduction map $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}^*_3$.*

(4) *$j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}^*_3)$ is σ-closed.*

Moreover, there exists a lifting of the supercompact embedding for κ_2 to $j\colon V[\mathbb{L}_2 * \mathbb{L}_3] \to M[j(\mathbb{L}_2 * \mathbb{L}_3)]$.

One of the main technical parts of the proof is to show that the forcing iteration has a good chain condition. The main work lies in the following lemma, which deals with the successor step of the iteration. Note that $\mathbb{L}_2 * \mathbb{L}_3$ with $\mathbb{L}_2 * \mathbb{L}^*_3$ as a subforcing of $j(\mathbb{L}_2 * \mathbb{L}_3)$ fulfills the requirements of the following lemma.

Lemma 4.12. *Assume $\mathbb{P} = \mathbb{L}_2 * \mathbb{L}_3 * \mathbb{P}_0$ is a forcing with $V[G(\mathbb{P})] \models \kappa_2 = \omega_2$ and $\mathbb{P}^* = \mathbb{L}_2 * \mathbb{L}^*_3 * \mathbb{P}^*_0$ is forcing equivalent to $\mathbb{P}$, and $\mathbb{P}^*$ is a regular subforcing of $j(\mathbb{P})$ and the sets of decisive conditions in $\mathbb{P}_0$ and in $\mathbb{P}^*_0$ are dense and nicely σ-closed, and $\mathbb{P}_0$ is forcing equivalent to $\mathbb{P}^*_0$. Further assume $\mathbb{P}^*_0$ is a regular subforcing of $j(\mathbb{P}_0)$ with reduction map $\pi\colon j(\mathbb{P}_0) \to \mathbb{P}^*_0$. Let $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ be a lifting of the supercompact embedding for κ_2 and $\mathbb{S} = \mathbb{S}_{\mathbb{P}}(\dot{T})$ a specializing forcing of a $\mathbb{P}$-name for a κ_2-Aronszajn tree $\dot{T}$. Then the following hold:*

(1) *There exists a regular subforcing $\mathbb{P}^*_0 * \mathbb{S}^*$ of $j(\mathbb{P}_0) * j(\mathbb{S})$ with a reduction map $\pi^*\colon j(\mathbb{P}_0) * j(\mathbb{S}) \to \mathbb{P}^*_0 * \mathbb{S}^*$ such that the first component of $\pi^*(p, s)$ extends $\pi(p)$.*

(2) *$|\mathbb{S}^*| = \kappa_2$.*

(3) $j(\mathbb{P}) \Vdash$ "$\mathbb{S}^*$ *is a regular subforcing of* $j(\mathbb{S})$" *and* $\mathbb{P} \Vdash$ "$\mathbb{S}$ *has the* κ_2*-c.c.*".

(4) $\mathbb{P} * \mathbb{S}$ *is forcing equivalent to* $\mathbb{P}^* * \mathbb{S}^*$.

(5) *The supercompact embedding* j *can be lifted to*

$$j\colon V[G(\mathbb{P} * \mathbb{S})] \to M[G(j(\mathbb{P}) * j(\mathbb{S}))].$$

(6) $\mathbb{P}_0 * \mathbb{S}$ *has a dense subset which is nicely* σ*-closed in* $V[\mathbb{L}_2 * \mathbb{L}_3]$ *and the quotient* $j(\mathbb{P}_0 * \mathbb{S})/G(\mathbb{P}^*_0 * \mathbb{S}^*)$ *is equivalent to a* σ*-closed forcing in* $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))][G(\mathbb{P}^*_0 * \mathbb{S}^*)]$.

Proof. The main work is to prove (1).

Proof of (1): We work in $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$. Let $(p, s) \in j(\mathbb{P}_0) * j(\mathbb{S})$. Let $p' \leq p, \pi(p)$ such that p' decides s, that means in $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$ there exists a countable partial function $f\colon \omega_1 \times j(\kappa_2) \to [\omega_1]^{\leq\omega}$ such that $p' \Vdash s = f$. If $p'' \leq \pi(p')$, then p'' is compatible with p' and therefore with $\pi(p)$, thus $\pi(p)$ and $\pi(p')$ are compatible in $j(\mathbb{P}_0)$. Since $\mathbb{P}^*_0$ is a regular subforcing of $j(\mathbb{P}_0)$, $\pi(p)$ and $\pi(p')$ are compatible in $\mathbb{P}^*_0$. Let $\hat{p} \in \mathbb{P}^*_0$ with $\hat{p} \leq \pi(p), \pi(p')$.

Continue working in $V[G(\mathbb{L}_2 * \mathbb{L}_3)] = V[G(\mathbb{L}_2 * \mathbb{L}^*_3)]$: choose a generic $G(\mathbb{P}^*_0)$ containing $\hat{p}$ and let $G(\mathbb{P}_0)$ be the corresponding generic for $\mathbb{P}_0$, i.e., $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}_0)] = V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}^*_0)]$; that is possible because $\mathbb{P}_0$ and $\mathbb{P}^*_0$ are forcing equivalent. Note that $p \in j(\mathbb{P}_0)/G(\mathbb{P}^*_0)$ because $\hat{p} \leq \pi(p)$ and $\pi(p)$ is a reduction of p.

Let $T := \dot{T}^{G(\mathbb{P}_0)}$. Since $T \in V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}_0)]$, it follows that $T \in V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}^*_0)]$. Let $\dot{T}^*$ be a $\mathbb{P}^*$-name for T and let $\mathbb{S}^* := \mathbb{S}_{\mathbb{P}^*}(\dot{T}^*)$, the specializing forcing of $\dot{T}^*$.

We assume that the nodes on the αth level T_α of T are elements of $\omega_1 \times \{\alpha\}$, and all the levels are of size $< \kappa_2$, therefore $T = j[T] = j(T) \upharpoonright \kappa_2$.

We can assume that for each $\sigma \in \operatorname{dom}(s) \cap j(T)_{>\kappa_2}$ there exists a $\sigma' \in \operatorname{dom}(s)$ on level κ_2 such that $p' \Vdash \sigma' \leq_T \sigma$.

Let $\bar{s} := s \upharpoonright T$, $\{\sigma_n \mid n \in \omega\} := \operatorname{dom}(s) \cap T_{\kappa_2}$ and $C_n := \bigcup\{s(\tau) \mid \tau \geq_T \sigma_n, \tau \in \operatorname{dom}(s)\}$ the set of colors which s assigns to nodes which are in $\operatorname{dom}(s)$ and equal to or above σ_n.

Let $\mathbb{Q} := j(\mathbb{P}_0)/G(\mathbb{P}^*_0)$. By assumption in $\mathbb{P}_0$ the set of decisive conditions is dense and nicely σ-closed and in $\mathbb{P}^*_0$ the set of decisive conditions is dense. By elementarity also in $j(\mathbb{P}_0)$ the set of decisive conditions is dense and nicely σ-closed. Further, note that the set D^* of decisive conditions in $\mathbb{P}^*_0$ is contained in the set $\bar{D}$ of decisive conditions in $j(\mathbb{P}_0)$. So by Lemma 3.6 D^* is a regular subforcing of $\bar{D}$. Therefore by Lemma 3.16, by working in the dense sets of decisive conditions, we can assume that $\mathbb{Q}$ is σ-closed. Define a tree $\mathcal{T}$ of height ω inductively. Each node t on level n will be of the form $(p_w, \tau^0_w, \ldots, \tau^n_w)$ for some

$w \in 2^n$ and $p_w \in \mathbb{Q}$ with $p_w \leq p$, and $p_w \Vdash$ "$\tau^k_w <_T \sigma_k$" for each $k \leq n$. The construction is as follows:

- The root of $\mathcal{T}$ is $(p_{\langle\rangle}, \tau^0_{\langle\rangle})$ where $p_{\langle\rangle} \in \mathbb{Q}$ with $p_{\langle\rangle} \leq p$ and $p_{\langle\rangle} \Vdash$ "$\tau^0_{\langle\rangle} <_T \sigma_0 \wedge \tau^0_{\langle\rangle} \in T$". So $\tau^0_{\langle\rangle}$ is just some node which is forced by $p_{\langle\rangle}$ to be below σ_0.
- Assume t is a node of $\mathcal{T}$ on level n, so t is of the form $(p_w, \tau^0_w, \ldots, \tau^n_w)$ for some $w \in 2^n$ and $p_w \in \mathbb{Q}$, and $p_w \Vdash$ "$\tau^k_w <_T \sigma_k$" for each $k \leq n$.

  Since T is an Aronszajn tree in $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}^*_0)]$, every cofinal branch through T in $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))][G(j(\mathbb{P}_0)/G(\mathbb{P}^*_0))]$ is new. Therefore there exist two conditions $p_{w^\frown 0} \leq p_w$ and $p_{w^\frown 1} \leq p_w$ which decide for every $k \leq n$ the nodes between τ^k_w and σ_k differently. We define two successors for t in $\mathcal{T}$:

  $$(p_{w^\frown 0}, \tau^0_{w^\frown 0}, \ldots, \tau^n_{w^\frown 0}, \tau^{n+1}_{w^\frown 0}) \text{ and } (p_{w^\frown 1}, \tau^0_{w^\frown 1}, \ldots, \tau^n_{w^\frown 1}, \tau^{n+1}_{w^\frown 1})$$

  where $p_{w^\frown i}$ and $\tau^k_{w^\frown i}$ for $k \leq n$ and $i \in \{0, 1\}$ are such that the following hold true: $p_{w^\frown i} \Vdash$ "$\tau^k_w \leq_T \tau^k_{w^\frown i} <_T \sigma_k$, $\tau^k_{w^\frown i} \in T$" and $\tau^k_{w^\frown 0}$ is incomparable with $\tau^k_{w^\frown 1}$ in T, and $p_{w^\frown i} \Vdash$ "$\tau^{n+1}_{w^\frown i} <_T \sigma_{n+1} \wedge \tau^{n+1}_{w^\frown i} \in T$".

For each branch b through $\mathcal{T}$ let p_b be stronger than all $p_{b\restriction k}$ and τ^n_b such that $p_b \Vdash$ "$\tau^n_{b\restriction k} \leq_T \tau^n_b \leq_T \sigma_n$ and τ^n_b is the limit of $\langle \tau^n_{b\restriction k} \rangle_{k \in \omega}$". Note that such τ^n_b exist in T, since the height of T is κ_2, and $\kappa_2 = \aleph_2$ in $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}^*_0)]$. Further note that τ^n_b and $\tau^n_{b'}$ are incomparable for all $b \neq b'$.

Let $s' := \bar{s} \cup \{(\tau^n_b, C_n) \mid n \in \omega, b \in K\}$, where K is the set of elements in 2^ω which have only boundedly many 1's. This is a condition in $\mathbb{S}$, because for each n the set C_n contains all the colors which appear at or above σ_n, so they don't appear at nodes below σ_n and therefore not at nodes below τ^n_b.

Let q be such that $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}^*_0)] \models q \in \mathbb{S} \wedge q \leq s'$. As a preparation for the definition of the reduction map, we show that in $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}^*_0)]$ there exists a $p' \in \mathbb{Q}$ such that $p' \Vdash q \not\perp s$. Let $c \in 2^\omega$ be such that no node in $\mathrm{dom}(q)$ extends a τ^n_c for any n. Note that $c \notin K$. Such a c exists, since 2^ω is uncountable and $\mathrm{dom}(q)$ is countable. Now $p_c \Vdash$ "$\tau^n_c \leq_T \sigma_n$" for all n, thus $p_c \Vdash$ "$\tau^n_b \not\leq_T \sigma_n$" for all n and all $b \in K$. Let $t \in \mathrm{dom}(q)$ and $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$. Since $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ there exists $n \in \omega$ with $\sigma_n \leq_T \tau$. By induction on n we define a decreasing sequence $\langle p^n_c \mid n \in \omega \rangle$ such that $p^{n+1}_c \Vdash$ "$(\tau, s(\tau))$ is compatible with $(t, q(t))$" (i.e., $p^{n+1}_c \Vdash \{(\tau, s(\tau))\} \cup \{(t, q(t))\} \in \mathbb{S}$) for all $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ with $\sigma_n \leq_T \tau$.

Let $p^0_c := p_c$.

Let $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ with $\sigma_n \leq_T \tau$.

Case 1: $p^n_c \Vdash t <_T \tau^n_c$. Since p_c forces that τ^n_c is the limit of some τ^n_w's, $p^n_c \leq p_c$, and for every $w \in 2^{<\omega}$ there exists a $b \in K$ which extends w, p^n_c forces that there exists some $b \in K$ with $t <_T \tau^n_b$. Therefore, since q is a condition and τ^n_b is in its domain, $p^n_c \Vdash q(t) \cap q(\tau^n_b) = \emptyset$, and since $q(\tau^n_b) = C_n \supseteq s(\tau)$, it follows that $p^n_c \Vdash$ "$(\tau, s(\tau))$ is compatible with $(t, q(t))$". Let $p^{n+1}_c := p^n_c$.

Case 2: $p^n_c \nVdash t <_T \tau^n_c$. On the other hand, $\tau^n_c \nleq_T t$ by the choice of c, thus there exists $p^{n+1}_c \leq p^n_c$ with $p^{n+1}_c \Vdash$ "τ^n_c is incomparable with t". Since $p_c \Vdash$ "$\tau^n_c \leq_T \sigma_n$", it follows that $p^{n+1}_c \Vdash$ "$t \nless_T \sigma_n$" and therefore $p^{n+1}_c \Vdash$ "$(\tau, s(\tau))$ is compatible with $(t, q(t))$".

Using the σ-closure of $\mathbb{Q}$, there exists a lower bound p'_c of $\langle p^n_c \mid n \in \omega \rangle$. Since p'_c forces for every $t \in \operatorname{dom}(q)$ and every $\tau \in \operatorname{dom}(s) \setminus \operatorname{dom}(s')$ that $(\tau, s(\tau))$ is compatible with $(t, q(t))$, together with the fact that $q \leq s'$, it follows that $p'_c \Vdash$ "q is compatible with s". Thus it holds in $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{P}^*_0)]$ that for every $q \leq s'$ there exists a $p' \leq p$ such that $p' \Vdash$ "q is compatible with s". Since $G(\mathbb{P}^*_0)$ is a filter, we can choose a condition $\bar{p} \in G(\mathbb{P}^*_0)$ below $\hat{p}$ which forces this.

Define $\pi^*(p, s) := (\bar{p}, s')$.

If $(p^*, s^*) \leq \pi^*(p, s)$ then $p^* \leq \pi(p)$ and therefore p^* is compatible with p and $p^* \leq \bar{p}$ and $p^* \Vdash s^* \leq s'$. Therefore p^* forces that some $p' \in \mathbb{Q}$, with $p' \leq p$, forces s^* to be compatible with s. Since $p^* \Vdash p' \in \mathbb{Q} = j(\mathbb{P}_0)/G(\mathbb{P}^*_0)$, it follows that there exists $p'' \leq p, p^*$ with $p'' \Vdash s \not\perp s^*$. So (p^*, s^*) is compatible with (p, s) and therefore π^* is a reduction map such that the first component of $\pi^*(p, s)$ extends $\pi(p)$.

To see that $\mathbb{P}^*_0 * \mathbb{S}^*$ is a regular subforcing of $j(\mathbb{P}_0) * j(\mathbb{S})$ we also have to show that if two conditions in $\mathbb{P}^*_0 * \mathbb{S}^*$ are compatible in $j(\mathbb{P}_0) * j(\mathbb{S})$, then they are compatible in $\mathbb{P}^*_0 * \mathbb{S}^*$. To see this, we show that the set D of conditions (p, s) with the following property is dense in $j(\mathbb{P}_0) * j(\mathbb{S})$: There exists s^* such that

1. $p \Vdash s \leq s^*$,
2. $p \Vdash s^* \in \mathbb{S}^*$,
3. if $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$.

If p decides s then (p, s) fulfills this property: Let s^* be s restricted to the nodes on levels below κ_2. So $p \Vdash s \leq s^* \wedge s^* \in \mathbb{S}^*$ and if $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$, because in this case $\bar{s} \subseteq s^*$. So the set D is dense.

Suppose now that (p^*_0, s^*_0) and (p^*_1, s^*_1) are in $\mathbb{P}^*_0 * \mathbb{S}^*$ and they are compatible in $j(\mathbb{P}_0) * j(\mathbb{S})$. Let (p, s) be a witness for the compatibility in the dense set D with witness s^*. So (p, s^*) is also below (p^*_0, s^*_0) and (p^*_1, s^*_1). Now $(\pi(p), s^*)$ is in $\mathbb{P}^*_0 * \mathbb{S}^*$ and stronger than (p^*_0, s^*_0) and (p^*_1, s^*_1): Since $p \Vdash s^* \in \mathbb{S}^* \wedge s^* \leq s^*_0, s^*_1$ and that depends only on $\mathbb{P}^*_0$, the same holds true for $\pi(p)$.

Proof of (2): Since $\mathbb{S}^* = \mathbb{S}_{\mathbb{P}^*}(\dot{T}^*)$ and $\dot{T}^*$ is a name for a κ_2-Aronszajn tree, we know that $|\mathbb{S}^*| = \kappa_2$.

Proof of (3): Let $G(\mathbb{P})$ be generic for $\mathbb{P}$ and $G(\mathbb{P}^*)$ the corresponding generic for $\mathbb{P}^*$. Let $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ be the lifting of the supercompact embedding for κ_2. In $V[G(\mathbb{P})]$ let A^* be a maximal antichain in $\mathbb{S}$. Since $V[G(\mathbb{P})] = V[G(\mathbb{P}^*)]$ and $\mathbb{S}$ in $V[G(\mathbb{P})]$ is the same as $\mathbb{S}^*$ in $V[G(\mathbb{P}^*)]$, we get that in $V[G(\mathbb{P}^*)]$ A^* is a maximal antichain in $\mathbb{S}^*$. On the other hand, also each maximal antichain in $\mathbb{S}^*$ is a maximal antichain in $\mathbb{S}$. By elementarity $M[G(j(\mathbb{P}))] \models$ "$j(A^*)$ is a maximal antichain in $j(\mathbb{S})$". Since j is the identity on $\mathbb{S}$ it follows that $A^* = j[A^*] \subseteq j(A^*)$. Let $G(j(\mathbb{P})/G(\mathbb{P}^*))$ be generic for $j(\mathbb{P})/G(\mathbb{P}^*)$ and assume $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models$ "$s \in j(\mathbb{S})$". Since $|\mathbb{S}^*| = \kappa_2$ and ${}^{\leq\kappa_3}M \subseteq M$ it follows that $A^* \in M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))]$.

Claim. $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models$ "*$\exists a \in A^*$ which is compatible with s*".

Proof. Let $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ be such that (p, s) is a condition. We show that the set of conditions which force that there exists $a \in A^*$ which is compatible with s is dense below p. Let $p' \leq p$ and let (p^*, s^*) be a reduction of (p', s) to $\mathbb{P}^* * \mathbb{S}^*$. Since A^* is maximal in $\mathbb{S}^*$ we know that p^* forces over $\mathbb{P}^*$ that there exists $a \in A^*$ which is compatible with s^* and we can pick a name $\dot{b}$ for the witness in $\mathbb{S}^*$. Now $(p^*, \dot{b}) \leq (p^*, s^*)$. Since (p^*, s^*) is a reduction of (p', s) we know that $(p^*, \dot{b})$ is compatible with (p', s). So there exists $\bar{p} \leq p^*, p'$ with $\bar{p} \Vdash$ "$\dot{b}$ is compatible with s", and since $\dot{b}$ is forced to be $\leq a$ by p^*, also $\bar{p} \Vdash$ "$a \in A^*$ is compatible with s".

