Calculus of Variations

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Global existence of Dirac-wave maps with curvature term on expanding spacetimes

Volker Branding[1] · Klaus Kröncke[2]

Abstract

We prove the global existence of Dirac-wave maps with curvature term with small initial data on globally hyperbolic manifolds of arbitrary dimension which satisfy a suitable growth condition. In addition, we also prove a global existence result for wave maps under similar assumptions.

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1 Introduction and results

Wave maps are among the fundamental variational problems in differential geometry. They are defined as critical points of the Dirichlet energy for a map between two manifolds, where one assumes that the domain manifold is Lorentzian and the target manifold is Riemannian. More precisely, let (M, h) be a globally hyperbolic Lorentzian manifold, (P, G) a Riemannian manifold and $\phi\colon M \to P$ a map. Squaring the norm of its differential gives rise to the Dirichlet energy, whose critical points are given by the *wave map equation*, which is $\tau(\phi) = 0$, where $\tau(\phi) := -h^{\alpha\beta}\nabla_{\partial_\alpha}d\phi(\partial_\beta)$.

The wave map equation is a second-order semilinear hyperbolic system, that is the natural analogue of the harmonic map equation for maps between Riemannian manifolds. Wave maps are also well-studied in the physics literature, they appear as critical points of the *Polyakov action* in bosonic string theory for a string with Lorentzian worldsheet.

There are many articles that study wave maps in the case that the domain is Minkowski space. We cannot give an exhaustive list of the results here, but want to mention the influ-

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ential works of Klainerman [15], Tataru [26], Klainerman and Selberg [17], Klainerman and Machedon [16], Tao [24,25] and Shatah and Struwe [22,23].

There are less articles that consider the case of a domain being a non-flat globally hyperbolic manifold. Here, we want to mention the works of Choquet-Bruhat [9,10] for wave maps on Robertson–Walker spacetimes and several recent articles that consider wave maps on non-flat backgrounds [11,14,18,20]. To obtain an overview on the current status of research on the wave map equation we refer to the recent book [12].

In modern quantum field theory one considers extensions of the wave map system, one of them being the supersymmetric nonlinear σ-model [1]. Recently, there has been a lot of interested in this model from a mathematical perspective. In mathematical terms, the model leads to a geometric variational problem that couples a map between manifolds to spinor fields.

Most of the mathematical research on this model so far is concerned with the case that both manifolds are Riemannian leading to the notion of Dirac-harmonic maps [8] and Dirac-harmonic map with curvature term [4,7], which represent semilinear elliptic problems. At present, many results regarding the geometric and analytic structure of Dirac-harmonic maps and Dirac-harmonic maps with curvature are known [5], but no existence result for these kind of equations could be achieved.

In case that the domain manifold has a Lorentzian metric the critical points of the supersymmetric nonlinear σ-model lead to a system of the wave map equation coupled to spinor fields. For this system two existence results are available [6,13] for the domain being two-dimensional Minkowski space.

In order to couple the wave map equation to spinor fields we have to recall some concepts from spin geometry on globally hyperbolic manifolds. We have to make the additional assumption that the manifold M is spin, which guarantees the existence of the spinor bundle SM. Sections in the spinor bundle are called spinors. Moreover, we fix a spin structure. The spinor bundle is a vector bundle on which we choose a metric connection compatible with the hermitian scalar product denoted by $\langle \cdot, \cdot \rangle_{SM}$. On the spinor bundle we have the Clifford multiplication of spinors with tangent vectors, which satisfies

$$\langle \psi, X \cdot \xi \rangle_{SM} = \langle X \cdot \psi, \xi \rangle_{SM}$$

for all $X \in TM$, $\psi, \xi \in \Gamma(SM)$. In addition, the Clifford relations

$$X \cdot Y + Y \cdot X = -2h(X, Y)$$

hold for all $X, Y \in TM$, where h represents the metric on M.

