BI-MODAL NAIVE SET THEORY

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ABSTRACT. This paper describes a modal conception of sets, according to which sets are 'potential' with respect to their members. A modal theory is developed, which invokes a naive comprehension axiom schema, modified by adding 'forward looking' and 'backward looking' modal operators. We show that this 'bi-modal' naive set theory can prove modalized interpretations of several ZFC axioms, including the axiom of infinity. We also show that the theory is consistent by providing an S5 Kripke model. The paper concludes with some discussion of the nature of the modalities involved, drawing comparisons with *noneism*, the view that there are some non-existent objects.

Contemporary set theory is based on the iterative conception of set. According to this conception, the universe of sets has a hierarchical structure, which can be divided into levels. It is a natural conception of set, a conception that one could come up with in the absence of any extensive formal training in set theory. If we start from the idea that a set is simply a collection of objects, then the iterative conception suggests itself quite readily. But there are other natural conceptions of set. The naive conception, for example, is a natural conception of set. According to the naive conception, for any predicate, there exists a set of all and only those things which satisfy that predicate. But the naive conception of set, as we all know, is inconsistent.

The inconsistency of naive set theory, combined with classical logic, explodes to become a trivial theory of sets, a theory that makes everything true. In order to avoid triviality, those hoping to save naive set theory have argued in favor of weaker, paraconsistent logics, in which contradictions do not explode into triviality. Much of the work on this approach is due to and inspired by Richard Routley (1977, 1980), along with others working in the paraconsistent tradition, such as Ross Brady (1989, 2006) and Graham Priest (1979).[1]

In addition to a paraconsistent approach to naive set theory, both Routley and Priest endorse a particular understanding of the existential quantifier, one that is not existentially committing. This understanding of the existential quantifier, which has roots in the work of Meinong, is captured by a view that Routley and Priest call *noneism*. In this paper, we present a novel application of the noneist approach, focusing specifically on the notion of existence in naive set theory. We do this by combining naive set theory with (classical) modal logic. What results is a natural theory of sets that is both proof-theoretically strong and classically consistent, and hence non-trivial.

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[1]See also the collection edited by Priest, Routley and Norman (1989). For more recent treatments of naive set theory, see Weber (2010a, 2010b).

The plan of this paper is as follows. The first section motivates and presents the naive modal theory of sets. In particular, we discuss how the tools from modal logic can be used to provide an interpretation of the noneist approach to existence. In the second section, we show that this theory avoids the familiar threats of inconsistency, and furthermore we present a model of the theory to show that it is consistent. This section also demonstrates the proof-theoretic strength of the theory, with regard to the kinds of sets it can prove to exist. The third and final section discusses the use of modalities in set theory, comparing the naive modal theory of sets presented here to other modal set theories available in the literature. The paper concludes by proposing a novel interpretation of the modality involved in modal set theory, an interpretation that is inspired by the noneist approach to existence.

1. A MODAL APPROACH TO SETS

Any conception or theory of sets should tell us what sets exist. The naive conception of set captures the idea that for any objects that have something in common, i.e., share a particular property or satisfy a particular condition, there exists a set of all and only those objects. This conception of set can be developed axiomatically. Most axiomatic treatments of naive set theory invoke the unrestricted comprehension axiom schema (UC):

Unrestricted Comprehension (UC). $\exists y \forall x[x \in y \leftrightarrow \Phi]$

For every formula Φ of the first-order language of set theory, there is a corresponding instance of unrestricted comprehension that naive set theory includes as an axiom. A restriction is often imposed on the formula Φ such that it should not contain the variable y free. For if it could contain y free, then inconsistency would easily result by taking Φ to be the formula: $x \notin y$.