Now, since $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$, there exists a $q \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ with $q \Vdash$ "$\exists a \in A^*$ which is compatible with s". □

Therefore it follows that A^* is a maximal antichain for $j(\mathbb{S})$ in the model $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))]$. Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. From the above it follows that every maximal antichain of $\mathbb{S}^*$ is a maximal antichain in $j(\mathbb{S})$, hence $\mathbb{S}^*$ is a regular subforcing of $j(\mathbb{S})$. For the second part of (3) note that $A^* \subseteq \mathbb{S}$ and $|\mathbb{S}| = \kappa_2$, so we have that $|j(A^*)| \leq \kappa_2 < j(\kappa_2)$ and by elementarity $|A^*| < \kappa_2$.

Proof of (4): $\mathbb{P}^*$ is forcing equivalent to $\mathbb{P}$, and $\mathbb{S}^*$ in $V[\mathbb{P}^*]$ is the same forcing as $\mathbb{S}$ in $V[\mathbb{P}]$.

Proof of (5): By the assumption of the lemma there exists $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$, a lifting of the supercompact embedding j. Since $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ we can replace $V[G(\mathbb{P})]$ by $V[G(\mathbb{P}^*)]$ and get $j\colon V[G(\mathbb{P}^*)] \to M[G(j(\mathbb{P}))]$. Let $G(j(\mathbb{S}))$ be generic for $j(\mathbb{S})$ over $M[G(j(\mathbb{P}))]$. Since by (3) $\mathbb{S}^*$ is a regular subforcing of $j(\mathbb{S})$ and $\mathbb{S}^* \subseteq j(\mathbb{S})$, $G(j(\mathbb{S}))$ contains a generic filter $G(\mathbb{S}^*)$ for $\mathbb{S}^*$. Thus, by the Lifting Lemma (Lemma 3.9), j can be lifted to an embedding $j\colon V[G(\mathbb{P}^*)][G(\mathbb{S}^*)] \to M[G(j(\mathbb{P}))][G(j(\mathbb{S}))]$. By (4) $\mathbb{P} * \mathbb{S}$ is equivalent to $\mathbb{P}^* * \mathbb{S}^*$, so we can replace $V[G(\mathbb{P}^*)][G(\mathbb{S}^*)]$ by $V[G(\mathbb{P})][G(\mathbb{S})]$ to get a lifting $j\colon V[G(\mathbb{P})][G(\mathbb{S})] \to M[G(j(\mathbb{P}))][G(j(\mathbb{S}))]$.

Proof of (6): By assumption, in $\mathbb{P}_0$ and $\mathbb{P}_0^*$ the sets of decisive conditions are dense and nicely σ-closed. By Lemma 4.6, $\mathbb{S}$ and $\mathbb{S}^*$ are σ-closed with weakest lower bounds and well-met, and weakest lower bounds of decided conditions are decided. Further, by the definition of the forcings to specialize names, $\mathbb{P}_0 \Vdash$ "$q \subseteq V$ and $|q| < \omega_1$ for all $q \in \mathbb{S}$" and $\mathbb{P}_0^* \Vdash$ "$q \subseteq V$ and $|q| < \omega_1$ for all $q \in \mathbb{S}^*$", and whenever $(p, q) \not\perp (p', q')$ are decisive conditions in $\mathbb{P}_0 * \mathbb{S}$, or in $\mathbb{P}_0^* * \mathbb{S}^*$, and $p^* \leq p, p'$, then $p^* \Vdash q \not\perp q'$. So by Lemma 3.23 in $\mathbb{P}_0 * \mathbb{S}$ the set D of decisive conditions, and in $\mathbb{P}_0^* * \mathbb{S}^*$ the set D^* of decisive conditions, are dense and nicely σ-closed. By elementarity the same holds for $j(\mathbb{P}_0 * \mathbb{S})$. Further, note that the set D^* of decisive conditions in $\mathbb{P}_0^* * \mathbb{S}^*$ is contained in the set $\bar{D}$ of decisive conditions in $j(\mathbb{P}_0 * \mathbb{S})$. So by Lemma 3.6 D^* is a regular subforcing of $\bar{D}$. Since $\bar{D}$ is equivalent to $j(\mathbb{P}_0 * \mathbb{S})$ and D^* is equivalent to $\mathbb{P}_0^* * \mathbb{S}^*$, the quotient $j(\mathbb{P}_0 * \mathbb{S})/G(\mathbb{P}_0^* * \mathbb{S}^*)$ is equivalent to $\bar{D}/G(D^*)$, and by Lemma 3.16 $\bar{D}/G(D^*)$ is σ-closed. □

Now we are ready to prove the κ_2-c.c. of $\mathbb{S}_{\kappa_3}$. In particular, we will consider the limit steps of the iteration. We prove the lemma for all iterations of length $< \kappa_3^+$ since we will need it for iterations longer than κ_3 in Lemma 4.14.

Lemma 4.13. *Let $\mathbb{L}_2 * \mathbb{L}_3^*$ be forcing equivalent to $\mathbb{L}_2 * \mathbb{L}_3$. Let $\pi\colon j(\mathbb{L}_2 * \mathbb{L}_3) \to \mathbb{L}_2 * \mathbb{L}_3^*$ be a reduction map and $j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3)] \to M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$ a lifting of the supercompact embedding for κ_2, and in $V[G(\mathbb{L}_2 * \mathbb{L}_3)]$ let $\mathbb{S}_\alpha$ be a countable support iteration of length $\alpha < \kappa_3^+$ of forcings to specialize names of κ_2-Aronszajn trees. Then there exists $\mathbb{S}_\alpha^*$ with the following properties:*

(1) $|\mathbb{S}_\alpha^*| \leq \kappa_3$,

(2) *there exists a reduction map $\pi_\alpha^*\colon j(\mathbb{S}_\alpha) \to \mathbb{S}_\alpha^*$,*

(3) *$j(\mathbb{S}_\alpha)/\mathbb{S}_\alpha^*$ is equivalent to a σ-closed forcing,*

(4) *$\mathbb{L}_2 * \mathbb{L}_3^* * \mathbb{S}_\alpha^*$ is forcing equivalent to $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha$,*

(5) *$\mathbb{S}_\alpha^*$ is a regular subforcing of $j(\mathbb{S}_\alpha)$,*

(6) *j can be lifted to an elementary embedding*

$$j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha)] \to M[G(j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha))],$$

(7) *in $V[G(\mathbb{L}_2 * \mathbb{L}_3)]$ the forcing $\mathbb{S}_\alpha$ has the κ_2-c.c..*

Proof. The proof is by induction on $\alpha < \kappa_3^+$.

For $\alpha = 0$ there is nothing to show.

$\alpha = \beta + 1$: By induction and by Lemma 4.7 $\mathbb{S}_\beta$, the first β steps of the iteration, fulfills the requirements of Lemma 4.12, so we can apply this lemma to $\mathbb{L}_2 * \mathbb{L}_3 *$

$\mathbb{S}_\beta * \mathbb{S}_{\mathbb{L}_2*\mathbb{L}_3*\mathbb{S}_\beta}(\dot T_\beta)$, from which it is easy to see that properties (1)–(7) hold true; for (7), note that the two-step iteration of two forcings which have the $\aleph_2$-c.c. has again the $\aleph_2$-c.c.. Let $\mathbb{S}^*(\dot T_\beta)$ be the regular subforcing of $\mathbb{S}_{\mathbb{L}_2*\mathbb{L}_3*\mathbb{S}_\beta}(\dot T_\beta)$ and $\pi^*_{\beta+1}$ the reduction map given by the lemma. Note that by (1) of Lemma 4.12 $\pi^*_{\beta+1}$ is coherent with π^*_β in the sense that for $(p,\dot q) \in \mathbb{S}_\beta * \mathbb{S}_{\mathbb{L}_2*\mathbb{L}_3*\mathbb{S}_\beta}(\dot T_\beta)$ we have $\pi^*_{\beta+1}(p,\dot q) \leq \pi^*_\beta(p)$. So, $\mathbb{S}^*_\alpha := \mathbb{S}^*_\beta * \mathbb{S}^*(\dot T_\beta)$ is the required regular subforcing of $j(\mathbb{S}_\alpha)$.

α limit: In $V[\mathbb{L}_2 * \mathbb{L}^*_3]$ let $\mathbb{S}^*_\alpha$ be the iteration $\mathbb{S}^*(\dot T_0) * \mathbb{S}^*(\dot T_1) * \mathbb{S}^*(\dot T_2) * \dots$ of length α with countable support, where the $\mathbb{S}^*(\dot T_\beta)$ are given by induction. We will prove that the properties (1)–(7) hold true.

Proof of (1): Since $|\mathbb{S}^*(\dot T_\beta)| = \kappa_2$ for each $\beta < \alpha$ and $\alpha < \kappa_3^+$, we know that $|\mathbb{S}^*_\alpha| \leq \kappa_3$.

Proof of (2): Let $p \in j(\mathbb{S}_\alpha)$ and let $\{\beta_i \mid i < \omega\}$ be increasing indices cofinal in the support of p. Let $\pi^*_{\beta_i}$ be the reduction map of the iteration of length β_i given by induction. Since these maps cohere, $\pi^*_{\beta_0}(p\restriction\beta_0) \geq \pi^*_{\beta_1}(p\restriction\beta_1) \geq \pi^*_{\beta_2}(p\restriction\beta_2) \geq \dots$ and since $\mathbb{S}^*(\dot T_0) * \mathbb{S}^*(\dot T_1) * \mathbb{S}^*(\dot T_2) * \dots$ is σ-closed (as it is a countable support iteration of σ-closed forcings), there exists a lower bound of these reductions; let $\pi^*_\alpha(p)$ be such a lower bound. It is easy to check that π^*_α is a reduction map which is coherent with the earlier π^*_β's.

Proof of (3): By Lemma 4.7 the sets D and D^* of decisive conditions in $\mathbb{S}_\alpha$ and in $\mathbb{S}^*_\alpha$, respectively, are dense and nicely σ-closed. Therefore by elementarity the set $\bar D$ of decisive conditions in $j(\mathbb{S}_\alpha)$ is dense and nicely σ-closed. Further, D^* is contained in $\bar D$. Therefore by Lemma 3.6 D^* is a regular subforcing of $\bar D$. So $\bar D/G(D^*)$ is σ-closed by Lemma 3.16 and equivalent to $j(\mathbb{S}_\alpha)/G(\mathbb{S}^*_\alpha)$.

Proof of (4): Since the iterands of the two iterations are forcing equivalent and the iterations are both countable support iterations, the two iterations are forcing equivalent.

Proof of (5): Next we show that if two conditions in $(\mathbb{L}_2 * \mathbb{L}^*_3) * \mathbb{S}^*_\alpha$ are compatible in $j((\mathbb{L}_2 * \mathbb{L}_3) * \mathbb{S}_\alpha)$, then they are compatible in $(\mathbb{L}_2 * \mathbb{L}^*_3) * \mathbb{S}^*_\alpha$. To see this, we show that the set D of conditions $(p, \vec s)$ with the following property is dense in $j((\mathbb{L}_2 * \mathbb{L}_3) * \mathbb{S}_\alpha)$: There exists $\vec{s^*}$ such that

1. $p \Vdash \vec s \leq \vec{s^*}$,
2. $p \Vdash \vec{s^*} \in \mathbb{S}^*_\alpha$,
3. if $p \Vdash \vec s \leq \vec{\bar s} \wedge \vec{\bar s} \in \mathbb{S}^*_\alpha$ then $p \Vdash \vec{s^*} \leq \vec{\bar s}$.

If p decides $\vec s$, then $(p, \vec s)$ fulfills this property: Let $\vec{s^*}$ be the tuple of coordinates of $\vec s$ restricted to the nodes on levels below κ_2. So $p \Vdash \vec s \leq \vec{s^*} \wedge \vec{s^*} \in \mathbb{S}^*_\alpha$ and if $p \Vdash \vec s \leq \vec{\bar s} \wedge \vec{\bar s} \in \mathbb{S}^*_\alpha$ then $p \Vdash \vec{s^*} \leq \vec{\bar s}$, because in this case every coordinate of $\vec{\bar s}$ is forced to be a subset of the corresponding coordinate of $\vec{s^*}$. So D is dense.

Suppose now that $(p_0^*, \vec{s}_0^*)$ and $(p_1^*, \vec{s}_1^*)$ are in $(\mathbb{L}_2 * \mathbb{L}_3^*) * \mathbb{S}_\alpha^*$ and they are compatible in $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha)$. Let $(p, \vec{s})$ be a witness for the compatibility in the dense set with witness $\vec{s}^*$. So $(p, \vec{s}^*)$ is also below $(p_0^*, \vec{s}_0^*)$ and $(p_1^*, \vec{s}_1^*)$. Now $(\pi(p), \vec{s}^*)$ is in $(\mathbb{L}_2 * \mathbb{L}_3^*) * \mathbb{S}_\alpha^*$ and stronger than $(p_0^*, \vec{s}_0^*)$ and $(p_1^*, \vec{s}_1^*)$: Since $p \Vdash \vec{s}^* \in \mathbb{S}_\alpha^* \wedge \vec{s}^* \leq \vec{s}_0^*, \vec{s}_1^*$ and that depends only on $\mathbb{L}_2 * \mathbb{L}_3^*$, the same holds for $\pi(p)$.

It follows that in $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$ two conditions $\vec{s}_0^*$ and $\vec{s}_1^*$ in $\mathbb{S}_\alpha^*$ which are compatible in $j(\mathbb{S}_\alpha)$ are compatible in $\mathbb{S}_\alpha^*$. Together with (2) it follows that $\mathbb{S}_\alpha^*$ is a regular subforcing of $j(\mathbb{S}_\alpha)$.

Proof of (6): By the same proof as the proof of (5) of Lemma 4.12 it follows that j can be lifted: Let $G(j(\mathbb{S}_\alpha))$ be generic for $j(\mathbb{S}_\alpha)$ over $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$. Since $\mathbb{S}_\alpha^*$ is a regular subforcing of $j(\mathbb{S}_\alpha)$, $G(j(\mathbb{S}_\alpha))$ contains a generic filter $G(\mathbb{S}_\alpha^*)$ for $\mathbb{S}_\alpha^*$. Thus, by the Lifting Lemma (Lemma 3.9), j can be lifted to an embedding from $V[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(\mathbb{S}_\alpha^*)]$ to $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))][G(j(\mathbb{S}_\alpha))]$. Since $\mathbb{L}_2 * \mathbb{L}_3^* * \mathbb{S}_\alpha^*$ is equivalent to $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha$ we can replace $V[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(\mathbb{S}_\alpha^*)]$ by $V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{S}_\alpha)]$ to get a lifting $j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3)][G(\mathbb{S}_\alpha)] \to M[G(j(\mathbb{L}_2 * \mathbb{L}_3))][G(j(\mathbb{S}_\alpha))]$.

Proof of (7): Now we show that $\mathbb{L}_2 * \mathbb{L}_3 \Vdash$ "$\mathbb{S}_\alpha$ has the κ_2-c.c.". This follows by the same argument as (3) of Lemma 4.12:

Let $G(\mathbb{L}_2 * \mathbb{L}_3^*)$ be generic for $\mathbb{L}_2 * \mathbb{L}_3^*$ and $G(\mathbb{L}_2 * \mathbb{L}_3)$ the corresponding generic for $\mathbb{L}_2 * \mathbb{L}_3$. Let $j\colon V[G(\mathbb{L}_2 * \mathbb{L}_3)] \to M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$ be a lifting of the supercompact embedding for κ_2. In $V[G(\mathbb{L}_2 * \mathbb{L}_3)]$ let A^* be a maximal antichain in $\mathbb{S}_\alpha$. Since $V[G(\mathbb{L}_2 * \mathbb{L}_3^*)] = V[G(\mathbb{L}_2 * \mathbb{L}_3)]$ and $\mathbb{S}_\alpha$ in $V[G(\mathbb{L}_2 * \mathbb{L}_3)]$ is the same as $\mathbb{S}_\alpha^*$ in $V[G(\mathbb{L}_2 * \mathbb{L}_3^*)]$, we get that A^* is also a maximal antichain in $\mathbb{S}_\alpha^*$. On the other hand, also each maximal antichain in $\mathbb{S}_\alpha^*$ is a maximal antichain in $\mathbb{S}_\alpha$. By elementarity $j(A^*)$ is a maximal antichain in $j(\mathbb{S}_\alpha)$ in $M[G(j(\mathbb{L}_2 * \mathbb{L}_3))]$. Since j is the identity on $\mathbb{S}_\alpha$ it follows that $A^* = j[A^*] \subseteq j(A^*)$. Let $G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))$ be generic for $j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*)$ and assume that $M[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))] \models \vec{s} \in j(\mathbb{S}_\alpha)$. Since $|\mathbb{S}_\alpha^*| \leq \kappa_3$ and ${}^{\leq\kappa_3}M \subseteq M$ it follows that $A^* \in M[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))]$.

Claim. $M[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))] \models$ "$\exists \vec{a} \in A^*$ *which is compatible with* $\vec{s}$".

Proof. Since by (5) $\mathbb{S}_\alpha^*$ is a regular subforcing of $j(\mathbb{S}_\alpha)$ there exists a reduction map from $j(\mathbb{S}_\alpha)$ to $\mathbb{S}_\alpha^*$. So there exists $\vec{s}'$ such that $M[G(\mathbb{L}_2 * \mathbb{L}_3^*)][G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))] \models$ "$\vec{s}' \in \mathbb{S}_\alpha^*$ is a reduction of $\vec{s}$".

Let $p \in G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))$ with $p \Vdash \vec{s} \in j(\mathbb{S}_\alpha)$. The following set is dense in $j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*)$ below p:

$$\{q \in j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*) \mid q \Vdash \text{“}\exists \vec{a} \in A^* \text{ which is compatible with } \vec{s}\text{”}\}.$$

Indeed, let $p' \leq p$. So $p' \Vdash$ "$\vec{s} \in j(\mathbb{S}_\alpha)$ and there exists a reduction $\vec{s}'$ of $\vec{s}$ in $\mathbb{S}_\alpha^*$". Therefore $p' \Vdash$ "$\exists \vec{a} \in A^*$ with $\vec{a} \not\perp \vec{s}'$". So there exists a name $\vec{a}$ and $q \leq p'$ such that $q \Vdash$ "$\vec{d} \not\perp \vec{s}'$ and $\vec{d} \in A^*$". Since $q \Vdash$ "$\vec{s}'$ is a reduction of $\vec{s}$", it follows that $q \Vdash$ "$\vec{d}$ is compatible with $\vec{s}$", showing that the above set is dense. Now, since $p \in G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))$, there exists a $q \in G(j(\mathbb{L}_2 * \mathbb{L}_3)/G(\mathbb{L}_2 * \mathbb{L}_3^*))$ with $q \Vdash$ "$\exists \vec{d} \in A^*$ which is compatible with $\vec{s}$". □

Thus it follows that A^* is a maximal antichain for $j(\mathbb{S}_\alpha)$. Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. Note that $A^* \subseteq \mathbb{S}_\alpha$ and $|\mathbb{S}_\alpha| \leq \kappa_3$, so we have that $|j(A^*)| \leq \kappa_3$. Thus $|j(A^*)| < j(\kappa_2)$ and by elementarity $|A^*| < \kappa_2$. □

A variant of the next lemma has been proven in [GH20, Lemma 2.5].