The natural operator acting on spinors is the *Dirac operator*, which is defined as the composition of applying the covariant derivative first and Clifford multiplication in the second step. More precisely, the Dirac operator acting on spinors is given by

$$\slashed{\partial} := h^{\alpha\beta} \partial_\alpha \cdot \nabla_{\partial_\beta}.$$

The Dirac operator is a linear first order hyperbolic differential operator. For more background on spin geometry on globally hyperbolic manifolds we refer to [2,3]. The Dirac operator itself is anti-self-adjoint with respect to the L^2-norm. However, the combination $i\slashed{\partial}$ yields a self-adjoint operator, that is

$$\int_M \langle i\slashed{\partial}\xi, \psi \rangle \, dV_h = \int_M \langle \xi, i\slashed{\partial}\psi \rangle \, dV_h.$$

The square of the Dirac operator satisfies the *Schrödinger–Lichnerowicz* formula

$$\partial\!\!\!/^2 = \Box + \frac{\text{scal}^M}{4},$$

where scal^M denotes the scalar curvature of the manifold M. Note that $\partial\!\!\!/^2$ is a wave-type operator.

In quantum field theory one usually considers spinors that are twisted by some additional vector bundle. Here, we consider the case that the spinor bundle is twisted with the pullback of the tangent bundle from the target manifold. More precisely, we are considering the bundle $SM \otimes \phi^*TP$ and sections in this bundle are called *vector spinors*. We obtain a connection on $SM \otimes \phi^*TP$ by setting

$$\nabla^{SM \otimes \phi^*TP} = \nabla^{SM} \otimes \mathbb{1}^{\phi^*TP} + \mathbb{1}^{SM} \otimes \nabla^{\phi^*TP}.$$

Note that Clifford multiplication on the twisted bundle is defined by acting only on the first factor. This allows us to define the Dirac operator acting on vector spinors as follows

$$\not{D} := h^{\alpha\beta} \partial_\alpha \cdot \nabla^{SM \otimes \phi^*TN}_{\partial_\beta}.$$

We assume that the connection on ϕ^*TP is metric and thus the operator $i\not{D}$ is also self-adjoint with respect to the L^2-norm. After these considerations we are ready to present the energy functional for Dirac-wave maps with curvature term

$$S\left(\phi, \tilde{\psi}, \tilde{h}\right) = \frac{1}{2} \int_M \left(|d\phi|^2 + \left\langle \tilde{\psi}, i\not{D}\tilde{\psi} \right\rangle - \frac{1}{6} \left\langle \tilde{\psi}, R^P\left(\tilde{\psi}, \tilde{\psi}\right)\tilde{\psi} \right\rangle \right) dV_{\tilde{h}}. \qquad (1.1)$$

In the last term the indices are contracted as follows

$$\left\langle \tilde{\psi}, R^P\left(\tilde{\psi}, \tilde{\psi}\right)\tilde{\psi} \right\rangle_{SM \otimes \phi^*TP} = R_{IJKL} \left\langle \tilde{\psi}^I, \tilde{\psi}^K \right\rangle_{SM} \left\langle \tilde{\psi}^J, \tilde{\psi}^L \right\rangle_{SM},$$

which ensures that the energy functional is real-valued. We would like to point out that in the physics literature (see e.g. [1]) one usually considers Grassmann-valued spinors in the analysis of (1.1). However, we want to use methods from the geometric calculus of variation and due to this reason we are employing standard spinors.

The critical points of (1.1) can easily be calculated as

$$\tau(\phi) = \frac{1}{2} h^{\alpha\beta} R^P\left(\tilde{\psi}, i\partial_\alpha \cdot \tilde{\psi}\right) d\phi(\partial_\beta) - \frac{1}{12} \left\langle \left(\nabla R^P\right)^\sharp \left(\tilde{\psi}, \tilde{\psi}\right)\tilde{\psi}, \tilde{\psi} \right\rangle, \qquad (1.2)$$

$$i\not{D}\tilde{\psi} = \frac{1}{3} R^P\left(\tilde{\psi}, \tilde{\psi}\right)\tilde{\psi}, \qquad (1.3)$$

where $\tau(\phi) := -h^{\alpha\beta} \nabla_{\partial_\alpha} d\phi(\partial_\beta)$ represents the wave map operator. The solutions of (1.2), (1.3) are called *Dirac-wave maps with curvature term*.