Even with the restriction on y's occurrence in Φ, however, the naive conception of set is still inconsistent, as it generates Russell's paradox, from which we can derive a contradiction. The contradiction follows from an instance of unrestricted comprehension in a relatively straightforward way. Take Φ to be the formula $x \notin x$. From this instance of unrestricted comprehension, we can prove the existence of a set, r, known as the Russell set. The Russell set is the set of all sets that are not members of themselves. The Russell set is then a member of itself if and only if it is not a member of itself: $r \in r \leftrightarrow r \notin r$. And so we have Russell's paradox. The paradox leads to contradiction in the presence of the law of excluded middle: either the Russell set is a member of itself, in which case it follows that it isn't. Or the Russell set is not a member of itself, in which case it follows that it is. In either case, we have a contradiction. From the contradiction, given the classically valid rule of explosion, which says that any sentence follows from a contradiction, naive set theory is a trivial theory of sets.

In the face of triviality, there have historically been two responses. The most widespread response has been to develop an alternative theory of sets, one that maintains classical logic, but which avoids contradiction. The most popular alternative theory comprises the Zermelo-Fraenkel axioms, with the axiom of choice (ZFC). ZFC aims to capture the iterative conception of set, according to which sets are built in stages.[2] At the first stage are all of the *urelemente*, the objects that are not collections. Often set theorists ignore the *urelemente* and start with the empty set, building up from there. Starting with the empty set, $\emptyset$, we then take all of the various collections of everything that we have so far, giving us the empty set $\emptyset$ and its singleton $\{\emptyset\}$. We continue in this way for a while, at each successor stage taking all of the possible collections of sets that were formed at the previous stage. At limit stages, we collect together everything that was formed at all of the previous stages. What results is called the cumulative hierarchy of sets.

The iterative conception provides us with an alternative to the naive conception of set, which, for all we know, is classically consistent. But the fact that it is consistent does not conclusively show that it is the *correct* conception of sets. Moreover, beyond its inconsistency, little has been said about what is wrong with the naive conception of set. After all, the naive conception is an extremely natural conception of set.

The other response to the paradoxes has been to defend the naive conception of set and embrace the contradiction, while avoiding triviality. Richard Routley and Graham Priest, among others, have opted for this approach, pointing out that naive set theory captures a wider and more inclusive conception of set than the iterative conception.

> [T]he genuine conception of set is that given by the unrestricted [comprehension] scheme, according to which a set is the extension of an arbitrary property or condition. The cumulative hierarchy is exactly what it appears to be with a little historical perspective: *a* consistent substructure of the inconsistent universe of sets, masquerading as the whole thing (Priest and Routley 1989, p. 199, emphasis in the original).

Unfortunately, when combined with classical logic, the inconsistency of naive set theory results in triviality and makes every sentence true. To avoid triviality, Routley and Priest, and others in this tradition, have opted to combine unrestricted comprehension with a paraconsistent logic, a logic that does not validate explosion.

We argue that there is a third approach that one can take to the paradoxes of naive set theory, one that preserves the use of classical logic, but which avoids contradiction. Furthermore, the approach can be seen as an interpretation of another view that both Routley (1980) and Priest (2005) endorse: *noneism*.

Noneism is a view about existence, a view that has its roots in Meinong. According to noneism, some objects, e.g., concrete objects, exist. Other objects, perhaps abstract objects, or merely possible objects, or impossible objects, do not exist. They are non-existent objects. Importantly, one can quantify over non-existent objects. In this respect, noneism departs from the classical Quinean view, according to which the things that exist are the things one can quantify over. On the noneist approach, quantifiers range over both existent and non-existent objects. In order to capture the noneist approach to existence in a formal language, the existential quantifier does not involve any existential commitments. The existence of an object is given through an existence predicate $\mathcal{E}$. To

[2]See Boolos (1971).

avoid confusion, the noneist reading of the existential quantifier interprets $\exists x\Phi$ as 'for some x, Φ holds', rather than 'there exists an x such that Φ holds'. The latter sentence, which is existentially committing, is represented by the formula $\exists x(\mathcal{E}x \wedge \Phi)$.

