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Generalized Hankel Operators with Conjugate Holomorphic Symbols on the Fock Space

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Abstract. In this paper, we consider generalized Hankel operators $\tilde{\mathrm{H}}_{\overline{f}} := (\mathrm{Id} - \mathrm{P}_1)\,\mathrm{M}_{\overline{f}} : A^2 \to L^2$, where P_1 denotes the orthogonal projection onto $A^{2,1}$ and $\mathrm{M}_{\overline{f}}$ denotes the multiplication with $\overline{f}$. The paper extends the results from [2], where the special case of the Hankel operators $\tilde{\mathrm{H}}_{\overline{z}^k}$ has been considered. Especially, we show that the Hankel operator $\tilde{\mathrm{H}}_{\overline{f}}$ is bounded if and only if $\overline{f}$ is a polynomial in $\overline{z}$ of degree less or equal than two.

1. **Introduction**

Remember that the Fock space A^2 is defined by

$$A^2 := \big\{ g : \mathbb{C} \to \mathbb{C} \,\big|\, g \text{ is entire and } \|g\|^2 < \infty \big\},$$

where

$$\|g\|^2 := \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^2} \, d\lambda(z)$$

and λ denotes the Lebesgue measure on $\mathbb{C}$. It is well known that A^2 is a closed subspace of the corresponding L^2-space given by

$$L^2 = \big\{ g : \mathbb{C} \to \mathbb{C} \,\big|\, g \text{ is measurable and } \|g\|^2 < \infty \big\}.$$

Therefore, A^2 is a Hilbert space equipped with the inner product defined by

$$\langle f \mid g \rangle := \int_{\mathbb{C}} f(z)\overline{g}(z) e^{-|z|^2}\, d\lambda(z).$$

Hence, an orthogonal projection, the Bergman projection, $\mathrm{P} : L^2 \to A^2$ exists. In the following we abbreviate

$$c_n^2 = \langle z^n \mid z^n \rangle = \int_{\mathbb{C}} |z^n|^2 e^{-|z|^2}\, d\lambda(z) = \pi n!$$

for $n \in \mathbb{N}$ for further convenience.

Remember that the Hankel operator, $\mathrm{H}_{\overline{f}} : A^2 \to A^{2\perp}$ ($A^{2\perp}$ denotes the orthogonal space of the Fock space), with symbol $\overline{f} = \sum_{k=0}^{\infty} b_k \overline{z}^k \in L^2$ ($b_k \in \mathbb{C}$) is defined by

$$\mathrm{H}_{\overline{f}} := (\mathrm{Id} - \mathrm{P})\, \mathrm{M}_{\overline{f}}\,,$$

where Id and $\mathrm{M}_{\overline{f}}$ denote the identity map and the multiplication operator with $\overline{f}$, respectively. Hence,

$$h \mapsto \mathrm{H}_{\overline{f}}(h) = (\mathrm{Id} - \mathrm{P})(\overline{f} h).$$

Operators of this form have been extensively studied, for instance in [4], [1], [5] and [6]. In this article, we want to consider slightly different Hankel operators. Let P_1 be the projection from L^2 to $A^{2,1}$, where

$$A^{2,1} := \mathrm{cl}\big(\mathrm{span}\big\{ \overline{z}^j z^n \,|\, n \in \mathbb{N} \text{ and } j \in \{0,1\} \big\}\big)\,,$$

i.e., the closure of the linear span of the set of monomials of the form $\overline{z}^j z^n$, where $n \in \mathbb{N}$ and $j \in \{0,1\}$. Then the generalized Hankel operator, $\tilde{\mathrm{H}}_{\overline{f}} : A^2 \to A^{2,1\perp}$, where $A^{2,1\perp}$ denotes the orthogonal space of the generalized Fock space, is defined by

$$\tilde{\mathrm{H}}_{\overline{f}} := (\mathrm{Id} - \mathrm{P}_1)\, \mathrm{M}_{\overline{f}}\,.$$

A detailed motivation to study such operators can be found in [2]. We just mention here that $\mathrm{H}_{\overline{z}^k}$ is a solution operator to the differential operator $\frac{\partial^k}{\partial \overline{z}^k}$. However, it is not the canonical solution operator. Obviously, $\big\{ \overline{z}^j z^n \,|\, n \in \mathbb{N} \text{ and } j \in \{0,1\} \big\}$ is in the kernel of $\frac{\partial^k}{\partial \overline{z}^k}$ for $k \geq 2$. For more details, we refer the reader to [2].