We are able to provide the first existence result for the Dirac-wave map with curvature term system which is as follows:

Theorem 1.1 *Let $\tilde{g}_t$ be a smooth family of complete Riemannian metrics on Σ^{n-1}, $N \in C^\infty(\mathbb{R} \times \Sigma)$ with $0 < A \le N \le B < \infty$ and $(M^n, \tilde{h}) = (\mathbb{R} \times \Sigma, -N^2 dt^2 + \tilde{g}_t)$ be a globally hyperbolic Lorentzian spin manifold that satisfies the following condition: There exists a monotonically increasing smooth function $s : \mathbb{R} \to \mathbb{R}_+$ with $\int_0^\infty s^{-1} dt < \infty$, such that the conformal metric*

$$h = (Ns)^{-2}\tilde{h} = -s^{-2}dt^2 + g_t$$

has bounded geometry, the metrics g_t have a uniform Sobolev constant and

$$\|N\|_{C^k(g_t)} + \|\nabla_\nu N\|_{C^k(g_t)} + \|\mathrm{II}\|_{C^k(g_t)} + s\|\mathrm{II}\|_{L^\infty} \leq C < \infty \tag{1.4}$$

for all $k \in \mathbb{N}$. Here, ν is the future-directed unit normal of the hypersurfaces $\{t\} \times \Sigma$ and II is their second fundamental form.

Then if in addition, the Riemannian manifold (P, G) has bounded geometry, there exists for each $r \in \mathbb{N}$ with $r > \frac{n-1}{2}$ an $\varepsilon > 0$ such that if the initial data (ϕ_0, ϕ_1, ψ_0) for the system (1.2), (1.3) *satisfies*

$$\|\phi_0\|_{H^{r+1}(\tilde{g}_0)} + \|\phi_1\|_{H^r(\tilde{g}_0)} + \|\psi_0\|_{H^r(\tilde{g}_0)} < \varepsilon, \tag{1.5}$$

the unique solution of the system (1.2), (1.3) *with initial data $\phi|_{t=0} = \phi_0, \partial_t\phi|_{t=0} = \phi_1, \psi|_{t=0} = \psi_0$ exists for all times $t \in [0, \infty)$ and satisfies*

$$\phi \in C^0\left([0, \infty), H^{r+1}(\Sigma, P)\right) \cap C^1\left([0, \infty), H^r(\Sigma, P)\right),$$
$$\psi \in C^0\left([0, \infty), H^r\left(M, SM \otimes \phi^*TP\right)\right) \cap C^1\left([0, \infty), H^{r-1}\left(M, SM \otimes \phi^*TP\right)\right).$$

Remark 1.2 Under the same assumptions, the proof of Theorem 1.1 also implies global existence of *Dirac-wave maps*, which satisfy an equation slightly simpler than (1.2), (1.3). In this case, the second term on the right hand side of (1.2) and the right-hand side of (1.3) both vanish.

Along the line of Theorem 1.1 we obtain the following result for the wave map equation generalizing the results from [9,10]:

Theorem 1.3 *Let $\tilde{g}_t$ be a smooth family of complete Riemannian metrics on Σ^{n-1}, $N \in C^\infty(\mathbb{R} \times \Sigma)$ with $0 < A \leq N \leq B < \infty$ and $(M^n, \tilde{h}) = (\mathbb{R} \times \Sigma, -N^2dt^2 + \tilde{g}_t)$ be a globally hyperbolic Lorentzian manifold that satisfies the following condition: There exists a monotonically increasing smooth function $s : \mathbb{R} \to \mathbb{R}_+$ with $\int_0^\infty s^{-1}dt < \infty$, such that for the conformal metric*

$$h = (Ns)^{-2}\tilde{h} = -s^{-2}dt^2 + g_t$$

the metrics g_t admit a uniform Sobolev constant and

$$\|R^{g_t}\|_{C^k(g_t)} + \|N\|_{C^k(g_t)} + \|\nabla_\nu N\|_{C^k(g_t)} + \|\mathrm{II}\|_{C^k(g_t)} \leq C < \infty \tag{1.6}$$

for all $k \in \mathbb{N}$. Here, ν is the future-directed unit normal of the hypersurfaces $\{t\} \times \Sigma$ and II is their second fundamental form.