This paper explores an alternative way to represent the noneist approach to existence in a formal language that uses tools from modal logic. Rather than introduce an existence predicate, we take the existential quantifier, $\exists x\Phi$, to be existentially committing, as in the classical Quinean tradition. Non-existence is represented in the formal language as 'possible' existence, using the standard possibility operator from modal logic, as in $\Diamond\exists x\Phi$. Of course, non-existence should mean more than possible existence, under the standard interpretation of possibility, as possible existence allows for actual existence. For our purposes, the diamond operator should be interpreted so as to include the condition of non-actuality. In fact, we will see that in the context of the naive set theory presented below, it must include the non-actuality condition on pain of contradiction. Semantically one might read $\Diamond\exists x\Phi$ as 'at some accessible point, w, there exists a non-actual x such that Φ'.[3]

Focusing on the case of naive set theory, we can apply the modal interpretation to a noneist approach to the existence of sets.[4] According to naive set theory, sets are given through the unrestricted comprehension axiom schema, each instance of which is an existentially quantified formula. Taking sets to be non-existent objects, the modal interpretation translates these instances of UC into formulas that provide for the 'possible' existence of sets. In effect, we can affix a possibility operator to the front of these instances of unrestricted comprehension. Further discussion of the modality that this possibility operator represents is postponed to section 3. For now, it will suffice to say that non-existence, in the noneist sense, can be understood as existence at a non-actual point in a Kripke frame.

In order to develop this modal theory of sets, we extend the first-order language of set theory with two modal operators. The first is the familiar possibility operator, $\Diamond$, which satisfies the usual conditions. Using the familiar possibility operator, we can translate the unrestricted comprehension axiom schema, which is an existence claim, into a possible existence claim:

Modal Unrestricted Comprehension (MUC). $\Diamond\exists y\forall x[x \in y \leftrightarrow \Phi]$

Modal unrestricted comprehension is a natural formal representation of the modal translation of the noneist approach to naive set theory sketched above. It is also interesting to note that some kind of modality is arguably explicit in Cantor's original discussion in the *Grundlagen* of the concept of sets as collections of well-defined objects into a whole.

> In general, by a 'manifold' or 'set' I understand every multiplicity which *can* be thought of as one, i.e., every aggregate of determinate elements which *can* be united into a whole by some law (Cantor 1996, p. 85, emphases added).

The passage suggests that sets are in some sense potential relative to their members. Some contemporary philosophers of mathematics, such as Charles Parsons, have picked up on this idea.

[3]Thanks to an anonymous referee for discussion of this point.

[4]Whether the modal interpretation works as a general approach to noneism is an open question that we do not consider here.

> Objects that exist together *can* constitute a set. However, we do have to distinguish between 'existing together' and 'constituting a set.' A multiplicity of objects that exist together *can* constitute a set, but it is not necessary that they *do*. Given the elements of a set, it is not necessary that the set exists together with them. If it is possible that there should be objects satisfying some conditions, then the realization of this possibility is not as such the realization *also* of the possibility that there be a set of such objects (Parsons 1977, pp. 355 - 6, emphases in the original).

The most obvious way to capture the modality that Parsons describes is to modify the unrestricted comprehension axiom schema with a single modal operator, as is done in MUC. But MUC doesn't quite capture Parsons' idea. What Parsons wants is to collect all the things that satisfy Φ at w into a set at some accessible point v. But MUC simply collects all the things that satisfy Φ at v into a set at v. Furthermore, this modal comprehension axiom schema, like the non-modal version of unrestricted comprehension (UC), is classically inconsistent. Russell's paradox still generates a contradiction, as the following semantic argument shows. Supposing $\Diamond\exists y\forall x[x \in y \leftrightarrow x \notin x]$ is true at a point, w, in a Kripke model, $\mathcal{M}$, there is a point, v, in $\mathcal{M}$ that is accessible from w, where v makes $\exists y\forall x[x \in y \leftrightarrow x \notin x]$ true. This will entail that the contradictory Russell set exists at v. This is contradictory, as every point in a classical Kripke model is classical.