Lemma 4.14. *Let $\alpha < \kappa_3$. In $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$ the forcing $\mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha) \times \mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha)$ has the κ_2-c.c..*

Proof. In $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$ let φ be a bookkeeping function such that $\varphi(\mathbb{P}_\beta)$ is a $\mathbb{P}_\beta$-name for an $\aleph_2$-Aronszajn tree (if there exists one) for every forcing $\mathbb{P}_\beta$ and $\dot{\mathbb{Q}}_\beta = \mathbb{S}_{\mathbb{P}_\beta}(\varphi(\mathbb{P}_\beta))$ is a forcing to specialize this name for a tree, and assume φ is a bookkeeping function which gives $\mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha)$ as an iteration. Let $\mathbb{S}'_\beta := \mathbb{S}_{\alpha+\beta}/G(\mathbb{S}_\alpha)$ for every $\beta < \kappa_3$. Now define a different bookkeeping function $\tilde{\varphi}$ and let $\tilde{\mathbb{S}}_\beta$ be the forcing iteration of length β, given by the bookkeeping $\tilde{\varphi}$. For $\beta < \kappa_3$ let $\tilde{\varphi}(\tilde{\mathbb{S}}_\beta) = \varphi(\mathbb{S}'_\beta)$. For $\beta = \kappa_3 + \gamma$ for some $\gamma < \kappa_3$, let $\tilde{\varphi}(\tilde{\mathbb{S}}_\beta) = \varphi(\mathbb{S}'_\gamma)$, i.e., we repeat the same iteration which was done between α and κ_3 between κ_3 and $\kappa_3 + \kappa_3$.

$\tilde{\mathbb{S}}_{\kappa_3+\kappa_3}$ has the $\aleph_2$-c.c. by Lemma 4.13, and since no new countable sets are added by $\mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha)$ in $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$ it holds true that $\mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha) \times \mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha) = \mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha) * \mathbb{S}_{\kappa_3}/G(\mathbb{S}_\alpha) = \tilde{\mathbb{S}}_{\kappa_3+\kappa_3}$. □

Lemma 4.15. *For every $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$-name $\dot{T}$ for an $\aleph_1$-tree with level α being $\{\alpha\} \times \omega$ for every $\alpha < \aleph_1$ there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}$ of $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$ with the following properties:*

(1) $\bar{\mathbb{L}} \Vdash |\bar{\mathbb{S}}| < \kappa_2$,

(2) $\bar{\mathbb{L}} \Vdash$ "$\bar{\mathbb{S}}$ *is ω-distributive*",

(3) $\bar{\mathbb{L}}$ *is a regular subforcing of* $\mathbb{L}_2 \times \{\mathbb{1}_{\mathbb{L}_3}\}$,

(4) $\mathbb{L}_2 * \mathbb{L}_3 \Vdash$ "$\bar{\mathbb{S}}$ *is a regular subforcing of* $\mathbb{S}_{\aleph_3}$ *and* $\mathbb{S}_{\aleph_3}/G(\bar{\mathbb{S}})$ *is equivalent to a σ-closed forcing*",

(5) *there exists an* $\bar{\mathbb{L}} * \bar{\mathbb{S}}$*-name* $\dot{T}'$ *such that* $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3} \Vdash \dot{T} = \dot{T}'$.

Proof. Using Corollary 4.11 and Lemma 4.13 we get the following. There exists a lifting of j to $j\colon V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}] \to M[j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3})]$ and a regular subforcing $\mathbb{L}^* * \mathbb{S}^*$ of $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3})$ of size $< j(\kappa_2)$ such that $\mathbb{S}^*$ is a regular subforcing of $j(\mathbb{S}_{\aleph_3})$, $j(\mathbb{S}_{\aleph_3})/G(\mathbb{S}^*)$ is equivalent to a σ-closed forcing in $M[j(\mathbb{L}_2 * \mathbb{L}_3)]$, $\mathbb{L}^*$ is a regular subforcing of $j(\mathbb{L}_2 * \mathbb{L}_3)$, and $\mathbb{L}^* * \mathbb{S}^*$ is equivalent to $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$. In particular $\mathbb{S}^*$ is a regular subforcing of an ω-distributive forcing, so it is ω-distributive.

Let $\dot{T}$ be an $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$-name for an $\aleph_1$-tree with level α being $\{\alpha\} \times \omega$ for every $\alpha < \aleph_1$. Let T be the evaluation of $\dot{T}$ in $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}]$. Since the critical point of j is κ_2, $T = j(T) \in M[j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3})]$. On the other hand, since $\mathbb{L}^* * \mathbb{S}^*$ is equivalent to $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$, there exists an $\mathbb{L}^* * \mathbb{S}^*$-name $\dot{T}^*$ such that $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}) \Vdash j(\dot{T}) = \dot{T}^*$. Thus we have that

- there exist regular subforcings $\mathbb{L}^*, \mathbb{S}^*$ of $j(\mathbb{L}_2 * \mathbb{L}_3)$, $j(\mathbb{S}_{\aleph_3})$ such that $\mathbb{S}^*$ is ω-distributive and $|\mathbb{S}^*| < j(\kappa_2)$,
- there exists an $\mathbb{L}^* * \mathbb{S}^*$-name $\dot{T}^*$ such that $j(\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}) \Vdash j(\dot{T}) = \dot{T}^*$, and
- $j(\mathbb{L}_2 * \mathbb{L}_3) \Vdash$ "$j(\mathbb{S}_{\aleph_3})/G(\mathbb{S}^*)$ is equivalent to a σ-closed forcing".

By elementarity of j the same holds for $\mathbb{L}_2$, $\mathbb{L}_3$ and $\mathbb{S}_{\aleph_3}$:

- there exist regular subforcings $\bar{\mathbb{L}}, \bar{\mathbb{S}}$ of $\mathbb{L}_2 * \mathbb{L}_3, \mathbb{S}_{\aleph_3}$ such that $\bar{\mathbb{S}}$ is ω-distributive and $|\bar{\mathbb{S}}| < \kappa_2$,
- there exists an $\bar{\mathbb{L}} * \bar{\mathbb{S}}$-name $\dot{T}'$ such that $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3} \Vdash \dot{T} = \dot{T}'$, and
- $\mathbb{L}_2 * \mathbb{L}_3 \Vdash$ "$\mathbb{S}_{\aleph_3}/G(\bar{\mathbb{S}})$ is equivalent to a σ-closed forcing". □

4.3 The final model

Theorem 4.16. *It follows from the consistency of a supercompact cardinal and an inaccessible cardinal above that it is consistent that all $\aleph_2$-Aronszajn trees are special, there are such, and there is no $\aleph_1$-Kurepa tree and no $\aleph_2$-Kurepa tree.*

Proof. Let $\kappa_2 < \kappa_3$ with κ_2 supercompact and κ_3 inaccessible. The model we use for the consistency is the extension by $\mathbb{P}_{\kappa_3}^{\aleph_2} = \mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\kappa_3}$. It has already been argued in Section 4.1 that in $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\kappa_3}]$ all $\aleph_2$-Aronszajn trees are special.

Next we show that there are no $\aleph_1$-Kurepa trees: Let $\dot{T}$ be an $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$-name for an $\aleph_1$-tree with level α equal to $\{\alpha\} \times \omega$ for every $\alpha < \aleph_1$. By Lemma 4.15 there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}$ of $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$, and an $\bar{\mathbb{L}} * \bar{\mathbb{S}}$-name $\dot{T}'$ such that $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3} \Vdash \dot{T} = \dot{T}'$. Furthermore $\bar{\mathbb{S}}$ is ω-distributive, $|\bar{\mathbb{S}}| < \kappa_2$ and $\bar{\mathbb{L}}$ is a regular subforcing of $\mathbb{L}_2 \times \{\mathbb{1}_{\mathbb{L}_3}\}$, $\bar{\mathbb{S}}$ is a regular subforcing of $\mathbb{S}_{\aleph_3}$ and $\mathbb{S}_{\aleph_3}/G(\bar{\mathbb{S}})$ is equivalent to a σ-closed forcing. Note that as $\bar{\mathbb{L}}$ is a regular subforcing of

$\mathbb{L}_2 \times \{\mathbb{1}_{\mathbb{L}_3}\}$, $\dot{T}'$ can also be regarded as an $\mathbb{L}_2 * \bar{\mathbb{S}}$-name and $\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}$ is equivalent to $\mathbb{L}_2 * \bar{\mathbb{S}} * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}/G(\bar{\mathbb{S}})$.

By Proposition 2.10 there exists no $\aleph_1$-Kurepa tree in $V[\mathbb{L}_2 * \bar{\mathbb{S}}]$. So $V[\mathbb{L}_2 * \bar{\mathbb{S}}] \models \|[\dot{T}']\| < \aleph_2$. Since $\mathbb{L}_3 * \mathbb{S}_{\aleph_3}/G(\bar{\mathbb{S}})$ is equivalent to a σ-closed forcing, by Lemma 2.8 it does not add cofinal branches to $\dot{T}'$ and thus "$\dot{T}'$ is not an $\aleph_1$-Kurepa tree and $\dot{T} = \dot{T}'$" holds true in $V[\mathbb{L}_2 * \bar{\mathbb{S}} * \mathbb{L}_3 * \mathbb{S}_{\aleph_3}/G(\bar{\mathbb{S}})]$.

Now we show that there are no $\aleph_2$-Kurepa trees: We work in $V[\mathbb{L}_2 * \mathbb{L}_3]$. Let $\dot{T}$ be an $\mathbb{S}_{\aleph_3}$-name for an $\aleph_2$-tree. Since $\mathbb{S}_{\aleph_3}$ has the $\aleph_3$-c.c. (indeed the $\aleph_2$-c.c. by Lemma 4.13), we can assume that $|\dot{T}| = \aleph_2$, hence there exists $\alpha < \aleph_3$ such that $\dot{T}$ is an $\mathbb{S}_\alpha$-name. So $T \in V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$. Note that $\mathbb{S}_\alpha$ is a forcing iteration of length $< \aleph_3$ of forcings of size $\leq \aleph_2$, so $|\mathbb{S}_\alpha| = \aleph_2$ and $\mathbb{S}_\alpha$ has the $\aleph_2$-c.c., hence by Proposition 2.10 there exists no $\aleph_2$-Kurepa tree in $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha]$. Therefore $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha] \models \|[T]\| < \aleph_3$. By Lemma 4.14 $(\mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha)) \times (\mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha))$ has the $\aleph_2$-c.c., so by Lemma 3.13 $\mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha)$ does not add cofinal branches to T and thus $V[\mathbb{L}_2 * \mathbb{L}_3 * \mathbb{S}_\alpha * \mathbb{S}_{\aleph_3}/G(\mathbb{S}_\alpha)] \models$ "T is not an $\aleph_2$-Kurepa tree".

After forcing with $\mathbb{L}_2 * \mathbb{L}_3$, CH holds, and since $\mathbb{S}_{\aleph_3}$ is σ-closed, CH holds in the final model, which implies the existence of a (special) $\aleph_2$-Aronszajn tree (see Proposition 2.13). □

Chapter 5

Trees for all $\aleph_n$

Now let us continue with the proof of the main result. Some of the proofs in this chapter are generalizations of proofs from the previous chapter.

From the existence of ω many supercompact cardinals, we prove that it is consistent that for all $0 < n < \omega$, all $\aleph_n$-Aronszajn trees are special, there are such, and there are no $\aleph_n$-Kurepa trees.

5.1 Definition of the forcing

Let $\langle \kappa_n \mid 1 < n < \omega \rangle$ be an increasing sequence of Laver indestructible supercompact cardinals; for simplicity of notation let $\kappa_0 = \aleph_0$ and $\kappa_1 = \aleph_1$. Let $\delta := (\sup_{n\in\omega} \kappa_n)^{++}$ and assume $2^{((\sup_{n\in\omega} \kappa_n)^+)} = \delta$.

For every $1 < n < \omega$ let $j_n \colon V \to M$ be a supercompact embedding for κ_n with $j_n(\kappa_n) > \delta$ and ${}^{\leq\delta}M \subseteq M$. We will often write j instead of j_n if it is clear from context which n is meant.

We define a forcing iteration of length δ to specialize all $\aleph_n$-Aronszajn trees as follows. We start with an iteration of Lévy collapses of all the supercompact cardinals. Inductively define for every $n \geq 2$ a forcing $\mathbb{L}_n$: let

$$\mathbb{L}_n := \mathrm{Col}(\kappa_{n-1}, {<}\kappa_n)^{V[\mathbb{L}_2 * \dot{\mathbb{L}}_3 * \cdots * \dot{\mathbb{L}}_{n-1}]}.$$

We will use the following notation. Let $\mathbb{L}_\omega := \mathbb{L}_2 * \dot{\mathbb{L}}_3 * \dot{\mathbb{L}}_4 * \dots$ with countable support, let $\dot{\mathbb{L}}_{>n} := \dot{\mathbb{L}}_{n+1} * \dot{\mathbb{L}}_{n+2} * \dot{\mathbb{L}}_{n+3} * \dots$ with countable support, and let $\mathbb{L}_{\leq n} := \mathbb{L}_2 * \dot{\mathbb{L}}_3 * \cdots * \dot{\mathbb{L}}_n$ and $\mathbb{L}_{<n} := \mathbb{L}_2 * \dot{\mathbb{L}}_3 * \cdots * \dot{\mathbb{L}}_{n-1}$.

To specialize the $\aleph_1$-Aronszajn trees, we use the classical forcing from [BMR70]:

Definition 5.1. Let T be an $\aleph_1$-Aronszajn tree. Let $\mathbb{S}(T)$ be the following forcing: Conditions are functions f satisfying

1. $\mathrm{dom}(f) \subseteq T$ is finite,
2. $\mathrm{range}(f) \subseteq \omega$,
3. if $s, t \in \mathrm{dom}(f)$ and $s <_T t$, then $f(s) \neq f(t)$.

The order is given by $g \leq f$ if $g \supseteq f$.

Lemma 5.2. *If T is an $\aleph_1$-Aronszajn tree, then $\mathbb{S}(T)$ has the c.c.c..*

Proof. See [BMR70] (or [Jec03, Lemma 16.19]). □

Following [GH20] we combine the specializing forcings for all the $\aleph_n$ as follows:

Definition 5.3 (Specializing names)**.** Assume that $\mathbb{P}$ is a forcing with $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{T}$ is an $\aleph_n$-Aronszajn tree with $\dot{T}_\xi = \{\xi\} \times \aleph_{n-1}$". Let $\mathbb{S}_{\mathbb{P}}(\dot{T})$ be the following forcing: Conditions are partial functions f from $\dot{T}$ to $[\aleph_{n-1}]^{<\aleph_{n-1}}$ such that $|\mathrm{dom}(f)| < \aleph_{n-1}$ and, for $s \neq t \in \mathrm{dom}(f)$, if $f(s) \cap f(t) \neq \emptyset$, then $\mathbb{1}_{\mathbb{P}} \Vdash$ "s is incomparable to t in $\dot{T}$". The order is given by $g \leq f$ if $g \supseteq f$.

Remark 5.4. Note that $\not\perp_{\mathbb{S}_{\mathbb{P}}(\dot{T})}$ is definable in V, more precisely, for $p \in \mathbb{P}$ and $f, g \in V$, $p \Vdash \check{f} \not\perp_{\mathbb{S}_{\mathbb{P}}(\dot{T})} \check{g}$ if and only if

- for all $s \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$, $f(s) = g(s)$, and
- for all $s \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)$ and $t \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)$, if $f(s) \cap g(t) \neq \emptyset$ then $\mathbb{1}_{\mathbb{P}} \Vdash$ "s is incomparable to t in $\dot{T}$".

Note that since $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{T}$ is an $\aleph_n$-tree", it follows that $|\mathbb{S}_{\mathbb{P}}(\dot{T})| \leq \aleph_n \cdot 2^{\aleph_{n-2}}$.

Let us now define the iteration of length δ. We start the iteration with the forcing $\mathbb{L}_\omega$ which collapses κ_n to $\aleph_n$ for every $n \geq 2$. Subsequently we use an iteration which specializes an Aronszajn tree (or a name for an Aronszajn tree) in each step. For $\aleph_1$ we use the usual forcing to specialize $\aleph_1$-Aronszajn trees (see Definition 5.1), and for $n > 1$ we use forcings to specialize names (see Definition 5.3). This is necessary, because we will look at a reordering of the iteration, where, for $n > 1$, $\aleph_n$-Aronszajn trees are considered for specialization before their names have been evaluated. To make sure that in the end all Aronszajn trees have been specialized, we consider each n cofinally often in the iteration.

- Let $\{A_n \mid 0 < n < \omega\}$ be a partition of δ such that every A_n is cofinal in δ. Since $2^{((\sup_{n\in\omega}\kappa_n)^+)} = \delta$ in V, $2^{\aleph_{\omega+1}} = \delta$ in $V[\mathbb{L}_\omega]$, hence $|H(\delta)| = \aleph_{\omega+2}$ in $V[\mathbb{L}_\omega]$. For each $n \in \omega$ enumerate $H(\delta)$ as $\langle x^n_\alpha \mid \alpha \in A_n\rangle$ such that each element of $H(\delta)$ is equal to x^n_α for cofinally many $\alpha \in A_n$.