Then if in addition, the Riemannian manifold (P, G) has bounded geometry, there exists for each $r \in \mathbb{N}$ with $r > \frac{n-1}{2}$ an $\varepsilon > 0$ such that if the initial data (ϕ_0, ϕ_1) for the wave map equation satisfies

$$\|\phi_0\|_{H^{r+1}(\tilde{g}_0)} + \|\phi_1\|_{H^r(\tilde{g}_0)} < \varepsilon, \tag{1.7}$$

the unique wave map with initial data $\phi|_{t=0} = \phi_0, \partial_t\phi|_{t=0} = \phi_1$ exists for all times $t \in [0, \infty)$ and satisfies

$$\phi \in C^0\left([0, \infty), H^{r+1}(\Sigma, P)\right) \cap C^1\left([0, \infty), H^r(\Sigma, P)\right).$$

Remark 1.4 If we compare the assumptions of Theorems 1.1 and 1.3 we observe that one also has to control the R_{0ij0}-components of the curvature tensor in the result for Dirac-wave maps with curvature term. This curvature contribution appears when computing the curvature of the spinor bundle. This explains why we have to assume bounded geometry of the Lorentzian manifold in Theorem 1.1 but only bounded geometry of the Riemannian slices in Theorem 1.3.

In addition, we also have to demand a decay of the second fundamental form (1.4) in Theorem 1.1 which originates from the choice of a positive definite scalar product on the spinor bundle that is no longer metric.

Remark 1.5 It seems that the class of Lorentzian manifolds that we are considering is the appropriate setting that guarantees a nice long-time behavior for solutions of various nonlinear wave equations with small initial data. We therefore think that our approach can be also used to establish existence results for a large class of other second order hyperbolic systems arising in mathematical physics.

Example 1.6 There are many spacetimes that satisfy the assumptions in Theorems 1.1 and 1.3. The simplest class is the class of Robertson–Walker spacetimes $-dt^2 + s^2(t)g$ with $s^{-1} \in L^1([0, \infty))$ which already contains the de-Sitter space and the power-law inflation metric. More generally, it is believed that generic future geodesically complete solutions of the Einstein equation with positive cosmological constant $\Lambda > 0$ satisfy the assumptions of our theorems.

Throughout this article we will employ the following notation: We will use small Greek letters α β γ for space-time indices, small Latin indices i j k for spatial derivatives and capital Latin indices I J K for indices on the Riemannian target. We will denote spatial derivative by D. Moreover, we will make use of the usual summation convention, that is we will always sum over repeated indices.

This article is organized as follows: In the second section we introduce a suitable conformal transformation that we will be using to prove our main results. In the third section we establish the necessary energy estimate which is the key tool to prove the main result.

2 Conformal Euler–Lagrange equations

In the following we calculate how the energy functional (1.1) transforms under a certain conformal transformation of the metric on M.

Lemma 2.1 *If we transform the metric $\tilde{h} = (Ns)^2 h$ and the vector spinors via*

$$\tilde{\psi} := (Ns)^{\frac{1-n}{2}} \psi$$

the energy functional (1.1) acquires the form

$$S(\phi, \psi, h) = \frac{1}{2} \int_M \left((Ns)^{n-2} |d\phi|^2 + \langle \psi, i \slashed{D} \psi \rangle - (Ns)^{2-n} \frac{1}{6} \langle \psi, R^P(\psi, \psi)\psi \rangle \right) dV_h.$$

$$(2.1)$$

Proof Under a conformal change of the metric $\bar{h} = (Ns)^2 h$ the volume elements transform as $dV_{\bar{h}} = (Ns)^n \, dV_h$. The Dirac operators transform as

$$\bar{\slashed{D}} \left((Ns)^{-\frac{n-1}{2}} \tilde{\psi} \right) = (Ns)^{-\frac{n+1}{2}} \overline{\slashed{D}\psi}.$$

Note that the twisted Dirac operator $\slashed{D}$ transforms in the same way as the standard Dirac operator $\slashed{\partial}$ since the twist bundle $\phi^* TP$ does not depend on the metric on M.