In connection with the investigation of operators of the form $\tilde{\mathrm{H}}_{\overline{f}}$, the following problem arises: if $h \in A^2$, then it is not clear that $\overline{f}h \in L^2$. Even the multiplication with $\overline{z}^n$ is only densely defined as an operator from A^2 to $L^2, \forall n \geq 1$. This can be easily illustrated with the following example (cf. [1]). Let $h = \sum_{j=1}^{\infty} a_j \overline{z}^j$ with $|a_j|^2 = \frac{1}{j^2 j!}$. Straightforward calculation gives

$$\|h\|^2 = \pi \sum_{j=1}^{\infty} \frac{1}{j^2} < +\infty \,,$$

but

$$\|\overline{z}^n h\|^2 = \pi \sum_{j=1}^{\infty} \frac{(n+j)\cdots(j+1)}{j^2} \geq \pi \sum_{j=1}^{\infty} \frac{1}{j} = +\infty.$$

Hence, $h \in L^2$, but $\overline{z}^n\, h \notin L^2\ \forall n \geq 1$. To ensure that $\tilde{\mathrm{H}}_{\overline{f}}$ is at least a densely defined operator we assume (as in [1]) that

$$\overline{f} z^n / c_n \in L^2 \quad \forall\, n \in \mathbb{N}\,. \tag{1}$$

Clearly, if the above condition is satisfied, we have

$$\tilde{\mathrm{H}}_{\overline{f}}(z^n/c_n) = \overline{f} z^n/c_n - \mathrm{P}_1(\overline{f} z^n/c_n) \in L^2.$$

In [2] the special generalized Hankel operators $\tilde{\mathrm{H}}_{\overline{z}^k}$ have been investigated. There the following result can be found.

Theorem 1. *On the generalized Fock space A^2 the generalized Hankel operator*

$$\tilde{\mathrm{H}}_{\overline{z}^k} : A^2 \to L^2$$

is compact for $k < 2$ and bounded for $k \leq 2$. For $k > 2$ it is unbounded.

Finally, it should be mentioned that quite similar results exist in the context of generalized Fock spaces. Remember that, for $m > 0$, the generalized Fock space A^2_m is defined by

$$A^2_m := \big\{ g : \mathbb{C} \to \mathbb{C} \,\big|\, g \text{ is entire and } \|g\|^2_m < \infty \big\},$$

where

$$\|g\|^2_m := \int_{\mathbb{C}} |g(z)|^2 e^{-|z|^m}\, d\lambda(z)\,.$$

Note that in the special case $m = 2$ the generalized Fock space coincides with the classic Fock space A^2. The corresponding L^2-space is denoted by L^2_m. The following two results can be found in [3].

Theorem 2. *On generalized Fock spaces A^2_m the generalized Hankel operator*

$$\tilde{\mathrm{H}}_{\overline{z}^k} : A^2_m \to L^2_m$$

is compact for $k < m$ and bounded for $k \le m$.

The case $k \ge m$ is considered in the following theorem (cf. [3]).

Theorem 3. *The following results hold:*

(i) The case $k = m$: There exists at most a finite number of integers k such that the generalized Hankel operator $\tilde{\mathrm{H}}_{\overline{z}^k} : A^2_k \to L^2_k$ is compact.

(ii) The case $k > m$: For $k > m$ we have almost everywhere unboundedness, i.e., for every m there exist at most finitely many integers k such that $\tilde{\mathrm{H}}_{\overline{z}^k} : A^2_m \to L^2_m$ is bounded.