In order to maintain classical logic and avoid contradiction, we introduce a second, dual possibility operator $\blacklozenge$. The most straightforward way to think about these two possibility operators is semantically, in terms of Kripke models. The familiar possibility operator, $\Diamond$, is a 'forward looking' operator: If $\Diamond\Phi$ is true at a point w, then there is a point v that is accessible from w, and Φ is true at v. In contrast, the dual possibility operator, $\blacklozenge$, is a 'backward looking' operator: If $\blacklozenge\Phi$ is true at a point w, then there is a point u such that w is accessible from u, and Φ is true at u.

Using the first-order language of set theory with two dual possibility operators, we introduce a 'bi-modal' version of naive comprehension:

Bi-Modal Unrestricted Comprehension (BMUC). $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\Phi]$

The axiom schema BMUC is a formal representation of the potential existence of sets, as described by Charles Parsons. The idea of potential existence can be understood as existence at a point in a Kripke model, and may therefore serve as a suitable way to capture the noneist idea of non-existent objects.

Admittedly, this formal interpretation is not completely faithful to Parsons' intuitive idea that sets are potential relative to the existence of their members. The intuitive idea is that for any things that exist (at a point w) satisfying a condition Φ, it is possible (from the perspective of w) that those things form a set. More precisely, the things in w that satisfy Φ form a set in some point v that is accessible from w. The axiom BMUC doesn't quite capture this idea. According to BMUC, what forms a set at v are all the things that satisfy Φ at *any* point that accesses v, not just those things at w.

While the formal interpretation does not perfectly match the intuitive idea, it is likely the closest one can get using the standard language and semantics of modal logic. Parsons (1983) himself offers an alternative formal interpretation that arguably comes closer, but it requires extending the language of modal logic in a nonstandard way with unfamiliar syntax. The syntax that Parsons introduces involves scoped formulas, which he uses in the following comprehension axiom schema:

Scoped Unrestricted Comprehension (SUC). $\Diamond\exists y\forall x(x \in y \leftrightarrow \Phi_0)$

In any instance of SUC, the formula Φ includes a subscript. The subscript notation Φ_n indicates that the formula Φ falls under the scope of the first n modal operators. In the comprehension axiom, Φ_0 does not fall under the scope of any modal operator, and acts much like an actuality operator. Read this way, SUC says: it's possible that there exists a set that contains all and only the things that actually satisfy Φ. But Parsons needs more than an actuality operator, as he allows for the option to say that SUC is a necessary truth, in which case an actuality operator would not do. The necessitated version of SUC would be formulated as:

Nec. Scoped Unrestricted Comprehension (□SUC). $\Box\Diamond\exists y\forall x(x \in y \leftrightarrow \Phi_1)$

For any formula Φ, the axiom says that at any point (not just the actual point), w, it's possible that there exists a set that contains all and only the things that satisfy Φ at w.

The use of scoped formulas allows Parsons to avoid the contradictions of naive set theory. However, the subscript notation adds additional unfamiliar syntax, which considerably complicates the semantics for the modal set theory that Parsons develops (see Parsons 1983, pp. 333-5). In comparison, the semantics for the dual modalities in BMUC are well-understood from modal temporal logics. They are simply the standard Kripkean possible worlds semantics, with a straightforward dual condition for the backward looking modal operator. This suggests that BMUC is conceptually easier to understand than SUC. Moreover, as we will see in the next section, the bi-modal naive set theory based on BMUC has significant proof-theoretic advantages over Parsons' modal theory of sets.