Inserting our choice for $\tilde{\psi}$ we get

$$\left\langle \tilde{\psi}, i \tilde{\slashed{D}} \tilde{\psi} \right\rangle = (Ns)^{-n} \langle \psi, i \slashed{D} \psi \rangle,$$

which completes the proof. □

For the sake of completeness we calculate the critical points of (2.1).

Lemma 2.2 (Critical points) *The critical points of* (2.1) *are given by*

$$\Box_h \phi = (n-2)(Ns)^{-1} \nabla_{\nabla(Ns)} \phi - \frac{1}{2}(Ns)^{2-n} h^{\alpha\beta} R^P \left(\psi, i\partial_\alpha \cdot \psi\right) d\phi(\partial_\beta)$$

$$+ \frac{1}{12}(Ns)^{4-2n} \left\langle \left(\nabla R^P\right)^\sharp (\psi, \psi)\psi, \psi \right\rangle,$$

$$i \not{D} \psi = (Ns)^{2-n} \frac{1}{3} R^P (\psi, \psi)\psi. \tag{2.2}$$

Here, $\Box := -h^{\alpha\beta} \nabla_{\partial_\alpha} d\phi(\partial_\beta)$ *denotes the wave map operator.*

Proof First, we vary the vector spinors ψ keeping the map ϕ fixed. More precisely, we consider a variation of $\psi: (-\varepsilon, \varepsilon) \times M \to SM \otimes \phi^* TP$ denoted by ψ_λ satisfying $\frac{\nabla \psi_\lambda}{\partial \lambda}\big|_{\lambda=0} = \xi$. We calculate

$$\frac{d}{d\lambda}\big|_{\lambda=0} \frac{1}{2} \int_M \left(\langle \psi_\lambda, i \not{D} \psi_\lambda \rangle - (Ns)^{2-n} \frac{1}{6} \left\langle \psi_\lambda, R^P (\psi_\lambda, \psi_\lambda) \psi_\lambda \right\rangle \right) dV_h$$

$$= \frac{1}{2} \int_M \left(\langle \xi, i \not{D} \psi \rangle + \langle \psi, i \not{D} \xi \rangle - (Ns)^{2-n} \frac{4}{6} \left\langle \xi, R^P (\psi, \psi)\psi \right\rangle \right) dV_h$$

$$= \int_M \left(\mathrm{Re} \langle \xi, i \not{D} \psi \rangle - (Ns)^{2-n} \frac{1}{3} \left\langle \xi, R^P (\psi, \psi)\psi \right\rangle \right) dV_h,$$

yielding the equation for the vector spinor.

Afterwards, we keep the vector spinors ψ fixed and consider a variation of $\phi: (-\varepsilon, \varepsilon) \times M \to P$ denoted by ϕ_λ satisfying $\frac{\partial \phi_\lambda}{\partial \lambda}\big|_{t=0} = \eta$. We calculate

$$\frac{d}{d\lambda}\big|_{\lambda=0} \frac{1}{2} \int_M (Ns)^{n-2} |d\phi_\lambda|^2 \, dV_h$$

$$= \int_M (Ns)^{n-2} \left\langle \nabla \frac{\partial \phi_\lambda}{\partial \lambda}, d\phi_\lambda \right\rangle dV_h \big|_{\lambda=0}$$

$$= \int_M \left((Ns)^{n-2} \langle \eta, \Box \phi \rangle - (n-2)(Ns)^{n-3} \nabla(Ns) \langle \eta, \nabla \phi \rangle \right) dV_h.$$