- For each $0 < n < \omega$ let $A_{\geq n} := \bigcup_{n \leq m < \omega} A_m$.
- For each $\alpha < \delta$ and $0 < n < \omega$ we will define $\dot{\mathbb{Q}}_\alpha$, $\mathbb{S}_\alpha$, $\mathbb{S}_\alpha^{\geq n}$ and $\mathbb{S}_\alpha^{<n}$ (in $V[\mathbb{L}_\omega]$) such that the following hold:
  (1) For $\alpha \in A_n$, $\dot{\mathbb{Q}}_\alpha$ is an $\mathbb{S}_\alpha^{\geq n}$-name for a $<\aleph_{n-1}$-closed forcing,
  (2) $\mathbb{S}_\alpha$ is forcing equivalent to $\mathbb{S}_\alpha^{\geq n} * \mathbb{S}_\alpha^{<n}$ for each $n < \omega$,
  (3) $\mathbb{S}_\alpha^{\geq n}$ is $<\aleph_{n-1}$-closed.
- We will define $\dot{\mathbb{Q}}_\alpha$ for every $\alpha < \delta$. The forcing iteration $\mathbb{P}_\delta$ is a mixed support iteration starting with $\mathbb{L}_\omega$ (i.e., $\mathbb{P}_0 = \mathbb{L}_\omega$) and then followed by the $\dot{\mathbb{Q}}_\alpha$'s. The support of the iteration is as follows: X is an allowed support if $|A_n \cap X| < \kappa_{n-1}$ for each $0 < n < \omega$.
- Let n be such that $0 \in A_n$. If x_0^n is an $\mathbb{L}_\omega$-name for an $\aleph_n$-Aronszajn tree, let $\dot{T}_0^n := x_0^n$. If $n = 1$, let T_0^1 be the evaluation of $\dot{T}_0^n$ in $V[\mathbb{L}_\omega]$ and let $\dot{\mathbb{Q}}_0 := \mathbb{S}(T_0^1)$. If $n > 1$, let $\dot{\mathbb{Q}}_0 := \mathbb{S}_{\mathbb{L}_\omega}(\dot{T}_0^n)$. If x_0^n is not an $\mathbb{L}_\omega$-name for an $\aleph_n$-Aronszajn tree, let $\dot{\mathbb{Q}}_0$ be a name for the trivial forcing.
- Next assume $\dot{\mathbb{Q}}_\beta$ has been defined for all $\beta < \alpha$.
  - Let $\dot{\mathbb{S}}_\alpha^{>n}$ be an $\mathbb{L}_\omega$-name for the iteration of all the $\dot{\mathbb{Q}}_\beta$ with $\beta < \alpha$ and $\beta \in \bigcup_{k>n} A_k$ (i.e., $\dot{\mathbb{Q}}_\beta$ is a forcing to specialize a name for an $\aleph_k$-Aronszajn tree for some $k > n$) with mixed support such that X is a possible support if $|A_k \cap X| < \kappa_{k-1}$ for each $n < k < \omega$.
  - Analogously define $\dot{\mathbb{S}}_\alpha^{\geq n}$.
  - For each $n > 1$, let $\dot{\mathbb{S}}_\alpha^n$ be an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha^{>n}$-name for the iteration of all the $\dot{\mathbb{Q}}_\beta$ with $\beta < \alpha$ and $\beta \in A_n$ (i.e., $\dot{\mathbb{Q}}_\beta$ is a forcing to specialize a name for an $\aleph_n$-Aronszajn tree) with $<\kappa_{n-1}$-support.
  - Let $\dot{\mathbb{S}}_\alpha^1$ be an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha^{>1}$-name for the iteration of all the $\dot{\mathbb{Q}}_\beta$ with $\beta < \alpha$ and $\beta \in A_1$ (i.e., $\dot{\mathbb{Q}}_\beta$ is a forcing to specialize an $\aleph_1$-Aronszajn tree) with finite support.
  - Let $\dot{\mathbb{S}}_\alpha^{<n}$ be an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha^{\geq n}$-name for the iteration of all the $\dot{\mathbb{Q}}_\beta$ with $\beta < \alpha$ and $\beta \in \bigcup_{k<n} A_k$ (i.e., $\dot{\mathbb{Q}}_\beta$ is a forcing to specialize a name for an $\aleph_k$-Aronszajn tree for some $k < n$) with mixed support such that X is a possible support if $|A_k \cap X| < \kappa_{k-1}$ for each $k < n$.
  - Finally, let $\dot{\mathbb{S}}_\alpha := \dot{\mathbb{S}}_\alpha^{\geq 1}$. Note that $\mathbb{P}_\alpha = \mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha$.
- Now we give the definition for $\dot{\mathbb{Q}}_\alpha$. Let n be such that $\alpha \in A_n$. If x_α^n is not an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha$-name for an $\aleph_n$-Aronszajn tree, let $\dot{\mathbb{Q}}_\alpha$ be an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha$-name for the trivial forcing.

– If $n > 1$ and x^n_α is an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha$-name for an $\aleph_n$-Aronszajn tree, let $\dot{T}^n_\alpha := x^n_\alpha$, take in $V[\mathbb{L}_\omega * \dot{\mathbb{S}}^{\geq n}_\alpha]$ the forcing $\mathbb{S}_{\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha}(\dot{T}^n_\alpha)$ to specialize the name $\dot{T}^n_\alpha$, i.e., partial functions in $V[\mathbb{L}_\omega * \dot{\mathbb{S}}^{\geq n}_\alpha]$ to specialize the name $\dot{T}^n_\alpha$, and let $\dot{\mathbb{Q}}_\alpha$ be an $\mathbb{L}_\omega * \dot{\mathbb{S}}^{\geq n}_\alpha$-name for it.

– If $n = 1$ and x^1_α is an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha$-name for an $\aleph_1$-Aronszajn tree, let T^1_α be the $\aleph_1$-Aronszajn tree in $V[\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha]$ given by x^1_α and let $\dot{\mathbb{Q}}_\alpha$ be an $\mathbb{L}_\omega * \dot{\mathbb{S}}_\alpha$-name for the forcing $\mathbb{S}(T^1_\alpha)$ to specialize T^1_α.

- In $V[\mathbb{L}_\omega]$ continue this iteration for length $\delta = \aleph_{\omega+2}$. Since each $x \in H(\delta)$ is enumerated cofinally in each A_n and each $\aleph_n$-Aronszajn tree in $V[\mathbb{P}_\delta]$ has a $\mathbb{P}_\delta$-name in $H(\delta)$, if $\dot{T}$ is an $\aleph_n$-Aronszajn tree it gets specialized in some step of the iteration.

Before we investigate how the forcings in the iteration specialize Aronszajn trees, we show that the forcings to specialize names of trees are closed.

Lemma 5.5. *Let $\mathbb{P}$ be a forcing with $\mathbb{1}_\mathbb{P} \Vdash$ "$\dot{T}$ is an $\aleph_n$-Aronszajn tree". Then $\mathbb{S}_\mathbb{P}(\dot{T})$ is $<\aleph_{n-1}$-closed with weakest lower bounds and well-met. In particular for $\lambda < \aleph_{n-1}$ and a decreasing sequence $\langle q_i \mid i < \lambda\rangle$, the union $\bigcup_{i<\lambda} q_i$ is a weakest lower bound of $\langle q_i \mid i < \lambda\rangle$. Moreover, weakest lower bounds of decided conditions in $\mathbb{P} * \mathbb{S}_\mathbb{P}(\dot{T})$ are decided (see Lemma 3.22).*

Proof. Let $\lambda < \aleph_{n-1}$ and let $\langle q_i \mid i < \lambda\rangle$ be a decreasing sequence in $\mathbb{S}_\mathbb{P}(\dot{T})$. Let $q^* := \bigcup_{i<\lambda} q_i$. It is easy to see that $q^* \in \mathbb{S}_\mathbb{P}(\dot{T})$, $q^* \leq q_i$ for every $i < \lambda$ and q^* is a weakest lower bound. Now let $q \not\perp q'$ in $\mathbb{S}_\mathbb{P}(\dot{T})$. Then clearly $q \cup q'$ is a weakest lower bound of q and q'.

It remains to show that weakest lower bounds of decided conditions are decided: Let $p \in \mathbb{P}$ and $q, q' \in V$ such that $p \Vdash \check{q}, \check{q}' \in \mathbb{S}_\mathbb{P}(\dot{T})$ and $(p, \check{q}) \not\perp (p, \check{q}')$. So there exists $p^* \leq p$ such that $p^* \Vdash \check{q} \not\perp \check{q}'$. Let $q^* := q \cup q'$. It follows that $q^* \in V$ and $p^* \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\check{q}$ and $\check{q}'$". So $|q^*| < \aleph_{n-1}$ and for s, t with $q^*(s) \cap q^*(t) \neq \emptyset$ we know that $\mathbb{1}_\mathbb{P} \Vdash$ "s is incomparable to t in $\dot{T}$". Since this does not depend on p^* it follows that $p \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\check{q}$ and $\check{q}'$" and p decides $\check{q}^*$, since $\not\perp_{\mathbb{S}_\mathbb{P}(\dot{T})}$ is definable in V.

Similarly, for $p \in \mathbb{P}$ and $q_i \in V$ for each $i < \lambda$ such that $p \Vdash$ "$\langle \check{q}_i \mid i < \lambda\rangle$ is decreasing in $\mathbb{S}_\mathbb{P}(\dot{T})$" let $q^* := \bigcup_{i\in\lambda} q_i$. Note that $|q^*| < \aleph_{n-1}$ since $\lambda < \aleph_{n-1}$. For $s, t \in \mathrm{dom}(q^*)$ with $q^*(s) \cap q^*(t) \neq \emptyset$ we know that there exists $i < \lambda$ such that $s, t \in \mathrm{dom}(q_i)$, therefore $\mathbb{1}_\mathbb{P} \Vdash$ "s is incomparable to t". So $q^* \in \mathbb{S}_\mathbb{P}(\dot{T})$ and $p \Vdash$ "$\check{q}^*$ is the weakest lower bound of $\langle \check{q}_i \mid i < \lambda\rangle$" and p decides $\check{q}^*$, since $\not\perp_{\mathbb{S}_\mathbb{P}(\dot{T})}$ is definable in V. □

Next, we show that the iterations $\mathbb{S}^{>k}_\alpha$ are not only $<\aleph_k$-closed, but even equivalent to nicely $<\aleph_k$-closed forcings.

Lemma 5.6. *Let $\alpha \leq \delta$ be a limit ordinal and let $\mathbb{P}$ be a forcing with $V[G(\mathbb{P})] \models \kappa_n = \aleph_n$ for each $n > k$. Then in $V[G(\mathbb{P})]$ the set of decisive conditions in $\mathbb{S}_\alpha^{>k}$ is dense and nicely $<\!\aleph_k$-closed.*

Proof. First we want to use Lemma 3.21. Note that by Lemma 5.5 for each iterand weakest lower bounds of decided conditions are decided for sequences of length $< \aleph_k$ and for pairs of conditions, hence the last requirement of Lemma 3.21 holds.

Now we argue that also the other requirements of Lemma 3.21 hold. $\mathbb{S}_\alpha^{>k}$ is an iteration where each iterand is $<\!\aleph_k$-closed, therefore also the iteration with the given mixed support (which is a combination of $<\!\aleph_n$-supports for all $n > k$) is $<\!\aleph_k$-closed, which shows the first requirement. The second requirement holds by the definition of the forcing $\mathbb{S}_\mathbb{P}(\dot{T})$. For the third requirement let $R(q', q)$ be defined as explained in Remark 5.4, i.e., for $p \in \mathbb{P}$ and $q, q' \in V$ such that $p \Vdash \check{q}, \check{q}' \in \mathbb{S}_\mathbb{P}(\dot{T})$, $p \Vdash \check{q}' \not\perp \check{q}$ if and only if $R(q', q)$ holds in V. So we can apply Lemma 3.21 and get that the set of decisive conditions is dense.

The proof of the nice $<\!\aleph_k$-closure is almost the same as the proof of Lemma 3.22, the only difference lies in the limit step for α with $\sup(\{\kappa_n \mid n \in \omega\}) > \operatorname{cf}(\alpha) > \kappa_k$. We give the adapted proof for these limits here.

Assume by induction that the dense sets D_β of decisive conditions in $\mathbb{S}_\beta^{>k}$ are closed for sequences of length λ with weakest lower bounds and well-met.

Let $\langle p_i \mid i < \lambda \rangle$ be a decreasing sequence in D_α. Let ℓ be such that $\kappa_\ell > \operatorname{cf}(\alpha) \geq \kappa_{\ell-1}$. Recall that $\mathbb{S}_\alpha^{>k} = \mathbb{S}_\alpha^{\geq \ell} * \mathbb{S}_\alpha^{\ell-1} * \mathbb{S}_\alpha^{\ell-2} * \cdots * \mathbb{S}_\alpha^{k+1}$. Let $\mathbb{S}_\alpha^{<\ell,>k} := \mathbb{S}_\alpha^{\ell-1} * \mathbb{S}_\alpha^{\ell-2} * \cdots * \mathbb{S}_\alpha^{k+1}$, so $\mathbb{S}_\alpha^{>k} = \mathbb{S}_\alpha^{\geq \ell} * \mathbb{S}_\alpha^{<\ell,>k}$ and $\langle p_i \mid i < \lambda \rangle = \langle (p_i', \dot{q}_i') \mid i < \lambda \rangle$ with $(p_i', \dot{q}_i') \in D_\alpha^{\geq \ell} * D_\alpha^{<\ell,>k}$, the iteration of the dense sets of decisive conditions in $\mathbb{S}_\alpha^{\geq \ell}$ and $\mathbb{S}_\alpha^{<\ell,>k}$. Considering the supports, $\mathbb{S}_\alpha^{\geq \ell}$ is an inverse limit, and by the same proof as in the proof of Lemma 3.22 for the case $\operatorname{cf}(\alpha) < \kappa$, we get a weakest lower bound p^* of $\langle p_i' \mid i < \lambda \rangle$. So p^* forces that $\langle \dot{q}_i' \mid i < \lambda \rangle$ is a decreasing sequence in $D_\alpha^{<\ell,>k}$, which is a bounded support limit. By the same proof as the case $\operatorname{cf}(\alpha) \geq \kappa$ in the proof of Lemma 3.22, we get $\dot{q}^*$ which is forced by p^* to be a weakest lower bound of $\langle \dot{q}_i' \mid i < \lambda \rangle$. It is straightforward to check that $(p^*, \dot{q}^*)$ is a weakest lower bound of $\langle (p_i', \dot{q}_i') \mid i < \lambda \rangle$.

The fact that D_α is well-met, follows by the same adaptation of the respective part of the proof of Lemma 3.22. Therefore, by Lemma 3.19 we get the nice $<\!\aleph_k$-closure. □

To complete the definition we have to prove that $\mathbb{S}_\alpha$ is forcing equivalent to $\mathbb{S}_\alpha^{\geq n} * \mathbb{S}_\alpha^{<n}$ for each $\alpha \leq \delta$ and each $n < \omega$.

Lemma 5.7. *Let $\alpha \leq \delta$ and $n < \omega$. Then $\mathbb{S}_\alpha$ is forcing equivalent to $\mathbb{S}_\alpha^{\geq n} * \mathbb{S}_\alpha^{<n}$.*

Proof. The proof is by induction on α. We show, for each $\alpha \leq \delta$, that the following set is dense in $\mathbb{S}_\alpha$:

$$D(\mathbb{S}_\alpha) := \{p \in \mathbb{S}_\alpha \mid \forall n \in \omega \ \forall \beta \in \alpha \cap A_n \ p(\beta) \text{ is an } \mathbb{S}_\beta^{\geq n}\text{-name}\}.$$

Then it follows that for each $n \in \omega$, $\mathbb{S}_\alpha$ is forcing equivalent to $\mathbb{S}_\alpha^{\geq n} * \mathbb{S}_\alpha^{<n}$, since for conditions $p \in D(\mathbb{S}_\alpha)$, $p \restriction A_{\geq n}$ is a condition in $\mathbb{S}_\alpha^{\geq n}$, and $\mathbb{S}_\alpha^{\geq n}$ does not change $\mathbb{S}_\alpha^{<n}$ due to its closure. In particular, for each $p \in \mathbb{S}_\alpha$ there exists $p' \leq p$ such that $(p' \restriction A_{\geq n}, p' \restriction A_{<n}) \in \mathbb{S}_\alpha^{\geq n} * \mathbb{S}_\alpha^{<n}$ for each n.

$\alpha = 0$: $\mathbb{S}_\alpha$ is the trivial forcing, and there is nothing to show.

Successor step: If $(p, \dot{q}) \in \mathbb{S}_\alpha * \dot{\mathbb{Q}}_\alpha$ and $\alpha \in A_n$, then there exists $p' \leq p$ such that $p' \Vdash \dot{q} = \dot{q}^*$ for some $\mathbb{S}_\alpha^{\geq n}$-name $\dot{q}^*$, because $p \Vdash \dot{q} \in V[\mathbb{L}_\omega * \mathbb{S}_\alpha^{\geq n}]$. Then by induction p' can be extended further to a $p^* \in D(\mathbb{S}_\alpha)$, and then $(p^*, \dot{q}^*)$ extends $(p, \dot{q})$ and $(p^*, \dot{q}^*) \in D(\mathbb{S}_{\alpha+1})$.

Limit step: Let α be a limit ordinal and assume for all $\beta < \alpha$ that $D(\mathbb{S}_\beta)$ is dense in $\mathbb{S}_\beta$. We proceed with a case distinction.

1. $\mathrm{cf}(\alpha) \geq \aleph_{\omega+1}$: The supports in $\mathbb{S}_\alpha$ are all bounded, therefore it follows directly by induction that $D(\mathbb{S}_\alpha)$ is dense.

2. $\mathrm{cf}(\alpha) < \aleph_{\omega+1}$, i.e., $\mathrm{cf}(\alpha) = \aleph_m$ for some $m \in \omega$:

   By definition, $\mathrm{supp}(p) \cap A_{<m+2}$ is bounded in α by some $\alpha' < \alpha$. Let $\langle \alpha_i \mid i < \aleph_m \rangle$ be increasing cofinal in α with $\alpha_0 = \alpha'$. By inductive hypothesis, we can extend p to $p' = p'_0$ so that $p' \restriction \alpha_0 \in D(\mathbb{S}_{\alpha_0})$ and $p' \restriction [\alpha_0, \alpha) = p \restriction [\alpha_0, \alpha)$. Then extend p'_0 to p'_1 so that $p'_1 \restriction \alpha_1 \in D(\mathbb{S}_{\alpha_1})$. We may assume that p'_1 agrees with p'_0 on $A_{<m+2}$ as replacing p'_1 on $A_{<m+2}$ by p'_0 on $A_{<m+2}$ yields a condition whose restriction to α_1 is in $D(\mathbb{S}_{\alpha_1})$. Continue building a descending sequence $\langle p'_i \mid i < \aleph_m \rangle$ such that $p'_i \restriction \alpha_i \in D(\mathbb{S}_{\alpha_i})$ and $p'_i \restriction \alpha_i$ agrees with p'_0 on $A_{<m+2}$ for each i. Note that $\mathrm{supp}(p'_i) \cap A_{<m+2} = \mathrm{supp}(p) \cap A_{<m+2}$ for each i. Since $\mathbb{S}_\alpha^{\geq m+2}$ is $<\aleph_{m+1}$-closed with weakest lower bounds and all the p'_i's agree on $A_{<m+2}$, it is straightforward to check that taking the weakest lower bound in each coordinate of the p'_i's gives a condition in $D(\mathbb{S}_\alpha)$ which extends p. □

Now we show that the generic filter of a forcing to specialize a name for an Aronszajn tree yields a specializing function in our setting.