2. Consistency and Strength

A straightforward semantic argument shows that Russell's paradox does not threaten the bi-modal comprehension axiom. Suppose that $\Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge x \notin x]$ is true at a point w. If follows that there is another point v that is accessible from w, and a set r at v that is like the familiar Russell set. The members of r at v are all and only those things that satisfy, at v, the formula $\blacklozenge(x \notin x)$. In particular, the set r contains all of the things that satisfy the formula $x \notin x$ at w. That is, it contains all of the non-self-membered sets at w. It also includes all of the non-self-membered sets at any other point that accesses v. So, if r itself is a non-self-membered set at any point that accesses v, then it will be a member of itself at v. That is, $r \in r \leftrightarrow \blacklozenge(r \notin r)$ is true at v. We are then faced with two cases: at v, either $r \notin r$ or $r \in r$. In the first case, we have $v \Vdash \neg\blacklozenge r \notin r$, which gives us $v \Vdash \blacksquare r \in r$, using the standard definition of the backward looking necessity operator. We will want the T axiom (i.e., reflexivity) for the backward looking modalities.[5] So this gives us $v \Vdash r \in r$, which is contradictory. But in the other case we have $v \Vdash r \in r$. It follows that $v \Vdash \blacklozenge(r \notin r)$,

[5]Reflexivity is used in the proof of Theorem 2 given later in this section.

and so there is a point u that access v, and $u \Vdash r \notin r$. But this is not contradictory. So while the existence of r shows that naive modal set theory is not well-founded, because $v \Vdash r \in r$, it does not show that the theory is inconsistent. There are perfectly coherent and quite elegant theories of non-well-founded sets, theories which include sets that are members of themselves (see, e.g., Aczel 1988).

Other familiar set-theoretic paradoxes, like Curry's paradox, are also blocked in this bi-modal naive set theory, the theory of sets as given by BMUC. In fact, one can show that this theory of sets is classically consistent, by providing a classical Kripke model of the theory.

Theorem 1. There is an S5 Kripke model that satisfies BMUC.

Proof. Let $\mathcal{M}$ be a universally accessible Kripke model, with a domain of points, $W = \{u, v\}$, consisting of two points that access themselves and each other, and a domain of quantification, $D = \{a\}$, consisting of a single object. Let $u \Vdash a \in a$ and $v \Vdash a \notin a$:

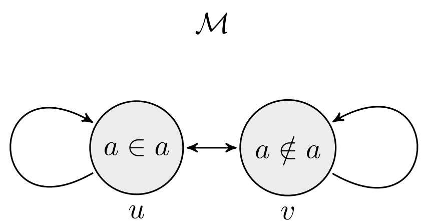

FIGURE 1. An S5 Kripke model of BMUC

To see that $\mathcal{M}$ is a model of BMUC, let Φ be given. There are two cases: (1) for some $w \in W, w \Vdash \Phi$, or (2) for all $w \in W, w \nVdash \Phi$. In the first case, it follows that $u \Vdash \blacklozenge\Phi$. As $u \Vdash a \in a$, we have $u \Vdash a \in a \leftrightarrow \blacklozenge\Phi$. In the second case, it follows that $v \Vdash \neg\blacklozenge\Phi$. As $v \Vdash a \notin a$, we have $v \Vdash a \in a \leftrightarrow \blacklozenge\Phi$. In both cases, because a is the only object in the domain of quantification, there is a point in the model that makes $\exists y \forall x[x \in y \leftrightarrow \blacklozenge\Phi]$ true. The other point in the model then makes $\Diamond\exists y \forall x[x \in y \leftrightarrow \blacklozenge\Phi]$ true.[6]

□

The existence of the universally accessible model $\mathcal{M}$ shows that BMUC is consistent assuming the modal logic S5, which is the strongest normal modal logic. It is therefore consistent in weaker modal logics, like S4, K4, B, etc. However, a consistent modal theory of sets is only interesting if it can be put to good use. Unfortunately, this particular model of naive modal set theory is not very interesting and looks rather trivial. So the question remains as to how strong the theory really is. One can show, however, that from a proof-theoretic perspective, BMUC has some interesting theorems.

[6]This proof is due to James Studd.