Moreover, we have

$$\frac{d}{d\lambda}\big|_{\lambda=0} \frac{1}{2} \int_M \left(\langle \psi, i \not{D} \psi \rangle - (Ns)^{2-n} \frac{1}{6} \left\langle \psi, R^P (\psi, \psi)\psi \right\rangle \right) dV_h$$

$$= \frac{1}{2} \int_M h^{\alpha\beta} \left\langle \psi, i R^P (\eta, d\phi(\partial_\alpha)) \partial_\beta \cdot \psi \right\rangle dV_h$$

$$+ \frac{1}{12} \int_M (Ns)^{2-n} \left\langle \left\langle \left(\nabla R^P\right)^\sharp (\psi, \psi)\psi, \psi \right\rangle, \eta \right\rangle dV_h,$$

where we used the equation for ψ. Finally, we have

$$h^{\alpha\beta} \left\langle \psi, i R^P (\eta, d\phi(\partial_\alpha)) \partial_\beta \cdot \psi \right\rangle = h^{\alpha\beta} \left\langle \eta, R^P (\psi, i\partial_\alpha \cdot \psi) d\phi(\partial_\beta) \right\rangle,$$

which completes the proof. $\Box$

We want to turn the system (2.2) into a system of two wave-type equations. To this end we recall the following Weitzenböck formula for the twisted Dirac operator $\not{D}$.

Lemma 2.3 *The square of the twisted Dirac operator $\not{D}$ satisfies*

$$\not{D}^2 = \Box + \frac{\mathrm{scal}^M}{4} + \frac{1}{2}h^{\alpha\gamma}h^{\beta\delta}\partial_\alpha \cdot \partial_\beta \cdot R^P\left(d\phi\left(\partial_\gamma\right), d\phi\left(\partial_\delta\right)\right)\psi. \tag{2.3}$$

Proof For a proof the we refer to [19, Theorem 8.17]. □

Making use of (2.3) we obtain the following rescaled Euler–Lagrange equations

$$\Box_h\phi = (n-2)(Ns)^{-1}\nabla_{\nabla(Ns)}\phi - \frac{1}{2}(Ns)^{2-n}h^{\alpha\beta}R^P(\psi, i\partial_\alpha \cdot \psi)d\phi(\partial_\beta)$$
$$+ \frac{1}{12}(Ns)^{4-2n}\left\langle\left(\nabla R^P\right)^{\sharp}(\psi,\psi)\psi,\psi\right\rangle, \tag{2.4}$$

$$\Box_h\psi = -\frac{\mathrm{scal}^M}{4}\psi - \frac{1}{2}h^{\alpha\gamma}h^{\beta\delta}\partial_\alpha \cdot \partial_\beta \cdot R^P\left(d\phi\left(\partial_\gamma\right), d\phi\left(\partial_\delta\right)\right)\psi$$
$$+ \frac{i}{3}\nabla\left(R^P(\psi,\psi)(Ns)^{2-n}\right)\cdot\psi$$
$$+ \frac{1}{9}(Ns)^{4-2n}R^P(\psi,\psi)R^P(\psi,\psi)\psi, \tag{2.5}$$

where $\Box_h$ denotes the corresponding wave-operator.

In terms of local coordinates this system acquires the form $(I = 1,\ldots,\dim P)$

$$s^2\partial_t^2\phi^I + s\dot{s}\partial_t\phi^I + \frac{1}{2}s^2\,\mathrm{tr}\,\dot{g}\,\partial_t\phi^I + D^*D\phi^I$$
$$= -(n-2)\dot{s}s\partial_t\phi^I - (n-2)s^2N^{-1}\partial_tN\partial_t\phi^I + (n-2)N^{-1}D_{\partial_i}ND_{\partial_i}\phi^I$$
$$- \frac{1}{2}(Ns)^{2-n}R^I_{JKL}h^{\alpha\beta}\left\langle\psi^K, i\partial_\alpha \cdot \psi^L\right\rangle\partial_\beta\phi^J$$
$$+ \frac{1}{12}(Ns)^{4-2n}G^{IJ}\nabla_J R_{KLMN}\left\langle\psi^K,\psi^M\right\rangle\left\langle\psi^L,\psi^N\right\rangle,$$