Lemma 5.8. *Let $\dot{T}$ be a $\mathbb{P}$-name for an $\aleph_n$-Aronszajn tree. For every $(\xi,\beta) \in \aleph_n \times \aleph_{n-1}$ the set $\{g \in \mathbb{S}_{\mathbb{P}}(\dot{T}) \mid (\xi,\beta) \in dom(g)\}$ is dense in $\mathbb{S}_{\mathbb{P}}(\dot{T})$.*

Proof. Let $f \in \mathbb{S}_{\mathbb{P}}(\dot{T})$, $\xi \in \aleph_n$ and $\beta \in \aleph_{n-1}$. Since $|\mathrm{dom}(f)| < \aleph_{n-1}$, and $|f(s)| < \aleph_{n-1}$ for every $s \in \mathrm{dom}(f)$, there exists $i \in \aleph_{n-1} \setminus \bigcup \mathrm{rng}(f)$. If $(\xi,\beta) \notin \mathrm{dom}(f)$, let $g := f \cup \{((\xi,\beta), \{i\})\}$. So $g \in \mathbb{S}_{\mathbb{P}}(\dot{T})$, $g \leq f$ and $(\xi,\beta) \in \mathrm{dom}(g)$. □

Lemma 5.9. *Let $\mathbb{P} * \dot{\mathbb{Q}}$ be a forcing with $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash$ "$\dot{T}$ is an $\aleph_n$-Aronszajn tree". Then $\mathbb{1}_{\dot{\mathbb{Q}}} \Vdash$ "$\dot{T}$ is special" over $V[\mathbb{P} * \mathbb{S}_{\mathbb{P}*\dot{\mathbb{Q}}}(\dot{T})]$.*

Proof. In $V[\mathbb{P}]$ let G be a generic filter for $\mathbb{S}_{\mathbb{P}*\dot{\mathbb{Q}}}(\dot{T})$. Let $F := \bigcup\{f \in \mathbb{S}_{\mathbb{P}*\dot{\mathbb{Q}}}(\dot{T}) \mid f \in G\}$. It follows from the above lemma that $\mathrm{dom}(F) = \aleph_n \times \aleph_{n-1}$. For distinct $s, t \in \aleph_n \times \aleph_{n-1}$ with $F(s) \cap F(t) \neq \emptyset$ we have that $\mathbb{1}_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash$ "s and t are incomparable in $\dot{T}$", hence $F(s) \cap F(t) = \emptyset$ if there exists a generic extension for $\mathbb{P} * \dot{\mathbb{Q}}$ in which $s <_T t$. In $V[\mathbb{P} * \mathbb{S}_{\mathbb{P}*\dot{\mathbb{Q}}}(\dot{T})]$ let $F'(s) := \min(F(s))$. So $F' \colon \aleph_n \times \aleph_{n-1} \to \aleph_{n-1}$ with $F'(s) \neq F'(t)$ if there exists a generic extension for $\dot{\mathbb{Q}}$ in which $s <_T t$. It follows that $\mathbb{1}_{\dot{\mathbb{Q}}} \Vdash$ "F' is a specializing function of $\dot{T}$". □

For the iteration, note that $\mathbb{L}_\omega \Vdash 2^{\aleph_{n-2}} \leq \aleph_{n-1}$ and $\mathbb{S}^{\geq n}_\alpha$ is $<\aleph_{n-1}$-closed. Therefore $\mathbb{L}_\omega * \mathbb{S}^{\geq n}_\alpha \Vdash 2^{\aleph_{n-2}} \leq \aleph_{n-1}$.

Since $\mathbb{S}^n_\delta$ is an iteration of length δ with $<\kappa_{n-1}$-support, it follows that in the final model $2^{\aleph_{n-1}} = \aleph_{\omega+2} = \delta$; in particular, $\aleph_\omega$ is not a strong limit.

We will show that all $\aleph_n$ are preserved by the forcing iteration after $\mathbb{L}_\omega$ and can thus, using Lemma 5.9, conclude that in the extension by $\mathbb{P}_\delta$, all $\aleph_n$-Aronszajn trees will be special for all $n > 0$.

5.2 Chain condition and regular subforcings

Lemma 5.10. *Let $\alpha \leq \delta$. In $V[\mathbb{L}_{<k}]$ let $j_k \colon V[\mathbb{L}_{<k}] \to M[j_k(\mathbb{L}_{<k})]$ be a supercompact embedding for κ_k such that $j_k(\kappa_k) > |\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha|$.*

There exists a regular subforcing $\mathbb{P}^*$ of $j_k(\mathbb{L}_k)/G(\mathbb{L}_k)$ which is forcing equivalent to $\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha$ with $|\mathbb{P}^*| < j_k(\kappa_k)$ such that $j_k(\mathbb{L}_k)/\mathbb{P}^*$ is equivalent to $j_k(\mathbb{L}_k)$ and $(j_k(\mathbb{L}_k)/G(\mathbb{L}_k))/\mathbb{P}^*$ is $<\kappa_{k-1}$-closed.

Proof. In $V[\mathbb{L}_{<k}]$ let $G(\mathbb{L}_k)$ be generic for $\mathbb{L}_k$. By applying Theorem 3.10 we get the following: $V[\mathbb{L}_{<k}][G(\mathbb{L}_k)] \models$ "$\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha$ is $<\kappa_{k-1}$-closed and $j_k(\kappa_k) > |\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha|$", thus there exists a regular embedding $\iota \colon \mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha \to \mathrm{Col}(\kappa_{k-1}, <j_k(\kappa_k))$ such that if $G(\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha)$ is a generic filter for $\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha$ over $V[\mathbb{L}_{<k}][G(\mathbb{L}_k)]$, then the collapse $\mathrm{Col}(\kappa_{k-1}, <j_k(\kappa_k))$ is equivalent to the quotient $\mathrm{Col}(\kappa_{k-1}, <j_k(\kappa_k))/\iota[G(\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha)]$, which can easily seen to be equal to $j_k(\mathbb{L}_k)/\iota[G(\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha)]$. It follows that in $V[\mathbb{L}_{<k}]$ there exists a regular embedding $\iota \colon \mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha \to j_k(\mathbb{L}_k)/G(\mathbb{L}_k)$ such that $j_k(\mathbb{L}_k)/G(\mathbb{L}_k)$ is equivalent to $(j_k(\mathbb{L}_k)/G(\mathbb{L}_k))/\iota[G(\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha)]$. Again in $V[\mathbb{L}_{<k}][G(\mathbb{L}_k)]$, note that $j_k(\mathbb{L}_k)/G(\mathbb{L}_k) = \mathrm{Col}(\kappa_{k-1}, <j_k(\kappa_k))$ is nicely $<\kappa_{k-1}$-closed by Lemma 3.15. So by Lemma 3.16 it follows that $(j_k(\mathbb{L}_k)/G(\mathbb{L}_k))/\iota[G(\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha)]$ is $<\kappa_{k-1}$-closed. $\mathbb{P}^* := \iota[\mathbb{L}_{>k} * \mathbb{S}^{>k}_\alpha]$ is the forcing we are looking for. □

Corollary 5.11. *In $V[\mathbb{L}_{<k}]$, there exists a reduction map $\pi \colon j_k(\mathbb{L}_{\geq k} * \mathbb{S}^{>k}_\delta) \to \mathbb{P}^*$.*

Proof. Clearly there exists a reduction map $\pi_1 \colon j_k(\mathbb{L}_k) * j_k(\mathbb{L}_{>k} * \mathbb{S}^{>k}_\delta) \to j_k(\mathbb{L}_k)$. By Lemma 5.10, $j_k(\mathbb{L}_k)$ has $\mathbb{P}^*$ as a regular subforcing in $V[\mathbb{L}_{<k}]$, hence there exists a reduction map $\pi \colon j_k(\mathbb{L}_k) * j_k(\mathbb{L}_{>k} * \mathbb{S}^{>k}_\delta) \to \mathbb{P}^*$. □

Lemma 5.12. *Let $G(\mathbb{L}_{\leq k})$ be generic for $\mathbb{L}_{\leq k}$ and $G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})$ be generic for $\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}$. Then there exist generic filters $G(j_k(\mathbb{L}_{\leq k}))$ and $G(j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}))$ for $j_k(\mathbb{L}_{\leq k})$ and $j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})$ such that the supercompact embedding j_k can be lifted to $j_k \colon V[G(\mathbb{L}_{\leq k})][G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \to M[G(j_k(\mathbb{L}_{\leq k}))][G(j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}))]$.*

Proof. First note that since $j_k(\mathbb{L}_{<k}) = \mathbb{L}_{<k}$ the embedding can clearly be lifted to $j_k \colon V[G(\mathbb{L}_{<k})] \to M[G(j_k(\mathbb{L}_{<k}))]$. Now let $G(\mathbb{L}_k)$ be generic for $\mathbb{L}_k$ over $V[G(\mathbb{L}_{<k})]$, let $G(\mathbb{L}_{\leq k})$ be such that $V[G(\mathbb{L}_{\leq k})] = V[G(\mathbb{L}_{<k})][G(\mathbb{L}_k)]$, and let $\iota \colon \mathbb{L}_{>k} * \mathbb{S}_\delta^{>k} \to j_k(\mathbb{L}_k)/G(\mathbb{L}_k)$ be a regular embedding as in Lemma 5.10. We can choose $G(j_k(\mathbb{L}_k))$ such that $G(j_k(\mathbb{L}_k)) \cap \mathrm{range}(\iota) = \iota[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})]$, thus $\iota[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \in V[G(\mathbb{L}_{<k})][G(j_k(\mathbb{L}_k))]$ and $G(\mathbb{L}_k) \subseteq G(j_k(\mathbb{L}_k))$; that is possible because $\iota[\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}]$ is a regular subforcing of $j_k(\mathbb{L}_k)$. Since ι and $j_k \restriction \mathbb{L}_k \in V[G(\mathbb{L}_{<k})][G(j_k(\mathbb{L}_k))]$ it follows that $j_k[G(\mathbb{L}_k)] \in V[G(\mathbb{L}_{<k})][G(j_k(\mathbb{L}_k))]$. Since M is closed under subsets of size $\leq \delta$ the same holds for $M[G(j_k(\mathbb{L}_{<k}))][G(j_k(\mathbb{L}_k))]$ and therefore $j_k[G(\mathbb{L}_k)] \in M[G(j_k(\mathbb{L}_{<k}))][G(j_k(\mathbb{L}_k))]$. So the embedding can be lifted to $j_k \colon V[G(\mathbb{L}_{\leq k})] \to M[G(j_k(\mathbb{L}_{\leq k}))]$ and $\iota[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \subseteq M[G(j_k(\mathbb{L}_{<k}))][G(j_k(\mathbb{L}_k))]$. Since $\iota, j_k \restriction \mathbb{L}_{>k} * \mathbb{S}_\delta^{>k} \in V[G(\mathbb{L}_{<k})][G(j_k(\mathbb{L}_k))]$ it follows that $j_k[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \in V[G(\mathbb{L}_{<k})][G(j_k(\mathbb{L}_k))]$. Since M is closed under subsets of size $\leq \delta$, i.e., ${}^{\leq\delta}M \subseteq M$, the same holds for $M[G(j_k(\mathbb{L}_{<k}))][G(j_k(\mathbb{L}_k))]$ and therefore $j_k[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \in M[G(j_k(\mathbb{L}_{<k}))][G(j_k(\mathbb{L}_k))]$.

$j_k[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \subseteq j_k[\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}] \subseteq j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})$, $j_k[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})]$ is a directed set of size $< j_k(\kappa_k)$ and $j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})$ is $<j_k(\kappa_k)$-directed closed, therefore there exists a master condition $p \in j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})$ for $j_k[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})]$. Let $G(j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}))$ be generic for $j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})$ with $p \in G(j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}))$. It follows that $j_k[G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \subseteq G(j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}))$.

Now we can use the Lifting Lemma (Lemma 3.9) to lift j_k to an embedding $j_k \colon V[G(\mathbb{L}_{\leq k})][G(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k})] \to M[G(j_k(\mathbb{L}_{\leq k}))][G(j_k(\mathbb{L}_{>k} * \mathbb{S}_\delta^{>k}))]$. □

One of the main technical parts of the proof is to show that the forcing iteration has a good chain condition. The main work lies in the following lemma, which deals with the successor step of the iteration. Note that $\mathbb{L}_k * \mathbb{L}_{>k} * \mathbb{S}_\delta^{\geq k+1}$ with $\mathbb{L} = \mathbb{L}_k * \mathbb{L}_{>k}$, $\mathbb{P}_0 = \mathbb{S}_\delta^{\geq k+1}$ and $\mathbb{L}_k * \mathbb{P}^*$ (with $\mathbb{P}^*$ from Lemma 5.10) as a subforcing of $j_k(\mathbb{L}_k * \mathbb{L}_{>k} * \mathbb{S}_\delta^{\geq k+1})$ fulfills the requirements of the following lemma.

Lemma 5.13. *Assume $\mathbb{P} = \mathbb{L} * \mathbb{P}_0$ is a forcing with $V[G(\mathbb{P})] \models \kappa_k = \aleph_k \wedge 2^{\aleph_{k-1}} \leq \kappa_k$ and $\mathbb{P}^* = \mathbb{L}^* * \mathbb{P}_0^*$ is forcing equivalent to $\mathbb{P}$, and $\mathbb{P}^*$ is a regular subforcing of $j_k(\mathbb{P})$ and the sets of decisive conditions in $\mathbb{P}_0$ and in $\mathbb{P}_0^*$ are dense and nicely $<\kappa_{k-1}$-closed, and $\mathbb{P}_0$ is forcing equivalent to $\mathbb{P}_0^*$. Further assume $\mathbb{P}_0^*$ is a regular subforcing of $j_k(\mathbb{P}_0)$ with reduction map $\pi \colon j_k(\mathbb{P}_0) \to \mathbb{P}_0^*$. Let $j_k \colon V[G(\mathbb{P})] \to M[G(j_k(\mathbb{P}))]$ be a lifting of the supercompact embedding for κ_k. Further let $\dot{\mathbb{P}}_1$ be a $\mathbb{P}$-name for a forcing with $j_k[\dot{\mathbb{P}}_1] = \dot{\mathbb{P}}_1$ and $\mathbb{S} = \mathbb{S}_{\mathbb{P} * \dot{\mathbb{P}}_1}(\dot{T})$ a specializing forcing of a $\mathbb{P} * \dot{\mathbb{P}}_1$-name for a κ_k-Aronszajn tree $\dot{T}$. Then the following hold:*

(1) *There exists a regular subforcing* $\mathbb{P}_0^* * \mathbb{S}^*$ *of* $j_k(\mathbb{P}_0) * j_k(\mathbb{S})$ *with a reduction map* $\pi^*\colon j_k(\mathbb{P}_0) * j_k(\mathbb{S}) \to \mathbb{P}_0^* * \mathbb{S}^*$ *such that the first component of* $\pi^*(p, s)$ *extends* $\pi(p)$.

(2) $|\mathbb{S}^*| \leq \kappa_k$.

(3) $j_k(\mathbb{P}) \Vdash$ "$\mathbb{S}^*$ *is a regular subforcing of* $j_k(\mathbb{S})$" *and* $\mathbb{P} \Vdash$ "$\mathbb{S}$ *has the* κ_k*-c.c.*".

(4) $\mathbb{P} * \mathbb{S}$ *is forcing equivalent to* $\mathbb{P}^* * \mathbb{S}^*$.

(5) *The supercompact embedding* j_k *can be lifted to*

$$j_k\colon V[G(\mathbb{P} * \mathbb{S})] \to M[G(j_k(\mathbb{P}) * j_k(\mathbb{S}))].$$

(6) $\mathbb{P}_0 * \mathbb{S}$ *has a dense subset which is nicely* $<\kappa_{k-1}$*-closed in* $V[\mathbb{L}]$ *and the quotient* $j_k(\mathbb{P}_0 * \mathbb{S})/G(\mathbb{P}_0^* * \mathbb{S}^*)$ *is equivalent to a* $<\kappa_{k-1}$*-closed forcing in* $M[G(j_k(\mathbb{L}))][G(\mathbb{P}_0^* * \mathbb{S}^*)]$.

Proof. The proof is a generalization of the corresponding proof in Chapter 4.

Proof of (1): We work in $M[G(j(\mathbb{L}))]$. Let $(p, s) \in j(\mathbb{P}_0) * j(\mathbb{S})$. Let $p' \leq p, \pi(p)$ such that p' decides s, that means in $M[G(j(\mathbb{L}))]$ there exists a partial function $f\colon \omega_{k-1} \times j(\kappa_k) \to [\omega_{k-1}]^{\leq\omega_{k-2}}$ of size $<\omega_{k-1}$ such that $p' \Vdash s = f$. If $p'' \leq \pi(p')$, then p'' is compatible with p' and therefore with $\pi(p)$, thus $\pi(p)$ and $\pi(p')$ are compatible in $j(\mathbb{P}_0)$. Since $\mathbb{P}_0^*$ is a regular subforcing of $j(\mathbb{P}_0)$, $\pi(p)$ and $\pi(p')$ are compatible in $\mathbb{P}_0^*$. Let $\hat{p} \in \mathbb{P}_0^*$ with $\hat{p} \leq \pi(p), \pi(p')$.

Continue working in $V[G(\mathbb{L})] = V[G(\mathbb{L}^*)]$: choose a generic $G(\mathbb{P}_0^*)$ containing $\hat{p}$ and let $G(\mathbb{P}_0)$ be the corresponding generic for $\mathbb{P}_0$, i.e., $V[G(\mathbb{L})][G(\mathbb{P}_0)] = V[G(\mathbb{L})][G(\mathbb{P}_0^*)]$; that is possible because $\mathbb{P}_0$ and $\mathbb{P}_0^*$ are forcing equivalent. Note that $p \in j(\mathbb{P}_0)/G(\mathbb{P}_0^*)$ because $\hat{p} \leq \pi(p)$ and $\pi(p)$ is a reduction of p. Let $\mathbb{P}_1 := \dot{\mathbb{P}}_1^{G(\mathbb{P}_0)}$. Since $\mathbb{P}_1 \in V[G(\mathbb{L})][G(\mathbb{P}_0)]$, it follows that $\mathbb{P}_1 \in V[G(\mathbb{L})][G(\mathbb{P}_0^*)]$. Let $\dot{\mathbb{P}}_1^*$ be a $\mathbb{P}^*$-name for $\mathbb{P}_1$. Now let $G(\mathbb{P}_0 * \dot{\mathbb{P}}_1)$ be generic for $\mathbb{P}_0 * \dot{\mathbb{P}}_1$ and $G(\mathbb{P}_0^* * \dot{\mathbb{P}}_1^*)$ the corresponding generic for $\mathbb{P}_0^* * \dot{\mathbb{P}}_1^*$.

Let $T := \dot{T}^{G(\mathbb{P}_0 * \dot{\mathbb{P}}_1)}$. Since $T \in V[G(\mathbb{L})][G(\mathbb{P}_0 * \dot{\mathbb{P}}_1)]$, it follows that $T \in V[G(\mathbb{L})][G(\mathbb{P}_0^* * \dot{\mathbb{P}}_1^*)]$. Let $\dot{T}^*$ be a $\mathbb{P}^* * \dot{\mathbb{P}}_1^*$-name for T and let $\mathbb{S}^* := \mathbb{S}_{\mathbb{P}^* * \dot{\mathbb{P}}_1^*}(\dot{T}^*)$, the specializing forcing of $\dot{T}^*$. Since $j_k[\dot{\mathbb{P}}_1] = \dot{\mathbb{P}}_1$, j_k can be lifted further to $j_k\colon V[G(\mathbb{P})][G(\dot{\mathbb{P}}_1)] \to M[G(j_k(\mathbb{P}))][G(\dot{\mathbb{P}}_1)]$.

We assume that the nodes on the αth level T_α of T are elements of $\omega_{k-1} \times \{\alpha\}$, and all the levels are of size $< \kappa_k$, therefore $T = j[T] = j(T) \restriction \kappa_k$.

We can assume that for each $\sigma \in \mathrm{dom}(s) \cap j(T)_{>\kappa_k}$ there exists a $\sigma' \in \mathrm{dom}(s)$ on level κ_k and $p_1 \in \mathbb{P}_1$ such that $(p', p_1) \Vdash \sigma' \leq_T \sigma$.