For example, while ZFC proves the existence of the empty set, BMUC proves the 'possible' existence of the empty set: BMUC $\vdash \Diamond\exists y\forall x(x \notin y)$. The possible existence of the empty set is given from the instance of BMUC that takes Φ to be '$x \neq x$'. According to the modal interpretation of noneism, this theorem provides for the non-existence of the empty set. Moving to more interesting sets, we have that for any set s, the power set of s possibly exists: for any s, BMUC $\vdash \Diamond\exists y\forall x[x \in y \leftrightarrow x \subseteq s]$. The non-existence of the power set of s is given by an instance of BMUC that takes Φ to be $x \subseteq s$.

While the non-existence of these sets is relatively straightforward, one of the more interesting features of bi-modal naive set theory is that it can prove the possible existence, or non-existence, of an infinite set.[7]

Theorem 2. BMUC $\vdash \Diamond\exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$

Proof. We show this semantically. Because BMUC proves the non-existence of the empty set, there is a point, v, in a Kripke model, $\mathcal{M}$, such that $\emptyset$ exists at v. We then assert the following instance of BMUC at v:

$$v \Vdash \Diamond\exists y\forall x[x \in y \leftrightarrow \blacklozenge\forall z(\emptyset \in z \wedge \forall u(u \in z \rightarrow \{u\} \in z) \rightarrow x \in z)]$$

There is then a point, w, which is accessible from v, and at w there is a set, which we'll call i, such that $w \Vdash \forall x[x \in i \leftrightarrow \blacklozenge\forall z(\emptyset \in z \wedge \forall u(u \in z \rightarrow \{u\} \in z) \rightarrow x \in z)]$. We want to show that $w \Vdash \emptyset \in i$, which follows iff $w \Vdash \blacklozenge\forall z(\emptyset \in z \wedge \forall u(u \in z \rightarrow \{u\} \in z) \rightarrow \emptyset \in z)$. To show this, assuming the T rule for $\blacklozenge$, i.e., that backward possibility is reflexive, we show that $w \Vdash \forall z(\emptyset \in z \wedge \forall u(u \in z \rightarrow \{u\} \in z) \rightarrow \emptyset \in z)$. Choosing an arbitrary object a, it's clear that $[\emptyset \in a \wedge \forall u(u \in a \rightarrow \{u\} \in a)] \rightarrow \emptyset \in a$ holds at w. So $w \Vdash \emptyset \in i$.

We now show that, at w, for any object, x, in i, its successor, $\{x\}$, is in i as well: $w \Vdash \forall x(x \in i \rightarrow \{x\} \in i)$. Choosing an arbitrary object b, we show that $w \Vdash b \in i \rightarrow \{b\} \in i$. Assuming $w \Vdash b \in i$, it follows from BMUC that there is a point u that accesses w, such that $u \Vdash \forall z(\emptyset \in z \wedge \forall u(u \in z \rightarrow \{u\} \in z) \rightarrow b \in z)$ [call this formula 1]. We now show that formula 1 also holds for $\{b\}$ at u: $u \Vdash \forall z(\emptyset \in z \wedge \forall u(u \in z \rightarrow \{u\} \in z) \rightarrow \{b\} \in z)$. Choosing an arbitrary object c, we assume that $u \Vdash \emptyset \in c \wedge \forall u(u \in c \rightarrow \{u\} \in c)$, and show that $u \Vdash \{b\} \in c$. Combining this assumption with formula 1, we have that $b \in c$. Taking the right conjunct of this assumption, it then follows that $b \in c \rightarrow \{b\} \in c$, and so by modus ponens $\{b\} \in c$. We have thus shown that an infinite set i exists at point w: $w \Vdash \exists y[\emptyset \in y \wedge \forall x(x \in y \rightarrow \{x\} \in y)]$. It follows that the possible existence, or the non-existence, of this set holds at v.[8]

□

[7] Note that this uses the Zermelo definition of successor, not the von Neumann definition.