$$s^2\nabla_t^2\psi^I + s\dot{s}\nabla_t\psi^I + \frac{1}{2}s^2\,\mathrm{tr}\,\dot{g}\,\nabla_t\psi^I + D^*D\psi^I$$
$$= -\frac{\mathrm{scal}^M}{4}\psi^I - \frac{1}{2}h^{\alpha\gamma}h^{\beta\delta}R^I_{JKL}\partial_\gamma\phi^K\partial_\delta\phi^L\partial_\alpha \cdot \partial_\beta \cdot \psi^J$$
$$+ \frac{i}{3}h^{\alpha\beta}\nabla_{\partial_\alpha}\left((Ns)^{2-n}R^I_{JKL}\left\langle\psi^J,\psi^L\right\rangle\partial_{y^I}\right)\partial_\beta \cdot \psi^K$$
$$+ \frac{1}{9}(Ns)^{4-2n}R^I_{JKL}R^K_{MRS}\left\langle\psi^J,\psi^L\right\rangle\left\langle\psi^M,\psi^S\right\rangle\psi^R.$$

Remark 2.4 Note that the system (1.2), (1.3) on the manifold $(M,\tilde{h})$ is equivalent to the system (2.2) on the conformally transformed manifold (M,h). From now on we will use the system (2.4), (2.5) which follows from (2.2).

3 The energy estimate

In this section we first develop several formulas that are useful in the study of energy estimates for sections in arbitrary vector bundles. Later on, we will apply these techniques to the cases of wave maps and Dirac-wave maps with curvature term.

3.1 Energies on general vector bundles

Let Σ^n be a manifold, $s : \mathbb{R} \to \mathbb{R}_+$ smooth and $g_t, t \in \mathbb{R}$ be a smooth family of Riemannian metrics on Σ. We consider the (globally hyperbolic) Lorentzian manifold

$$(M, h) = \left(\mathbb{R} \times \Sigma, -s(t)^{-2}dt^2 + g_t\right).$$

The non-vanishing Christoffel symbols of this metric are given by

$$\Gamma(h)_{00}^0 = -\frac{\dot{s}}{s}, \qquad \Gamma(h)_{ij}^0 = \frac{1}{2}s^2\dot{g}_{ij}, \qquad \Gamma(h)_{i0}^j = \Gamma(h)_{0i}^j = \frac{1}{2}g^{jk}\dot{g}_{ik}, \qquad \Gamma(h)_{ij}^k = \Gamma(g)_{ij}^k.$$

Let V be a Riemannian vector bundle over M which is equipped with a metric connection ∇. As usual, iterating yields a map

$$\nabla^k : \Gamma(V) \to \Gamma\left(T^*M^{\otimes k} \otimes V\right).$$

We define the associated wave operator $\square : \Gamma(V) \to \Gamma(V)$ by the sign convention such that

$$\square\xi = -h^{\alpha\beta}\nabla^2_{\alpha\beta}\xi$$

for $\xi \in \Gamma(V)$. The covariant derivative ∇ restricts for each $t \in \mathbb{R}$ to the spatial covariant derivative D which yields a map

$$D : \Gamma(V) \to \Gamma\left(\pi^*\left(T^*\Sigma\right) \otimes V\right),$$

where $\pi : M \to \Sigma$ is the canonical projection. In the following, we will write $T^*\Sigma$ instead of $\pi^*(T^*\Sigma)$ for notational convenience. The covariant derivative naturally extends as a map

$$D : \Gamma\left(T^*\Sigma^{\otimes k} \otimes V\right) \to \Gamma\left(T^*\Sigma^{\otimes k+1} \otimes V\right)$$

by defining

$$(D_X\xi)(X_1, \ldots, X_k) = {}^V\nabla_X(\xi(X_1, \ldots, X_k)) - \sum_{i=1}^k \xi(X_1, \ldots, D_X X_i, \ldots, X_k),$$

where for each $t \in \mathbb{R}$, $D_X X_i$ denotes the covariant derivative of g_t. Furthermore, we define its formal adjoint $D^* : \Gamma(T^*\Sigma^{\otimes k+1} \otimes V) \to \Gamma(T^*\Sigma^{\otimes k} \otimes V)$ by

$$D^*\xi(X_1, \ldots, X_k) = -g^{ij}D_{\partial_i}\xi(\partial_j, X_1, \ldots, X_k).$$