Let $\bar{s} := s \restriction T$, $\{\sigma_\alpha \mid \alpha \in \omega_{k-2}\} := \mathrm{dom}(s) \cap T_{\kappa_k}$ and $C_\alpha := \bigcup\{s(\tau) \mid \tau \geq_T \sigma_\alpha, \tau \in \mathrm{dom}(s)\}$ the set of colors which s assigns to nodes which are in $\mathrm{dom}(s)$ and equal to or above σ_α.

In $V[G(\mathbb{L})][G(\mathbb{P}_0^* * \dot{\mathbb{P}}_1^*)]$ let $\mathbb{Q} := j(\mathbb{P}_0)/G(\mathbb{P}_0^*)$. By assumption in $\mathbb{P}_0$ the set of decisive conditions is dense and nicely $<\kappa_{k-1}$-closed and in $\mathbb{P}_0^*$ the set of decisive conditions is dense. By elementarity also in $j(\mathbb{P}_0)$ the set of decisive conditions is dense and nicely $<\kappa_{k-1}$-closed. Further, note that the set D^* of decisive conditions in $\mathbb{P}_0^*$ is contained in the set $\bar{D}$ of decisive conditions in $j(\mathbb{P}_0)$. So by Lemma 3.6 D^* is a regular subforcing of $\bar{D}$. Therefore by Lemma 3.16, by working in the dense sets of decisive conditions, we can assume that $\mathbb{Q}$ is $<\kappa_{k-1}$-closed.

Define a tree $\mathcal{T}$ of height ω_{k-2} inductively. Each node t on level α will be of the form $(p_w, (\tau_w^\beta \mid \beta < \alpha))$ for some $w \in 2^\alpha$ and $p_w \in \mathbb{Q}$ with $p_w \leq p$, and $p_w \Vdash$ "$\tau_w^\beta <_T \sigma_\beta$" for each $\beta < \alpha$. The construction is as follows:

- The root of $\mathcal{T}$ is $(p_{\langle\rangle}, ())$ where $p_{\langle\rangle} \in \mathbb{Q}$ with $p_{\langle\rangle} \leq p$.
- Assume t is a node of $\mathcal{T}$ on level α, so t is of the form $(p_w, (\tau_w^\beta \mid \beta < \alpha))$ for some $w \in 2^\alpha$ and $p_w \in \mathbb{Q}$, and $p_w \Vdash$ "$\tau_w^\beta <_T \sigma_\beta$" for each $\beta < \alpha$.

  Since T is an Aronszajn tree in $V[G(\mathbb{L})][G(\mathbb{P}_0^*)][G(\dot{\mathbb{P}}_1^*)]$, every cofinal branch through T in $M[G(j(\mathbb{L}))][G(\mathbb{P}_0^*)][G(\dot{\mathbb{P}}_1^*)][G(j(\mathbb{P}_0)/G(\mathbb{P}_0^*))]$ is new. Therefore there exist two conditions $p_{w^\frown 0} \leq p_w$ and $p_{w^\frown 1} \leq p_w$ which decide for every $\beta < \alpha$ the nodes between τ_w^β and σ_β differently. We define two successors for t in $\mathcal{T}$:

  $$(p_{w^\frown 0}, (\tau^0_{w^\frown 0}, \dots, \tau^\alpha_{w^\frown 0})) \text{ and } (p_{w^\frown 1}, (\tau^0_{w^\frown 1}, \dots, \tau^\alpha_{w^\frown 1}))$$

  where $p_{w^\frown i}$ and $\tau^\beta_{w^\frown i}$ for $\beta < \alpha$ and $i \in \{0, 1\}$ are such that the following hold true: $p_{w^\frown i} \Vdash$ "$\tau_w^\beta \leq_T \tau^\beta_{w^\frown i} <_T \sigma_\beta, \tau^\beta_{w^\frown i} \in T$" and $\tau^\beta_{w^\frown 0}$ is incomparable with $\tau^\beta_{w^\frown 1}$ in T, and $p_{w^\frown i} \Vdash$ "$\tau^\alpha_{w^\frown i} <_T \sigma_\alpha \wedge \tau^\alpha_{w^\frown i} \in T$".
- For limit levels α of $\mathcal{T}$, assume that p_w and τ_w^β have been defined for every $w \in 2^{<\alpha}$ and every $\beta < |w|$. For $w \in 2^\alpha$ let p_w be a lower bound of $\langle p_{w\restriction\beta} \mid \beta < \alpha\rangle$, which exists because $\mathbb{Q}$ is $<\kappa_{k-1}$-closed. It follows that $p_w \Vdash$ "$\tau^\beta_{w\restriction\gamma} <_T \tau^\beta_{w\restriction\gamma'} <_T \sigma_\beta$" for all $\beta < \gamma < \gamma' < \alpha$. Therefore, for every $\beta < \alpha$ there exists the limit τ_w^β such that $p_w \Vdash$ "$\tau^\beta_{w\restriction\gamma} <_T \tau^\beta_w <_T \sigma_\beta$" for every $\beta < \gamma < \alpha$. Define the nodes on level α by $(p_w, (\tau_w^\beta \mid \beta < \alpha))$ for each $w \in 2^\alpha$.

For each branch b through $\mathcal{T}$ let p_b be stronger than all $p_{b\restriction\beta}$ and τ_b^α such that $p_b \Vdash$ "$\tau^\alpha_{b\restriction\beta} \leq_T \tau^\alpha_b \leq_T \sigma_\alpha$ and τ_b^α is the limit of $\langle\tau^\alpha_{b\restriction\beta}\rangle_{\beta\in\omega_{k-2}}$". Note that such τ_b^α exist in T, since the height of T is κ_k, and $\kappa_k = \aleph_k$ in $V[G(\mathbb{L})][G(\mathbb{P}_0^*)][G(\dot{\mathbb{P}}_1^*)]$. Further note that τ_b^α and $\tau_{b'}^\alpha$ are incomparable for all $b \neq b'$.

Let $s' := \bar{s} \cup \{(\tau_b^\alpha, C_\alpha) \mid \alpha \in \omega_{k-2}, b \in K\}$, where K is the set of elements in $2^{\omega_{k-2}}$ which have only boundedly many 1's. This is a condition in $\mathbb{S}$, because for each α the set C_α contains all the colors which appear at or above σ_α, so they don't appear at nodes below σ_α and therefore not at nodes below τ_b^α.

Let q be such that $V[G(\mathbb{L})][G(\mathbb{P}_0^*)] \models q \in \mathbb{S} \wedge q \leq s'$. As a preparation for the definition of the reduction map, we show that in $V[G(\mathbb{L})][G(\mathbb{P}_0^*)][G(\dot{\mathbb{P}}_1^*)]$ there exists a $p' \in \mathbb{Q}$ such that $p' \Vdash q \not\perp s$. Let $c \in 2^{\omega_{k-2}}$ be such that no node in $\mathrm{dom}(q)$ extends a τ_c^α for any α. Note that $c \notin K$. Such a c exists, since $2^{\omega_{k-2}}$ is larger than $\mathrm{dom}(q)$. Now $p_c \Vdash$ "$\tau_c^\alpha \leq_T \sigma_\alpha$" for all α, thus $p_c \Vdash$ "$\tau_b^\alpha \not\leq_T \sigma_\alpha$" for all α and all $b \in K$. Let $t \in \mathrm{dom}(q)$ and $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$. Since $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ there exists $\alpha \in \omega_{k-2}$ with $\sigma_\alpha \leq_T \tau$. By induction on α we define a decreasing sequence $\langle p_c^\alpha \mid \alpha \in \omega_{k-2}\rangle$ such that $p_c^{\alpha+1} \Vdash$ "$(\tau, s(\tau))$ is compatible with $(t, q(t))$" (i.e., $p_c^{\alpha+1} \Vdash \{(\tau, s(\tau))\} \cup \{(t, q(t))\} \in \mathbb{S}$) for all $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ with $\sigma_\alpha \leq_T \tau$.

Let $p_c^0 := p_c$.

For successors $\alpha + 1$ we use the following construction. Case 1: $p_c^\alpha \Vdash t <_T \tau_c^\alpha$. Since p_c forces that τ_c^α is the limit of some τ_w^α's, $p_c^\alpha \leq p_c$, and for every $w \in 2^{<\omega_{k-2}}$ there exists a $b \in K$ which extends w, p_c^α forces that there exists some $b \in K$ with $t <_T \tau_b^\alpha$. Therefore, since q is a condition and τ_b^α is in its domain, $p_c^\alpha \Vdash q(t) \cap q(\tau_b^\alpha) = \emptyset$, and since $q(\tau_b^\alpha) = C_\alpha \supseteq s(\tau)$, it follows that $p_c^\alpha \Vdash$ "$(\tau, s(\tau))$ is compatible with $(t, q(t))$". Let $p_c^{\alpha+1} := p_c^\alpha$.

Case 2: $p_c^\alpha \nVdash t <_T \tau_c^\alpha$. On the other hand, $\tau_c^\alpha \not\leq_T t$ by the choice of c, thus there exists $p_c^{\alpha+1} \leq p_c^\alpha$ with $p_c^{\alpha+1} \Vdash$ "τ_c^α is incomparable with t". Since $p_c \Vdash$ "$\tau_c^\alpha \leq_T \sigma_\alpha$", it follows that $p_c^{\alpha+1} \Vdash$ "$t \not<_T \sigma_\alpha$" and therefore $p_c^{\alpha+1} \Vdash$ "$(\tau, s(\tau))$ is compatible with $(t, q(t))$".

For limit ordinals α we use the closure of $\mathbb{Q}$ to find a lower bound p_c^α of $\langle p_c^\beta \mid \beta < \alpha\rangle$.

Again using the closure of $\mathbb{Q}$, there exists a lower bound p_c' of $\langle p_c^\alpha \mid \alpha \in \omega_{k-2}\rangle$. Since p_c' forces for every $t \in \mathrm{dom}(q)$ and every $\tau \in \mathrm{dom}(s) \setminus \mathrm{dom}(s')$ that $(\tau, s(\tau))$ is compatible with $(t, q(t))$, together with the fact that $q \leq s'$, it follows that $p_c' \Vdash$ "q is compatible with s". Thus it holds in $V[G(\mathbb{L})][G(\mathbb{P}_0^*)][G(\dot{\mathbb{P}}_1^*)]$ that for every $q \leq s'$ there exists a $p' \leq p$ such that $p' \Vdash$ "q is compatible with s". Since $G(\mathbb{P}_0^*)$ and $G(\dot{\mathbb{P}}_1^*)$ are filters, we can choose a condition $\bar{p} \in G(\mathbb{P}_0^*)$ below $\hat{p}$ and $p_1 \in G(\dot{\mathbb{P}}_1^*)$ such that $(\bar{p}, p_1)$ forces this.

Define $\pi^*(p, s) := (\bar{p}, s')$.

If $(p^*, s^*) \leq \pi^*(p, s)$ then $p^* \leq \pi(p)$ and therefore p^* is compatible with p and $p^* \leq \bar{p}$ and $p^* \Vdash s^* \leq s'$. Therefore p^* forces that some $p' \in \mathbb{Q}$, with $p' \leq p$, forces s^* to be compatible with s. Since $(p^*, p_1) \Vdash p' \in \mathbb{Q} = j(\mathbb{P}_0)/G(\mathbb{P}_0^*)$, it follows that there exists $p'' \leq p, p^*$ with $p'' \Vdash s \not\perp s^*$. So (p^*, s^*) is compatible with (p, s) and therefore π^* is a reduction map such that the first component of $\pi^*(p, s)$ extends $\pi(p)$.

To see that $\mathbb{P}_0^* * \mathbb{S}^*$ is a regular subforcing of $j(\mathbb{P}_0) * j(\mathbb{S})$ we also have to show that if two conditions in $\mathbb{P}_0^* * \mathbb{S}^*$ are compatible in $j(\mathbb{P}_0) * j(\mathbb{S})$, then they are compatible in $\mathbb{P}_0^* * \mathbb{S}^*$. To see this, we show that the set D of conditions (p, s) with the following property is dense in $j(\mathbb{P}_0) * j(\mathbb{S})$: There exists s^* such that

1. $p \Vdash s \leq s^*$,

2. $p \Vdash s^* \in \mathbb{S}^*$,

3. if $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$.

If p decides s then (p, s) fulfills this property: Let s^* be s restricted to the nodes on levels below κ_k. So $p \Vdash s \leq s^* \wedge s^* \in \mathbb{S}^*$ and if $p \Vdash s \leq \bar{s} \wedge \bar{s} \in \mathbb{S}^*$ then $p \Vdash s^* \leq \bar{s}$, because in this case $\bar{s} \subseteq s^*$. So the set D is dense.

Suppose now that (p_0^*, s_0^*) and (p_1^*, s_1^*) are in $\mathbb{P}_0^* * \mathbb{S}^*$ and they are compatible in $j(\mathbb{P}_0) * j(\mathbb{S})$. Let (p, s) be a witness for the compatibility in the dense set D with witness s^*. So (p, s^*) is also below (p_0^*, s_0^*) and (p_1^*, s_1^*). Now $(\pi(p), s^*)$ is in $\mathbb{P}_0^* * \mathbb{S}^*$ and stronger than (p_0^*, s_0^*) and (p_1^*, s_1^*): Since $p \Vdash s^* \in \mathbb{S}^* \wedge s^* \leq s_0^*, s_1^*$ and that depends only on $\mathbb{P}_0^*$, the same holds true for $\pi(p)$.

Proof of (2): $\mathbb{P} \Vdash |\mathbb{S}^*| \leq \aleph_k \cdot 2^{\aleph_{k-2}} \wedge \kappa_k = \aleph_k \wedge 2^{\aleph_{k-1}} \leq \kappa_k$, hence $\mathbb{P} \Vdash |\mathbb{S}^*| \leq \kappa_k$.

Proof of (3): Let $G(\mathbb{P})$ be generic for $\mathbb{P}$ and $G(\mathbb{P}^*)$ the corresponding generic for $\mathbb{P}^*$. Let $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ be the lifting of the supercompact embedding for κ_k. In $V[G(\mathbb{P})]$ let A^* be a maximal antichain in $\mathbb{S}$. Since $V[G(\mathbb{P})] = V[G(\mathbb{P}^*)]$ and $\mathbb{S}$ in $V[G(\mathbb{P})]$ is the same as $\mathbb{S}^*$ in $V[G(\mathbb{P}^*)]$, we get that in $V[G(\mathbb{P}^*)]$ A^* is a maximal antichain in $\mathbb{S}^*$. On the other hand, also each maximal antichain in $\mathbb{S}^*$ is a maximal antichain in $\mathbb{S}$. By elementarity $M[G(j(\mathbb{P}))] \models$ "$j(A^*)$ is a maximal antichain in $j(\mathbb{S})$". Since j is the identity on $\mathbb{S}$ it follows that $A^* = j[A^*] \subseteq j(A^*)$. Let $G(j(\mathbb{P})/G(\mathbb{P}^*))$ be generic for $j(\mathbb{P})/G(\mathbb{P}^*)$ and assume $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models$ "$s \in j(\mathbb{S})$". Since $|\mathbb{S}^*| \leq \kappa_k$ and ${}^{\leq\kappa_k}M \subseteq M$ it follows that $A^* \in M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))]$.

Claim. $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models$ "*$\exists a \in A^*$ which is compatible with s*".

Proof. Let $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ be such that (p, s) is a condition. We show that the set of conditions which force that there exists $a \in A^*$ which is compatible with s is dense below p. Let $p' \leq p$ and let (p^*, s^*) be a reduction of (p', s) to $\mathbb{P}^* * \mathbb{S}^*$. Since A^* is maximal in $\mathbb{S}^*$ we know that p^* forces over $\mathbb{P}^*$ that there exists $a \in A^*$ which is compatible with s^* and we can pick a name $\dot{b}$ for the witness in $\mathbb{S}^*$. Now $(p^*, \dot{b}) \leq (p^*, s^*)$. Since (p^*, s^*) is a reduction of (p', s) we know that $(p^*, \dot{b})$ is compatible with (p', s). So there exists $\bar{p} \leq p^*, p'$ with $\bar{p} \Vdash$ "$\dot{b}$ is compatible with s", and since $\dot{b}$ is forced to be $\leq a$ by p^*, also $\bar{p} \Vdash$ "$a \in A^*$ is compatible with s".

Now, since $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$, there exists a $q \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ with $q \Vdash$ "$\exists a \in A^*$ which is compatible with s". □

Therefore it follows that A^* is a maximal antichain for $j(\mathbb{S})$ in the model $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))]$. Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. From the above it follows that every maximal antichain of $\mathbb{S}^*$ is a maximal antichain in $j(\mathbb{S})$, hence $\mathbb{S}^*$ is a regular subforcing of $j(\mathbb{S})$. For the second part of (3) note that $A^* \subseteq \mathbb{S}$ and $|\mathbb{S}| \leq \kappa_k$, so we have that $|j(A^*)| \leq \kappa_k < j(\kappa_k)$ and by elementarity $|A^*| < \kappa_k$.

Proof of (4): $\mathbb{P}^*$ is forcing equivalent to $\mathbb{P}$, and $\mathbb{S}^*$ in $V[\mathbb{P}^*]$ is the same forcing as $\mathbb{S}$ in $V[\mathbb{P}]$.

Proof of (5): By the assumption of the lemma there exists $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$, a lifting of the supercompact embedding j. Since $\mathbb{P}^*$ is equivalent to $\mathbb{P}$ we can replace $V[G(\mathbb{P})]$ by $V[G(\mathbb{P}^*)]$ and get $j\colon V[G(\mathbb{P}^*)] \to M[G(j(\mathbb{P}))]$. Let $G(j(\mathbb{S}))$ be generic for $j(\mathbb{S})$ over $M[G(j(\mathbb{P}))]$. Since by (3) $\mathbb{S}^*$ is a regular subforcing of $j(\mathbb{S})$ and $\mathbb{S}^* \subseteq j(\mathbb{S})$, $G(j(\mathbb{S}))$ contains a generic filter $G(\mathbb{S}^*)$ for $\mathbb{S}^*$. Thus, by the Lifting Lemma (Lemma 3.9), j can be lifted to an embedding $j\colon V[G(\mathbb{P}^*)][G(\mathbb{S}^*)] \to M[G(j(\mathbb{P}))][G(j(\mathbb{S}))]$. By (4) $\mathbb{P} * \mathbb{S}$ is equivalent to $\mathbb{P}^* * \mathbb{S}^*$, so we can replace $V[G(\mathbb{P}^*)][G(\mathbb{S}^*)]$ by $V[G(\mathbb{P})][G(\mathbb{S})]$ to get a lifting $j\colon V[G(\mathbb{P})][G(\mathbb{S})] \to M[G(j(\mathbb{P}))][G(j(\mathbb{S}))]$.