[8] The existence of an infinite set can be proven more directly if the variable y can occur in Φ, by taking Φ to be the formula $x \subseteq y$. Note that the consistency proof of BMUC does not prevent y from occurring in Φ.

In this respect, bi-modal naive set theory has a notable advantage over the modal set theory proposed by Charles Parsons (1983), which uses scoped formulas. Parsons needs more than his comprehension axiom SUC to secure the strength of his modal theory of sets. The theory can show that if a set x exists at a point, then there is a point where its successor exists. But the theory cannot show that all of the successors exist together at one point. In order to do this, Parsons must invoke a reflection principle, allowing that any situation realized by the entire system of points is realized at a particular point.[9]

Given that BMUC proves the possible existence, or the non-existence, of an infinite set, one may wonder how this result squares with the consistency proof given previously. The consistency proof gives a model $\mathcal{M}$ of BMUC that only has one object, a, in its domain. Where, then, is the infinite set? The apparent tension is resolved by recognizing that one of the points in the model makes the formula $a \in a$ true. At that point, a is a non-well-founded set. In fact, at that point $a = \{a\}$. So, not only do we have $a \in a$, but also $\{a\} \in a$, and $\{\{a\}\} \in a$, etc., because they're all identical. The non-well-founded set is then 'infinite' because for every object in a, its successor is in a as well. It's just that each object is identical to its successor.

It is hard not to see this explanation as something of a disappointment. It is worth remembering, however, that axioms and theorems don't always do what they are intended to do. In our case, the fact the BMUC can prove the possible existence of an infinite set does not guarantee that there will be a set in every model with infinitely many members. In a similar way, the axiom of foundation in ZFC does not guarantee that every set is well-founded, as there are non-well-founded models of ZFC (assuming ZFC is consistent).

Further comfort can be found in the fact that naive modal set theory has some interesting features. Most immediately, it is encouraging to see that there is a straightforward interpretation of a natural conception of set that is both classically consistent and proof-theoretically strong. Indeed, the modal interpretation captures what was arguably a feature of naive set theory from the beginning: the potential nature of sets relative to the existence of their members. And this potentiality of sets can subsequently serve as an interpretation of the noneist view that sets are non-existent objects. Or rather, as will be suggested in section 3, the noneist approach to existence can serve as an interpretation of the modality in modal set theory.

Perhaps even more importantly, there are considerations having to do with relative interpretability, which suggest that BMUC is worthy of further study. We have seen that BMUC proves straightforward modal translations of some of the axioms of ZFC, including the Power Set Axiom and the Axiom of Infinity. It's possible that these translations can serve as the basis for modal interpretations of all of the axioms of ZFC. Though this possibility cannot be fully explored here, if one can show that bi-modal naive set theory, which is consistent, proves modal interpretations of the ZFC axioms, this may have interesting consequences for the question of the consistency of ZFC.

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[9] A system similar to Parsons, but which uses plural logic, is developed in Linnebo (2013). Linnebo's system must also invoke a reflection principle to prove the existence of infinite sets.

3. Set-Theoretic Modality

There has been a lot of recent interest in applying tools from modal logic to set theory. One of the most fruitful applications of modal logic to set theory has been in the study of forcing techniques.[10] This work is fascinating, but is not directly relevant to the modal theory of sets presented here. More relevant applications of modal logic to set theory include work by Charles Parsons (1983) discussed above, as well as more recent work by Øystein Linnebo (2013). Both Parsons and Linnebo formulate a set theory in the standard language of modal logic that includes forward looking modal operators only. James Studd (2013) also has a modal theory of sets, a theory that includes forward and backward looking modal operators. However, all three authors appeal to some form of reflection principle in order to prove the existence of infinite sets. The bi-modal naive set theory presented here, based on the BMUC axiom schema, therefore has a conceptual advantage over these other modal set theories, as it can prove a modal version of the Axiom of Infinity without appealing to any further principles.[11]

We have shown that the BMUC axiom schema is both classically consistent and proof-theoretically strong. But we have not said much as to how one should interpret the modalities involved. It is generally accepted that any modality involved with set theory is not metaphysical modality.[12] The main reason for rejecting this interpretation comes from the idea that pure sets are taken to exist at all metaphysically possible worlds. The existence of all of the members of a set without the set is then metaphysically impossible.