2. The main result

Here, we proof the main result of this article, i.e., that the Hankel operator $\tilde{\mathrm{H}}_{\overline{f}}$, with weight $\overline{f} = \sum_{k=1}^{\infty} b_k \overline{z}^k$, where $b_k \in \mathbb{C}$, is bounded if and only if $b_k = 0$ for $k > 2$. The idea of the proof of this result is to derive a connection between boundedness of the operators $\tilde{\mathrm{H}}_{\overline{f}}$ and $\tilde{\mathrm{H}}_{\overline{z}^k}$. The following proposition is necessary and derives a more pleasant form of $\mathrm{H}_{\overline{f}}(z^n/c_n)$. The following arguments combine ideas from [1] and [2].

Proposition 1. *Let* $\overline{f} := \displaystyle\sum_{k=0}^{\infty} b_k \overline{z}^k \in L^2$ $(b_k \in \mathbb{C})$. *The generalized Hankel operator with symbol $\overline{f}$, $\tilde{\mathrm{H}}_{\overline{f}} : A^{2,1} \to {A^{2,1}}^{\perp}$, evaluated at z^n/c_n is calculated to be*

$$\begin{aligned}&\tilde{\mathrm{H}}_{\overline{f}}(z^n/c_n)\\ &= \frac{1}{c_n}\sum_{k\le n} b_k(\overline{z}^k z^n - a^k_{n-k,0} z^{n-k} - a^k_{n-k,1}\overline{z} z^{n-k+1}) + \frac{1}{c_n}\sum_{k>n} b_k \overline{z}^k z^n,\end{aligned}$$

where

$$a^k_{n,0} = \frac{c^2_{n+2}c^2_{n+k} - c^2_{n+1}c^2_{n+k+1}}{c^2_n c^2_{n+2} - c^4_{n+1}}$$

and

$$a^k_{n,1} = \frac{c^2_n c^2_{n+k+1} - c^2_{n+1}c^2_{n+k}}{c^2_n c^2_{n+2} - c^4_{n+1}}.$$

Proof. We remember (cf. [2]) that for each $n \in \mathbb{N}$

$$\mathrm{P}_1(\overline{z}^k z^{n+k}) = a^k_{n,0} z^n + a^k_{n,1} \overline{z} z^{n+1} \tag{2}$$

is valid, where

$$a^k_{n,0} = \frac{c^2_{n+2} c^2_{n+k} - c^2_{n+1} c^2_{n+k+1}}{c^2_n c^2_{n+2} - c^4_{n+1}}$$

and

$$a^k_{n,1} = \frac{c^2_n c^2_{n+k+1} - c^2_{n+1} c^2_{n+k}}{c^2_n c^2_{n+2} - c^4_{n+1}}.$$

Furthermore, using the same reasoning one can see that

$$\mathrm{P}_1(\overline{z}^k z^n) = 0 \tag{3}$$

for $n < k$. The above conditions can be rewritten as

$$\begin{aligned}
\mathrm{P}_1(\overline{z}^k z^n) =& a^k_{n-k,0} z^{n-k} + a^k_{n-k,1} \overline{z} z^{n-k+1} \\
=& \left(\frac{c^2_{n-k+2} c^2_n - c^2_{n-k+1} c^2_{n+1}}{c^2_{n-k} c^2_{n-k+2} - c^4_{n-k+1}} \right) z^{n-k} \\
&+ \left(\frac{c^2_{n-k} c^2_{n+1} - c^2_{n-k+1} c^2_n}{c^2_{n-k} c^2_{n-k+2} - c^4_{n-k+1}} \right) \overline{z} z^{n-k+1}
\end{aligned}$$

for $n \geq k$, i.e., equation (2), and $\mathrm{P}_1(\overline{z}^k z^n) = 0$ otherwise. Therefore,

$$\mathrm{P}_1(\overline{f} z^n / c_n)(z) = \frac{1}{c_n} \sum_{k \leq n} b_k (a^k_{n-k,0} z^{n-k} + a^k_{n-k,1} \overline{z} z^{n-k+1})$$

and consequently

$$\begin{aligned}
\tilde{\mathrm{H}}_{\overline{f}}(z^n / c_n) &= \overline{f} \frac{z^n}{c_n} - \mathrm{P}_1(\overline{f} z^n) \\
&= \overline{f} \frac{z^n}{c_n} - \frac{1}{c_n} \sum_{k \leq n} b_k (a^k_{n-k,0} z^{n-k} + a^k_{n-k,1} \overline{z} z^{n-k+1}) \\
&= \frac{1}{c_n} \sum_{k \leq n} b_k (\overline{z}^k z^n - a^k_{n-k,0} z^{n-k} + a^k_{n-k,1} \overline{z} z^{n-k+1}) \\
&\quad + \frac{1}{c_n} \sum_{k > n} b_k \overline{z}^k z^n.
\end{aligned}$$