Proof of (6): By assumption, in $\mathbb{P}_0$ and $\mathbb{P}_0^*$ the sets of decisive conditions are dense and nicely $<\kappa_{k-1}$-closed. By Lemma 5.5, $\mathbb{S}$ and $\mathbb{S}^*$ are $<\kappa_{k-1}$-closed with weakest lower bounds and well-met, and weakest lower bounds of decided conditions are decided. Further, by the definition of the forcings to specialize names, $\mathbb{P}_0 \Vdash$ "$q \subseteq V$ and $|q| < \kappa_{k-1}$ for all $q \in \mathbb{S}$" and $\mathbb{P}_0^* \Vdash$ "$q \subseteq V$ and $|q| < \kappa_{k-1}$ for all $q \in \mathbb{S}^*$", and whenever $(p,q) \not\perp (p',q')$ are decisive conditions in $\mathbb{P}_0 * \mathbb{S}$, or in $\mathbb{P}_0^* * \mathbb{S}^*$, and $p^* \leq p, p'$, then $p^* \Vdash q \not\perp q'$. So by Lemma 3.23 in $\mathbb{P}_0 * \mathbb{S}$ the set D of decisive conditions, and in $\mathbb{P}_0^* * \mathbb{S}^*$ the set D^* of decisive conditions, are dense and nicely $<\kappa_{k-1}$-closed. By elementarity the same holds for $j(\mathbb{P}_0 * \mathbb{S})$. Further, note that the set D^* of decisive conditions in $\mathbb{P}_0^* * \mathbb{S}^*$ is contained in the set $\bar{D}$ of decisive conditions in $j(\mathbb{P}_0 * \mathbb{S})$. So by Lemma 3.6 D^* is a regular subforcing of $\bar{D}$. Since $\bar{D}$ is equivalent to $j(\mathbb{P}_0 * \mathbb{S})$ and D^* is equivalent to $\mathbb{P}_0^* * \mathbb{S}^*$, the quotient $j(\mathbb{P}_0 * \mathbb{S})/G(\mathbb{P}_0^* * \mathbb{S}^*)$ is equivalent to $\bar{D}/G(D^*)$, and by Lemma 3.16 $\bar{D}/G(D^*)$ is $<\kappa_{k-1}$-closed. □

Next we look at the forcing iteration, in particular the limit steps, and prove some important properties of it.

Lemma 5.14. *Let $\mathbb{P}$ be a forcing with $V[G(\mathbb{P})] \models \kappa_k = \aleph_k$. Let $\mathbb{P}^*$ be a regular subforcing of $j_k(\mathbb{P})$, forcing equivalent to $\mathbb{P}$ and $\pi^*\colon j_k(\mathbb{P}) \to \mathbb{P}^*$ a reduction map. Let $j_k(\mathbb{P})/G(\mathbb{P}^*)$ be $<\kappa_{k-1}$-closed and $j_k\colon V[G(\mathbb{P})] \to M[G(j_k(\mathbb{P}))]$ a lifting of the supercompact embedding for κ_k. Let $\mathbb{S}$ be an iteration of limit length $\alpha \leq \delta$ of forcings to specialize names for κ_k-Aronszajn trees with $<\kappa_{k-1}$-support. Then for every $\beta \leq \alpha$ in $V[G(\mathbb{P})]$ there exists $\mathbb{S}_\beta^*$ with the following properties:*

(1) *$|\mathbb{S}_\beta^*| \leq \delta$,*

(2) *there exists a reduction map $\pi_\beta^*\colon j_k(\mathbb{S} \upharpoonright \beta) \to \mathbb{S}_\beta^*$ such that $\pi_\beta^*(p \upharpoonright \beta) \leq \pi_\gamma^*(p \upharpoonright \gamma)$ for every $\gamma \leq \beta$,*

(3) *the set of decisive conditions in* $j_k(\mathbb{S}\upharpoonright\beta)$ *is dense and nicely* $<\kappa_{k-1}$*-closed and the set of decisive conditions in* $\mathbb{S}^*_\beta$ *is dense, so, in particular,* $j_k(\mathbb{S}\upharpoonright\beta)/\mathbb{S}^*_\beta$ *is equivalent to a* $<\kappa_{k-1}$*-closed forcing,*

(4) $\mathbb{P}^* * \mathbb{S}^*_\beta$ *is forcing equivalent to* $\mathbb{P} * \mathbb{S}\upharpoonright\beta$*,*

(5) $\mathbb{S}^*_\beta$ *is a regular subforcing of* $j_k(\mathbb{S}\upharpoonright\beta)$*,*

(6) *the supercompact embedding* j_k *can be lifted to an elementary embedding*

$$j_k\colon V[G(\mathbb{P})][G(\mathbb{S}\upharpoonright\beta)] \to M[G(j_k(\mathbb{P}))][G(j_k(\mathbb{S}\upharpoonright\beta))],$$

(7) *in* $V[G(\mathbb{P})]$ *the forcing* $\mathbb{S}\upharpoonright\beta$ *has the* κ_k*-c.c..*

Proof. The proof is by induction on $\beta \leq \alpha$.

For $\beta = 0$ there is nothing to show.

$\beta = \gamma + 1$: By induction and by Lemma 5.6 $\mathbb{S}\upharpoonright\gamma$, the first γ steps of the iteration, fulfills the requirements of Lemma 5.13 with $\dot{\mathbb{P}}_1 := \mathbb{S}^{<k}_{\gamma'}$ for suitable γ', so we can apply this lemma to $\mathbb{P} * \mathbb{S}\upharpoonright\gamma * \mathbb{S}_{\mathbb{P}*\mathbb{S}\upharpoonright\gamma*\dot{\mathbb{P}}_1}(\dot{T}_\gamma)$, from which it easily follows that properties (1)–(7) hold true; for (7), note that the two-step iteration of two forcings which have the κ_k-c.c. has again the κ_k-c.c.. Let $\mathbb{S}^*(\dot{T}_\gamma)$ be the regular subforcing of $\mathbb{S}_{\mathbb{P}*\mathbb{S}\upharpoonright\gamma*\dot{\mathbb{P}}_1}(\dot{T}_\gamma)$ and $\pi^*_{\gamma+1}$ the reduction map given by the lemma. Note that by (1) of Lemma 5.13 $\pi^*_{\gamma+1}$ is coherent with π^*_γ in the sense that for $(p,\dot{q}) \in \mathbb{S}\upharpoonright\gamma * \mathbb{S}_{\mathbb{P}*\mathbb{S}\upharpoonright\gamma*\dot{\mathbb{P}}_1}(\dot{T}_\gamma)$ we have $\pi^*_{\gamma+1}(p,\dot{q}) \leq \pi^*_\gamma(p)$.

β limit: In $V[\mathbb{P}^*]$ let $\mathbb{S}^*_\beta$ be the iteration $\mathbb{S}^*(\dot{T}_0) * \mathbb{S}^*(\dot{T}_1) * \mathbb{S}^*(\dot{T}_2) * \ldots$ of length β with $<\kappa_{k-1}$-support, where the $\mathbb{S}^*(\dot{T}_\gamma)$ are given by induction. We will prove that the properties (1)–(7) hold true.

Proof of (1): Since $|\mathbb{S}^*(\dot{T}_\gamma)| \leq \kappa_k$ for each $\gamma < \beta$ and $\beta \leq \alpha \leq \delta$, it follows that $|\mathbb{S}^*_\beta| \leq \delta$.

Proof of (2): Let $p \in j(\mathbb{S}\upharpoonright\beta)$ and let $\lambda \leq \kappa_{k-2}$ and $\{\gamma_i \mid i < \lambda\}$ be increasing indices cofinal in the support of p. Let $\pi^*_{\gamma_i}$ be the reduction map of the iteration of length γ_i given by induction. Since these maps cohere, $\pi^*_{\gamma_0}(p\upharpoonright\gamma_0) \geq \pi^*_{\gamma_1}(p\upharpoonright\gamma_1) \geq \pi^*_{\gamma_2}(p\upharpoonright\gamma_2) \geq \ldots$ and since $\mathbb{S}^*(\dot{T}_0) * \mathbb{S}^*(\dot{T}_1) * \mathbb{S}^*(\dot{T}_2) * \ldots$ is $<\kappa_{k-1}$-closed (as it is a $<\kappa_{k-1}$-support iteration of $<\kappa_{k-1}$-closed forcings), there exists a lower bound of these reductions; let $\pi^*_\beta(p)$ be such a lower bound. It is easy to check that π^*_β is a reduction map which is coherent with the earlier π^*_γ's.

Proof of (3): By Lemma 5.6 the sets D and D^* of decisive conditions in $\mathbb{S}\upharpoonright\beta$ and in $\mathbb{S}^*_\beta$, respectively, are dense and nicely $<\kappa_{k-1}$-closed. Therefore by elementarity the set $\bar{D}$ of decisive conditions in $j_k(\mathbb{S}\upharpoonright\beta)$ is dense and nicely $<\kappa_{k-1}$-closed (κ_{k-1} is below the critical point of j_k). Further, D^* is contained in $\bar{D}$. Therefore by Lemma 3.6 D^* is a regular subforcing of $\bar{D}$. So $\bar{D}/G(D^*)$ is $<\kappa_{k-1}$-closed by Lemma 3.16 and equivalent to $j_k(\mathbb{S}\upharpoonright\beta)/G(\mathbb{S}^*_\beta)$.

Proof of (4): Since the iterands of the two iterations are forcing equivalent and the iterations are both $<\kappa_{k-1}$-support iterations, the two iterations are forcing equivalent.

Proof of (5): Next we show that if two conditions in $\mathbb{P}^* * \mathbb{S}^*_\beta$ are compatible in $j(\mathbb{P}) * j(\mathbb{S}\restriction\beta)$, then they are compatible in $\mathbb{P}^* * \mathbb{S}^*_\beta$. To see this, we show that the set D of conditions $(p, \vec{s})$ with the following property is dense in $j(\mathbb{P} * \mathbb{S}\restriction\beta)$: There exists $\vec{s^*}$ such that

1. $p \Vdash \vec{s} \leq \vec{s^*}$,
2. $p \Vdash \vec{s^*} \in \mathbb{S}^*_\beta$,
3. if $p \Vdash \vec{s} \leq \vec{\bar{s}} \wedge \vec{\bar{s}} \in \mathbb{S}^*_\beta$ then $p \Vdash \vec{s^*} \leq \vec{\bar{s}}$.

If p decides $\vec{s}$, then $(p, \vec{s})$ fulfills this property: Let $\vec{s^*}$ be the tuple of coordinates of $\vec{s}$ restricted to the nodes on levels below κ_k. So $p \Vdash \vec{s} \leq \vec{s^*} \wedge \vec{s^*} \in \mathbb{S}^*_\beta$ and if $p \Vdash \vec{s} \leq \vec{\bar{s}} \wedge \vec{\bar{s}} \in \mathbb{S}^*_\beta$ then $p \Vdash \vec{s^*} \leq \vec{\bar{s}}$, because in this case every coordinate of $\vec{\bar{s}}$ is forced to be a subset of the corresponding coordinate of $\vec{s^*}$. So D is dense.

Suppose now that $(p^*_0, \vec{s^*_0})$ and $(p^*_1, \vec{s^*_1})$ are in $\mathbb{P}^* * \mathbb{S}^*_\beta$ and they are compatible in $j(\mathbb{P} * \mathbb{S}\restriction\beta)$. Let $(p, \vec{s})$ be a witness for the compatibility in the dense set with witness $\vec{s^*}$. So $(p, \vec{s^*})$ is also below $(p^*_0, \vec{s^*_0})$ and $(p^*_1, \vec{s^*_1})$. Now $(\pi(p), \vec{s^*})$ is in $\mathbb{P}^* * \mathbb{S}^*_\beta$ and stronger than $(p^*_0, \vec{s^*_0})$ and $(p^*_1, \vec{s^*_1})$: Since $p \Vdash \vec{s^*} \in \mathbb{S}^*_\beta \wedge \vec{s^*} \leq \vec{s^*_0}, \vec{s^*_1}$ and that depends only on $\mathbb{P}^*$, the same holds for $\pi(p)$.

It follows that in $M[G(j(\mathbb{P}))]$ two conditions $\vec{s^*_0}$ and $\vec{s^*_1}$ in $\mathbb{S}^*_\beta$ which are compatible in $j(\mathbb{S}\restriction\beta)$ are compatible in $\mathbb{S}^*_\beta$. Together with (2) it follows that $\mathbb{S}^*_\beta$ is a regular subforcing of $j(\mathbb{S}\restriction\beta)$.

Proof of (6): By the same proof as the proof of (5) of Lemma 5.13 it follows that j can be lifted: Let $G(j(\mathbb{S}\restriction\beta))$ be generic for $j(\mathbb{S}\restriction\beta)$ over $M[G(j(\mathbb{P}))]$. Since $\mathbb{S}^*_\beta$ is a regular subforcing of $j(\mathbb{S}\restriction\beta)$, $G(j(\mathbb{S}\restriction\beta))$ contains a generic filter $G(\mathbb{S}^*_\beta)$ for $\mathbb{S}^*_\beta$. Thus, by the Lifting Lemma (Lemma 3.9), j can be lifted to an embedding from $V[G(\mathbb{P}^*)][G(\mathbb{S}^*_\beta)]$ to $M[G(j(\mathbb{P}))][G(j(\mathbb{S}\restriction\beta))]$. Since $\mathbb{P}^* * \mathbb{S}^*_\beta$ is equivalent to $\mathbb{P} * \mathbb{S}\restriction\beta$ we can replace $V[G(\mathbb{P}^*)][G(\mathbb{S}^*_\beta)]$ by $V[G(\mathbb{P})][G(\mathbb{S}\restriction\beta)]$ to get a lifting $j\colon V[G(\mathbb{P})][G(\mathbb{S}\restriction\beta)] \to M[G(j(\mathbb{P}))][G(j(\mathbb{S}\restriction\beta))]$.

Proof of (7): Now we show that $\mathbb{P} \Vdash$ "$\mathbb{S}\restriction\beta$ has the κ_k-c.c.". This follows by the same argument as (3) of Lemma 5.13:

Let $G(\mathbb{P}^*)$ be generic for $\mathbb{P}^*$ and $G(\mathbb{P})$ the corresponding generic for $\mathbb{P}$. Let $j\colon V[G(\mathbb{P})] \to M[G(j(\mathbb{P}))]$ be a lifting of the supercompact embedding for κ_k. In $V[G(\mathbb{P})]$ let A^* be a maximal antichain in $\mathbb{S}\restriction\beta$. Since $V[G(\mathbb{P}^*)] = V[G(\mathbb{P})]$ and $\mathbb{S}\restriction\beta$ in $V[G(\mathbb{P})]$ is the same as $\mathbb{S}^*_\beta$ in $V[G(\mathbb{P}^*)]$, we get that A^* is also a maximal antichain in $\mathbb{S}^*_\beta$. On the other hand, also each maximal antichain in $\mathbb{S}^*_\beta$ is a maximal antichain in $\mathbb{S}\restriction\beta$. By elementarity $j(A^*)$ is a maximal antichain in $j(\mathbb{S}\restriction\beta)$ in $M[G(j(\mathbb{P}))]$. Since j is the identity on $\mathbb{S}\restriction\beta$ it follows that $A^* = j[A^*] \subseteq j(A^*)$. Let $G(j(\mathbb{P})/G(\mathbb{P}^*))$ be generic for $j(\mathbb{P})/G(\mathbb{P}^*)$ and assume that $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models \vec{s} \in j(\mathbb{S}\restriction\beta)$. Since $|\mathbb{S}^*_\beta| \leq \delta$ and ${}^{\leq\delta}M \subseteq M$ it follows that $A^* \in M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))]$.

Claim. *$M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models$ "$\exists \vec{a} \in A^*$ which is compatible with $\vec{s}$".*

Proof. Since by (5) $\mathbb{S}^*_\beta$ is a regular subforcing of $j(\mathbb{S}\restriction\beta)$ there exists a reduction map from $j(\mathbb{S}\restriction\beta)$ to $\mathbb{S}^*_\beta$. So there exists $\vec{s}'$ such that $M[G(\mathbb{P}^*)][G(j(\mathbb{P})/G(\mathbb{P}^*))] \models$ "$\vec{s}' \in \mathbb{S}^*_\beta$ is a reduction of $\vec{s}$".

Let $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ with $p \Vdash \vec{s} \in j(\mathbb{S}\restriction\beta)$. The following set is dense in $j(\mathbb{P})/G(\mathbb{P}^*)$ below p:

$$\{q \in j(\mathbb{P})/G(\mathbb{P}^*) \mid q \Vdash \text{"}\exists \vec{a} \in A^* \text{ which is compatible with } \vec{s}\text{"}\}.$$

Indeed, let $p' \leq p$. So $p' \Vdash$ "$\vec{s} \in j(\mathbb{S}\restriction\beta)$ and there exists a reduction $\vec{s}'$ of $\vec{s}$ in $\mathbb{S}^*_\beta$". Therefore $p' \Vdash$ "$\exists \vec{a} \in A^*$ with $\vec{a} \not\perp \vec{s}'$". So there exists a name $\vec{a}$ and $q \leq p'$ such that $q \Vdash$ "$\vec{a} \in A^* \wedge \vec{a} \not\perp \vec{s}'$". Since $q \Vdash$ "$\vec{s}'$ is a reduction of $\vec{s}$", it follows that $q \Vdash$ "$\vec{a}$ is compatible with $\vec{s}$", showing that the above set is dense. Now, since $p \in G(j(\mathbb{P})/G(\mathbb{P}^*))$, there exists a $q \in G(j(\mathbb{P})/G(\mathbb{P}^*))$ with $q \Vdash$ "$\exists \vec{a} \in A^*$ which is compatible with $\vec{s}$". □

Thus it follows that A^* is a maximal antichain for $j(\mathbb{S}\restriction\beta)$. Since $j(A^*)$ is an antichain and $A^* \subseteq j(A^*)$ it follows that $A^* = j(A^*)$. Note that $A^* \subseteq \mathbb{S}\restriction\beta$ and $|\mathbb{S}\restriction\beta| \leq \delta$. Thus $|j(A^*)| \leq \delta < j(\kappa_k)$ and by elementarity $|A^*| < \kappa_k$. □

Corollary 5.15. *In $V[\mathbb{L}_\omega * \mathbb{S}^{>k}_\delta]$ the forcing $\mathbb{S}^k_\delta$ has the κ_k-c.c..*

Proof. We work in $V[\mathbb{L}_{<k}]$. By Lemma 5.10, Corollary 5.11 and Lemma 5.12 $\mathbb{L}_{\geq k} * \mathbb{S}^{>k}_\delta$ fulfills (as $\mathbb{P}$) the requirements of Lemma 5.14, so it forces that $\mathbb{S}^k_\delta$ has the κ_k-c.c.. □

Corollary 5.16. *In $V[\mathbb{L}_\omega]$ the forcing $\mathbb{S}_\delta$ preserves every $\aleph_k$.*

Proof. For every $0 < k < \omega$ the forcing $\mathbb{S}_\delta = \mathbb{S}^{>k}_\delta * \mathbb{S}^{\leq k}_\delta$. This is an iteration of a forcing which is $<\kappa_k$-closed and a forcing which has the κ_k-c.c., therefore it does not collapse $\aleph_k = \kappa_k$. □

Lemma 5.17. *In $V[\mathbb{L}_{<k}]$ the following holds. For every $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$-name $\dot{T}$ for an $\aleph_{k-1}$-tree with level α being $\{\alpha\} \times \aleph_{k-2}$ for every $\alpha < \aleph_{k-1}$ there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$ of $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$ with the following properties:*

(1) *$|\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$,*

(2) *$\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-distributive,*

(3) $\bar{\mathbb{L}}$ *is a regular subforcing of* $\mathbb{L}_k$ *and* $\mathbb{L}_{\geq k}/\bar{\mathbb{L}}$ *is* $<\kappa_{k-1}$*-closed,*

(4) $\mathbb{L}_{\geq k} \Vdash$ "$\bar{\mathbb{S}}^{\geq k}$ *is a regular subforcing of* $\mathbb{S}_\delta^{\geq k}$*, the set of decisive conditions in* $\mathbb{S}_\delta^{\geq k}$ *is dense and nicely* $<\kappa_{k-1}$*-closed, and* $\mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k}$ *is equivalent to a* $<\kappa_{k-1}$*-closed forcing",*

(5) *there exists an* $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$*-name* $\dot{T}'$ *such that* $\mathbb{L}_{\geq k} * \mathbb{S}_\delta \Vdash \dot{T} = \dot{T}'$*, and*

(6) $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} \Vdash \bar{\mathbb{S}}^{<k} = \mathbb{S}_\delta^{<k}$.