As an alternative, one might take the modality to be temporal. One often encounters the analogy, especially in discussions of the iterative conception, of the process of building sets in a series of stages (e.g., Boolos 1971). This suggests a temporal dimension, where stages correspond to moments of time. But the analogy is weak, because we are not really building anything, and sets are generally thought to exist outside of space and time.

A more plausible alternative comes from the work of Kit Fine (2005, 2006), and is also discussed in more recent applications of modal tools to the iterative conception (e.g., Linnebo 2013, Studd 2013). The idea is that the modality is somehow connected to an ever expanding domain of mathematical objects. Possibility claims are about what may hold in some permissible expansion of mathematical ontology, while necessity claims are about what holds in every permissible expansion. This interpretation may be appropriate for the modal representation of the iterative conception, but it does not quite fit the theory presented here. In particular, it requires an expanding domain interpretation of set-formation, which may not entirely agree with the naive conception of sets. On a modal approach to the naive conception, there is nothing to prevent sets from coming into, and going out of, existence as one moves along the accessibility relation.

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[10]See Smullyan and Fitting (1996), as well as Hamkins and Löwe (2008).

[11]For a nice summary of recent work in modal set theory, see Menzel (2018).

[12]See Fine (1981), Linnebo (2013), Potter (2004), and Studd (2013). Parsons, however, may be open to this interpretation. See Parsons (1977), pp. 355 - 6.

A full discussion of the modality involved in naive modal set theory cannot be addressed here. But it seems that the noneist approach to existence offers a novel interpretation of set-theoretic modality. Introducing modality into a language is supposed to allow for different ways in which sentences can be true. Applied to existence claims, modality allows for different ways for something to exist, e.g., possible existence or necessary existence. With a little stretch of the imagination, one can understand the noneist conception of non-existence as a way for something to exist. That is, one can give a noneist interpretation of the modality in modal set theory: merely possible existence can be understood as non-existence.

On a modal approach to the naive conception, as one moves from one point to the next, new sets are created according to instances of the bi-modal comprehension scheme. These are sets that do not exist at the original point; they are non-existent objects. In fact, their non-existence at the original point is required, as otherwise the theory may be threatened by Russell's paradox. Recall that, starting from some point, w, BMUC allows for the existence of the Russell set at some accessible point v. This point v should not be identical to the original point w, because if the Russell set exists at w, then one can derive a contradiction. So the set that is created at v should be a non-existent object from the perspective of the original point w.

The noneist interpretation of the modality in BMUC captures this idea intuitively. In fact, one can understand both the forward and backward looking modalities along noneist lines. As one moves in both directions along the accessibility relation, sets may come into and go out of existence. Given two points, such that w accesses v, there are sets that exist at v that do not exist at w; they are non-existent objects from the perspective of w. The existence of some of these sets at v is determined by instances of BMUC that hold at w. Equally, however, there may be some sets at w that do not exist at v; they are non-existent objects from the perspective of v. One could develop principles, perhaps similar to BMUC, that determine, from the perspective of v, the existence of these objects at w. These principles will be guided by how the details of the noneist interpretation of the modalities are filled in.

Whether or not the language of modal logic provides a completely suitable formal representation of a general noneist approach to existence, outside of bi-modal naive set theory, remains to be seen. And whether or not a complete understanding of the modalities in bi-modal naive set theory can be given is of secondary importance. The intuitive idea behind introducing modal operators is to give a more accurate characterization of a particular conception of set and the principles that govern this conception. What results is a natural theory of sets, bi-modal naive set theory, which is both proof-theoretically interesting and classically consistent.

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