This finishes the proof.

$\square$

Now we come to our main result.

Theorem 4. *Regard the Hankel operator* $\tilde{\mathrm{H}}_{\overline{f}} : A^{2,1} \to A^{2,1^\perp}$, *where* $\overline{f} = \sum_{i=0}^{\infty} b_k \overline{z}^k$ $(b_k \in \mathbb{C})$ *satisfies the regularity condition* (1), *mentioned in the introduction. Then* $\tilde{\mathrm{H}}_{\overline{f}}$ *is bounded if and only if* $\overline{f}$ *is a polynomial in* $\overline{z}$ *of degree less or equal than two, i.e., if and only if* $b_k = 0$ *for* $k > 2$.

Proof. A straightforward calculation shows that

$$\begin{aligned}\left(\overline{z}^k z^n - a^k_{n-k,0} z^{n-k} - a^k_{n-k,1} \overline{z} z^{n-k+1}\right) \\ \perp \left(\overline{z}^l z^n - a^k_{n-l,0} z^{n-l} - a^k_{n-l,1} \overline{z} z^{n-l+1}\right)\end{aligned}$$

for $k \neq l$. Hence,

$$\begin{aligned}\left\| \tilde{\mathrm{H}}_{\overline{f}} \Big(\frac{z^n}{c_n}\Big) \right\|^2 &= \left\langle \tilde{\mathrm{H}}_{\overline{f}} \left(\frac{z^n}{c_n}\right) \mid \tilde{\mathrm{H}}_{\overline{f}} \left(\frac{z^n}{c_n}\right) \right\rangle \\ &= \sum_{k\le n} |b_k|^2 \left\| \frac{1}{c_n} \left(\overline{z}^k z^n - a^k_{n-k,0} z^{n-k} - a^k_{n-k,1} \overline{z} z^{n-k+1}\right) \right\|^2 \\ &\quad + \sum_{k>n} |b_k|^2 \frac{c^2_{n+k}}{c^2_n}.\end{aligned} \tag{4}$$

Using Proposition 1 for $\overline{f} = z^k$ we obtain

$$\begin{aligned}\left\| \tilde{\mathrm{H}}_{\overline{z}^k} \left(\frac{z^n}{c_n}\right) \right\|^2 &= \left\| \frac{1}{c_n} \left(\overline{z}^k z^n - a_{n-k,0} z^{n-k} - a_{n-k,1} \overline{z} z^{n-k+1}\right) \right\|^2 \\ &= \left\| \frac{1}{c_n} \overline{z}^k z^n \right\|^2 - \left\| \frac{1}{c_n} \left(a_{n-k,0} z^{n-k} + a_{n-k,1} \overline{z} z^{n-k+1}\right) \right\|^2,\end{aligned}$$

where we used the fact that P_1 is a projection, and hence $\|(\mathrm{Id} - \mathrm{P}_1)(g)\|^2 = \|g\|^2 - \|\mathrm{P}_1(g)\|^2$. Moreover, straightforward calculation yields

$$\begin{aligned}\left\| \tilde{\mathrm{H}}_{\overline{z}^k} \left(\frac{z^n}{c_n}\right) \right\|^2 &= \frac{2c^2_{n-k+1} c^2_n c^2_{n+1} + c^2_{n-k} c^2_{n-k+2} c^2_{n+k}}{c^2_n \left(c^2_{n-k} c^2_{n-k+2} - c^4_{n-k+1}\right)} \\ &\quad - \frac{c^4_{n-k+1} c^2_{n+2} + c^2_{n-k+2} c^4_n + c^2_{n-k} c^4_{n+1}}{c^2_n \left(c^2_{n-k} c^2_{n-k+2} - c^4_{n-k+1}\right)}.\end{aligned} \tag{5}$$