Proof. Using Lemma 5.10, Corollary 5.11 and Lemma 5.12 we know that $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{>k}$ fulfills (as $\mathbb{P}$) the requirements for Lemma 5.14, so there exists a lifting of j_k to $j_k : V[\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k}] \to M[j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k})]$. The critical point of j_k is κ_k and ${}^{\leq\delta}M \subseteq M$, and the forcing $\mathbb{S}_\delta^{<k}$ is a $<\kappa_k$-support iteration and each iterand is invariant under j_k, so it is easy to lift j_k further to $j_k : V[\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}] \to M[j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k})]$.

Let $\dot{T}$ be an $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$-name for an $\aleph_{k-1}$-tree with level α being $\{\alpha\} \times \omega_{k-2}$. Since the critical point of j_k is κ_k, $j_k(\dot{T})$ is a $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta)$-name for an $\aleph_{k-1}$-tree.

Let $\mathbb{L}_{\geq k}^*$ be the regular subforcing of $j_k(\mathbb{L}_k)$ which is equivalent to $\mathbb{L}_{\geq k}$ with $|\mathbb{L}_{\geq k}^*| < j_k(\kappa_k)$, so $\mathbb{L}_{\geq k}^*$ is also regular in $j_k(\mathbb{L}_{\geq k})$. Further let $\mathbb{S}^*$ be the regular subforcing of $j_k(\mathbb{S}_\delta^{\geq k})$ as in Lemma 5.14. So there exists an $\mathbb{L}_{\geq k}^* * \mathbb{S}^*$-name $\tilde{\mathbb{S}}^{<k}$ such that $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k}) \Vdash \tilde{\mathbb{S}}^{<k} = \mathbb{S}_\delta^{<k}$ and $\mathbb{L}_{\geq k}^* * \mathbb{S}^* \Vdash |\tilde{\mathbb{S}}^{<k}| < j_k(\kappa_k)$. So $\mathbb{L}_{\geq k}^* * \mathbb{S}^* * \tilde{\mathbb{S}}^{<k}$ is a regular subforcing of $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k})$ which is equivalent to $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$ with $|\mathbb{L}_{\geq k}^* * \mathbb{S}^* * \tilde{\mathbb{S}}^{<k}| < j_k(\kappa_k)$, so there exists an $\mathbb{L}_{\geq k}^* * \mathbb{S}^* * \tilde{\mathbb{S}}^{<k}$-name $\dot{T}^*$ such that $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}) \Vdash j_k(\dot{T}) = \dot{T}^*$.

Thus we have that there exist regular subforcings $\mathbb{L}_{\geq k}^*$, $\mathbb{S}^*$, $\tilde{\mathbb{S}}^{<k}$ of $j_k(\mathbb{L}_{\geq k})$, $j_k(\mathbb{S}_\delta^{\geq k})$, $j_k(\mathbb{S}_\delta^{<k})$ such that $\mathbb{L}_{\geq k}^* * \mathbb{S}^*$ is $<\kappa_{k-1}$-distributive, $|\mathbb{L}_{\geq k}^* * \mathbb{S}^* * \tilde{\mathbb{S}}^{<k}| < j_k(\kappa_k)$, and there exists an $\mathbb{L}_{\geq k}^* * \mathbb{S}^* * \tilde{\mathbb{S}}^{<k}$-name $\dot{T}^*$ for $j_k(\dot{T})$; moreover, $\mathbb{L}_{\geq k}^*$ is a regular subforcing of $j_k(\mathbb{L}_k)$, and $j_k(\mathbb{L}_{\geq k})/\mathbb{L}_{\geq k}^*$ is $<\kappa_{k-1}$-closed, $j_k(\mathbb{L}_{\geq k}) \Vdash$ "$j_k(\mathbb{S}_\delta^{\geq k})/\mathbb{S}^*$ is equivalent to a $<\kappa_{k-1}$-closed forcing" and $j_k(\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k}) \Vdash \tilde{\mathbb{S}}^{<k} = j_k(\mathbb{S}_\delta^{<k})$.

By elementarity of j_k the same holds for $\mathbb{L}_{\geq k} * \mathbb{S}_\delta$: There exist regular subforcings $\bar{\mathbb{L}}$, $\bar{\mathbb{S}}^{\geq k}$, $\bar{\mathbb{S}}^{<k}$ of $\mathbb{L}_{\geq k}$, $\mathbb{S}_\delta^{\geq k}$, $\mathbb{S}_\delta^{<k}$ such that $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-distributive, $|\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$, $\mathbb{L}_{\geq k}/\bar{\mathbb{L}}$ is $<\kappa_{k-1}$-closed, $\mathbb{L}_{\geq k} \Vdash$ "$\mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k}$ is equivalent to a $<\kappa_{k-1}$-closed forcing", and there exists an $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$-name $\dot{T}'$ such that $\mathbb{L}_{\geq k} * \mathbb{S}_\delta \Vdash \dot{T} = \dot{T}'$; moreover, $\bar{\mathbb{L}}$ is a regular subforcing of $\mathbb{L}_k$, and $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} \Vdash \bar{\mathbb{S}}^{<k} = \mathbb{S}_\delta^{<k}$. □

5.3 The final model

Now we are ready to finish the proof of the main theorem.

Theorem 5.18. *It follows from the consistency of* ω *many supercompact cardinals that it is consistent that for all* $0 < k < \omega$*, all* $\aleph_k$*-Aronszajn trees are special, there are such, and there is no* $\aleph_k$*-Kurepa tree.*

To prove the theorem, we analyze the forcing extension by $\mathbb{L}_\omega * \mathbb{S}_\delta$. We show that $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$ "For all $0 < k \in \omega$

there exists an $\aleph_k$-Aronszajn tree,

all $\aleph_k$-Aronszajn trees are special,

and there exists no $\aleph_k$-Kurepa tree."

We have already shown right after the definition of the forcing that all $\aleph_k$-Aronszajn trees are special in this model. The following two lemmata conclude the proof. Recall that $\langle \kappa_k \mid 1 < k < \omega \rangle$ is the increasing sequence of Laver indestructible supercompact cardinals which we fixed above and $\kappa_1 = \aleph_1$.

Lemma 5.19. $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$ *"there exists a special $\aleph_k$-Aronszajn tree" for every $0 < k \in \omega$.*

Proof. In the extension by $\mathbb{L}_\omega$ GCH holds, hence by Proposition 2.13 there exists a special $\aleph_k$-Aronszajn tree, for every $0 < k \in \omega$. Since the specializing forcing $\mathbb{S}_\delta$ does not collapse $\aleph_k$ by Corollary 5.16, these special $\aleph_k$-Aronszajn trees are preserved. □

Lemma 5.20. $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$ *"there does not exist an $\aleph_{k-1}$-Kurepa tree" for every $1 < k \in \omega$.*

Proof. Assume that $V[\mathbb{L}_\omega * \mathbb{S}_\delta] \models$"There exists an $\aleph_{k-1}$-Kurepa tree" and recall that $\mathbb{L}_\omega * \mathbb{S}_\delta = \mathbb{L}_{<k} * \mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$.

Now we work in $V[\mathbb{L}_{<k}]$. Let $\dot{T}$ be an $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$-name for an $\aleph_{k-1}$-tree with level α equal to $\{\alpha\} \times \aleph_{k-2}$. By Lemma 5.17, there exists a regular subforcing $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$ of $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} * \mathbb{S}_\delta^{<k}$ with the following properties:

1. $|\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$,
2. $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k}$ is $<\kappa_{k-1}$-distributive,
3. $\bar{\mathbb{L}}$ is a regular subforcing of $\mathbb{L}_{\geq k}$ and $\mathbb{L}_{\geq k}/\bar{\mathbb{L}}$ is $<\kappa_{k-1}$-closed,
4. $\mathbb{L}_{\geq k} \Vdash$ "$\bar{\mathbb{S}}^{\geq k}$ is a regular subforcing of $\mathbb{S}_\delta^{\geq k}$, the set of decisive conditions in $\mathbb{S}_\delta^{\geq k}$ is dense and nicely $<\kappa_{k-1}$-closed, and $\mathbb{S}_\delta^{\geq k}/\bar{\mathbb{S}}^{\geq k}$ is equivalent to a $<\kappa_{k-1}$-closed forcing",
5. there exists an $\bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}$-name $\dot{T}'$ such that $\mathbb{L}_{\geq k} * \mathbb{S}_\delta \Vdash \dot{T} = \dot{T}'$, and
6. $\mathbb{L}_{\geq k} * \mathbb{S}_\delta^{\geq k} \Vdash \bar{\mathbb{S}}^{<k} = \mathbb{S}_\delta^{<k}$.

Since $|\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}| < \kappa_k$, and κ_k is a Laver indestructible supercompact cardinal, $V[\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}] \models$ "κ_k is inaccessible". So $V[\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k}] \models |[T']| < \kappa_k$.

To argue that the number of cofinal branches is still small in the final model, first note that $\mathbb{L}_{<k} * \mathbb{L}_{\geq k} * \mathbb{S}^{\geq k}_{\delta} * \mathbb{S}^{<k}_{\delta} = \mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k} * \mathbb{L}_{\geq k}/\bar{\mathbb{L}} * \mathbb{S}^{\geq k}_{\delta}/\bar{\mathbb{S}}^{\geq k} * \mathbb{S}^{<k}_{\delta}/\bar{\mathbb{S}}^{<k}$.

Since $\mathbb{L}_{\geq k}/\bar{\mathbb{L}}$ is $<\kappa_{k-1}$-closed, by Lemma 2.8 it does not add cofinal branches to T'. Further, $\mathbb{S}^{\geq k}_{\delta}/\bar{\mathbb{S}}^{\geq k}$ is equivalent to a $<\kappa_{k-1}$-closed forcing and hence it does not add cofinal branches to T' by Lemma 2.8.

Since $\mathbb{L}_{<k} * \mathbb{L}_{\geq k} * \mathbb{S}^{\geq k}_{\delta} = \mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k} * \mathbb{L}_{\geq k}/\bar{\mathbb{L}} * \mathbb{S}^{\geq k}_{\delta}/\bar{\mathbb{S}}^{\geq k} \Vdash \mathbb{S}^{<k}_{\delta} = \bar{\mathbb{S}}^{<k}$, the last iterand $\mathbb{S}^{<k}_{\delta}/\bar{\mathbb{S}}^{<k}$ of the forcing iteration is equivalent to the trivial forcing and can be ignored.

Therefore, $\dot{T} = \dot{T}'$ in $V[\mathbb{L}_{<k} * \bar{\mathbb{L}} * \bar{\mathbb{S}}^{\geq k} * \bar{\mathbb{S}}^{<k} * \mathbb{L}_{\geq k}/\bar{\mathbb{L}} * \mathbb{S}^{\geq k}_{\delta}/\bar{\mathbb{S}}^{\geq k}] = V[\mathbb{L}_{\omega} * \mathbb{S}_{\delta}]$ is not an $\aleph_{k-1}$-Kurepa tree. $\square$

Chapter 6

Trees for all successors of regular cardinals

In this chapter, we generalize our result and show that it follows from the existence of a proper class of supercompact cardinals that it is consistent that for all successors of regular cardinals, all Aronszajn trees are special, and there exist such, while there exist no Kurepa trees on these cardinals.

Lemma 6.1. *Let α be a limit ordinal and $\langle \kappa_n \mid 1 < n < \omega \rangle$ an increasing sequence of Laver indestructible supercompact cardinals. Then there exists a forcing $\mathbb{R}^\alpha$ with the following properties:*

(1) $\mathbb{R}^\alpha$ *is* $<\aleph_{\alpha+1}$*-directed closed,*

(2) $|\mathbb{R}^\alpha| = (\sup_{1<n<\omega} \kappa_n)^{++} =: \delta^\alpha$,

(3) $\mathbb{R}^\alpha \Vdash$ "$\aleph_{\alpha+n} = \kappa_n$ *for every* $1 < n < \omega$",

(4) $\mathbb{R}^\alpha \Vdash$ "$2^{\aleph_{\alpha+n}} = \aleph_{\alpha+\omega+2} = \delta^\alpha$ *for every* $0 < n < \omega$",

(5) $\mathbb{R}^\alpha \Vdash$ *"all* $\aleph_{\alpha+n}$*-Aronszajn trees are special and there exist some for every* $1 < n < \omega$",

(6) $\mathbb{R}^\alpha \Vdash$ *"there exists no* $\aleph_{\alpha+n}$*-Kurepa tree for all* $0 < n < \omega$".

Proof. Let us define the forcing $\mathbb{R}^\alpha$. For every $0 < n < \omega$ let $\mathbb{L}^\alpha_{n+1} := \mathrm{Col}(\aleph_{\alpha+n}, < \kappa_{n+1})$ in $V[\mathbb{L}^\alpha_2 * \cdots * \mathbb{L}^\alpha_n]$ and let $\mathbb{L}^\alpha_\omega := \mathbb{L}^\alpha_2 * \mathbb{L}^\alpha_3 * \mathbb{L}^\alpha_4 * \ldots$ be the countable support iteration. Let $\mathbb{R}^\alpha_0 := \mathbb{L}^\alpha_\omega$. Let $\{A_n \mid 1 < n < \omega\}$ be a partition of δ^α such that every A_n is cofinal in δ^α.

As in the case of specializing all $\aleph_n$-Aronszajn trees, using $\mathbb{R}^\alpha_0$ as first step of the iteration, continue the iteration for length δ^α such that all $\aleph_{\alpha+n}$-Aronszajn trees for $n > 1$ get specialized: In every step β take the forcing to specialize the name of an $\aleph_{\alpha+n}$-Aronszajn tree given by a bookkeeping function (where the n depends on the A_n to which β belongs).

Analogously to the case of specializing all $\aleph_n$-Aronszajn trees, the forcing $\mathbb{R}^\alpha$ fulfills items (1)–(6). $\square$

Now we can combine all the forcings $\mathbb{R}^\alpha$ in an Easton support iteration to specialize all Aronszajn trees for all successors of regular cardinals:

Theorem 6.2. *If there is a proper class of supercompact cardinals with no inaccessible limit, then there is an extension in which for all successors of regular cardinals, all Aronszajn trees are special, there exist such, and for all regular uncountable cardinals there are no Kurepa trees.*

Proof. Let $\mathbb{R}$ be the Easton support iteration of the $\mathbb{R}^\alpha$. The supercompact cardinals get collapsed by $\mathbb{R}$ to the $\aleph_{\alpha+2+n}$, where α is 0 or a limit ordinal. The successors of a limit of supercompact cardinals and $\aleph_1$ are preserved.

The forcing $\mathbb{R}^\alpha$ fulfills Lemma 6.1 in $V[\mathbb{R}^{<\alpha}]$, therefore, as in the case of specializing all $\aleph_n$-Aronszajn trees, $\mathbb{R}^{\leq\alpha} \Vdash$ "all $\aleph_{\alpha+n}$-Aronszajn trees are special for all $1 < n < \omega$ and there exist such and there exist no $\aleph_{\alpha+n}$-Kurepa trees for $0 < n < \omega$". Furthermore $\mathbb{R}^{>\alpha}$ is $<\aleph_{\alpha+\omega+1}$-closed so $\mathbb{R}^{>\alpha}$ does not add new subsets of $\aleph_{\alpha+\omega}$, therefore it does not add new $\aleph_{\alpha+n}$-trees and it does not add new cofinal branches to such trees which already exist, therefore there are no $\aleph_{\alpha+n}$-Kurepa trees in the extension by $\mathbb{R}$ and all $\aleph_{\alpha+n}$-Aronszajn trees are special and there exists one. $\square$

Chapter 7

Questions

In this chapter we state some continuing questions on the topic of special Aronszajn trees and Kurepa trees.

Question 7.1. *Can we specialize trees of height $\aleph_n$ which do not have cofinal branches but levels of size $\geq \aleph_n$? Is it possible to specialize these trees and control the existence of $\aleph_n$-Kurepa trees at the same time?*

This question cannot be solved by the same technique as the one in our construction, because we use that the levels are of size $< \aleph_n$ and therefore do not get changed under the supercompact embedding. New ideas are necessary to overcome this issue.

We can also consider models in which Kurepa trees exist and be more precise about the number of branches of Kurepa trees:

Question 7.2. *Can we control the exact number of cofinal branches of the $\aleph_n$-Kurepa trees in a model in which all $\aleph_n$-Aronszajn trees are special?*

In our model from Theorem 6.2 all the limit cardinals are not strong limits and there are no inaccessible cardinals. Actually $2^{\aleph_{\alpha+n}} = \aleph_{\alpha+\omega+2}$ for each α, and the only regular cardinals are $\aleph_0$ and successor cardinals.

Question 7.3. *Is it possible to specialize all $\aleph_n$-Aronszajn trees for all $0 < n < \omega$ while keeping $\aleph_\omega$ a strong limit? Is it possible to specialize all κ^+-Aronszajn trees for all regular cardinals κ while keeping limit cardinals strong limit?*

Question 7.4. *Is it possible to specialize all κ^+-Aronszajn trees for all regular cardinals κ such that there are inaccessibles in the resulting model?*

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Zusammenfassung

Meine Doktorarbeit behandelt spezielle Aronszajn-Bäume und Kurepa-Bäume. Als erstes zeige ich, dass aus der Existenz einer superkompakten Kardinalzahl und einer unerreichbaren Kardinalzahl darüber folgt, dass konsistenterweise alle $\aleph_2$-Aronszajn-Bäume speziell sind und es welche gibt, und keine $\aleph_1$-Kurepa-Bäume und keine $\aleph_2$-Kurepa-Bäume existieren.

Danach zeige ich, unter der Annahme von ω vielen superkompakten Kardinalzahlen, dass es konsistent ist, dass für alle $0 < n < \omega$ alle $\aleph_n$-Aronszajn-Bäume speziell sind und es welche gibt, und keine $\aleph_n$-Kurepa-Bäume existieren.

Schließlich erweitere ich dieses Resultat zu einer globalen Version über alle Aronszajn-Bäume auf Nachfolgern von regulären Kardinalzahlen und allen Kurepa-Bäumen auf regulären Kardinalzahlen; dazu verwende ich eine echte Klasse von superkompakten Kardinalzahlen.