Therefore, equation (4) can be rewritten as

$$\left\| \tilde{\mathrm{H}}_{\overline{f}} \left(\frac{z^n}{c_n}\right) \right\|^2 = \sum_{k\le n} |b_k|^2 \left\| \tilde{\mathrm{H}}_{\overline{z}^k} \left(\frac{z^n}{c_n}\right) \right\|^2 + \sum_{k>n} |b_k|^2 \frac{c^2_{n+k}}{c^2_n}.$$

If the Hankel operator $\tilde{\mathrm{H}}_{\overline{f}}$ is bounded, we necessarily need

$$\left\| \tilde{\mathrm{H}}_{\overline{f}} \left(\frac{z^n}{c_n} \right) \right\|^2 \leq K$$

for some $K < \infty$ and all n and therefore

$$\left\| \tilde{\mathrm{H}}_{\overline{z}^k} \left(\frac{z^n}{c_n} \right) \right\|^2 < \frac{K}{|b_k|^2}$$

for each k with $b_k \neq 0$ and all n. Since the operators $\tilde{\mathrm{H}}_{\overline{z}^k}$ are diagonal operators (this can be seen via direct calculation or via reference to [2]) the above implies that the operators $\tilde{\mathrm{H}}_{\overline{z}^k}$ must be bounded. It is known from [2] that $\tilde{\mathrm{H}}_{\overline{z}^k}$ is bounded only for $k \leq 2$, which can alternatively be shown using equation (5). Hence, we have $b_k = 0$ for $k > 2$.

If $\overline{f} = b_1 \overline{z} + b_2 \overline{z}^2$ it follows immediately from the above that $\tilde{\mathrm{H}}_{\overline{f}}$ is bounded. This finishes the proof.

□

Remark: It would be interesting to generalize the results of [2] and the work here to Hankel operators of the form

$$\tilde{\mathrm{H}}^l_{\overline{f}} := (\mathrm{Id} - \mathrm{P}_l)\, \mathrm{M}_{\overline{f}}\,,$$

where P_l is the projection onto $A^{2,l}$ with

$$A^{2,l} := \mathrm{cl}\big(\mathrm{span}\{\overline{z}^j z^n \,|\, j, n \in \mathbb{N} \text{ and } 0 \leq j \leq l\}\big)$$

that is the closure of the linear span of the set of all monomials $\overline{z}^j z^n$, where $j, n \in \mathbb{N}$ and $0 \leq j \leq l$. However, such a generalization seems much more complicated than the present result.

Remark: A similar result than the one in Theorem 4 is also possible for the case of generalized Fock spaces A^2_m. The proof of the main result of this paper can be easily adopted for this generalized setting.

References

[1] W. Knirsch and G. Schneider, Continuity and Schatten-von Neumann $p-$class membership of Hankel operators with anti-holomorphic symbols on (generalized) Fock spaces, J. Math. Anal. Appl. 320 (2006) 403-414.

[2] W. Knirsch and G. Schneider, Hankel operators and the generalized solution operator to $\overline{\partial}$ on the Fock space and on the Bergman space of the unit disc, Math. Nachr. 279 (2006) 1684-1694.

[3] W. Knirsch and G. Schneider, Compactness of generalized Hankel operators with unbounded and anti-holomorphic symbols on generalized Fock spaces, Rev. Roum. 52 (2007) 213-229.

[4] G. Schneider, Hankel-operators with anti-holomorphic symbols on the Fock-space, Proc. Amer. Math. Soc. 132 (2004) 2399-2409.

[5] J. Xia, On the Schatten class membership of Hankel operators on the unit ball, Illinois J. Math., 46, (2002), no. 3, 913-928.

[6] K. Zhu, Hilbert-Schmidt Hankel-operators on the Bergman space. Proc. Amer. Math. Soc. 109 (1990), 721-730.

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