Finally, we define a covariant time-derivative $\nabla_t : \Gamma(T^*\Sigma^{\otimes k} \otimes V) \to \Gamma(T^*\Sigma^{\otimes k} \otimes V)$ by

$$(\nabla_t\xi)(X_1, \ldots, X_k) = \nabla^V_{\partial_t}(\xi(X_1, \ldots, X_k)) - \sum_{i=1}^k \xi(X_1, \ldots, \nabla_{\partial_t}X_i, \ldots, X_k).$$

Observe that this definition makes sense as $\nabla_{\partial_t}X_i \in \Gamma(TM)$ is always tangential to Σ due to the structure of the metric h. A quick computation shows that the wave operator can be written as

$$\square\xi = s^2\nabla_t\nabla_t\xi + \dot{s}s\nabla_t\xi + \frac{1}{2}s^2\,\mathrm{tr}_g\,\dot{g}\cdot\nabla_t\xi + D^*D\xi. \tag{3.1}$$

Lemma 3.1 *We use the notations from above. Let $\langle .,. \rangle$ be the natural t-dependent scalar product induced on $\Gamma(T^*\Sigma^{\otimes k} \otimes V)$. Let $\xi, \eta \in \Gamma(V)$. Then we have the product rule*

$$\partial_t\left\langle D^k\xi, D^k\eta\right\rangle = \left\langle \nabla_t D^k\xi, D^k\eta\right\rangle + \left\langle D^k\xi, \nabla_t D^k\eta\right\rangle.$$

Proof Let t_0 be a fixed time and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of (Σ, g_{t_0}) at $x \in \Sigma$. Then the scalar product can also be written as

$$\left\langle D^k \xi, D^k \eta \right\rangle (t_0, x) = \sum_{i_1, \ldots, i_k = 1}^{n} \left\langle D^k_{e_{i_1}, \ldots, e_{i_k}} \xi, D^k_{e_{i_1}, \ldots, e_{i_k}} \eta \right\rangle (t_0, x),$$

where the scalar products on the right hand side are the ones on V. Now think of $\{e_1, \ldots, e_n\}$ as an orthonormal system in $T_{(t_0, x)} M$. Parallel transport along the curve $t \mapsto (t, x)$ yields orthonormal systems $\{e_1, \ldots, e_n\}$ for each $T_{(t, x)} M$. Because $\nabla_{\partial_t} \partial_t = -\frac{\dot{s}}{s} \partial_t$, $h(\partial_t, e_i) \equiv 0$ for each $i \in \{1, \ldots, n\}$ so that we in fact obtain orthonormal bases of $T_x \Sigma$ with respect to the metric g_t for each $t \in \mathbb{R}$. Therefore we get

$$\partial_t \left\langle D^k \xi, D^k \eta \right\rangle = \sum_{i_1, \ldots, i_k = 1}^{n} \left\langle \nabla_t \left(D^k_{e_{i_1}, \ldots, e_{i_k}} \xi \right), D^k_{e_{i_1}, \ldots, e_{i_k}} \eta \right\rangle$$

$$+ \sum_{i_1, \ldots, i_k = 1}^{n} \left\langle D^k_{e_{i_1}, \ldots, e_{i_k}} \xi, \nabla_t \left(D^k_{e_{i_1}, \ldots, e_{i_k}} \eta \right) \right\rangle$$

$$= \sum_{i_1, \ldots, i_k = 1}^{n} \left\langle \nabla_t D^k_{e_{i_1}, \ldots, e_{i_k}} \xi, D^k_{e_{i_1}, \ldots, e_{i_k}} \eta \right\rangle + \sum_{i_1, \ldots, i_k = 1}^{n} \left\langle D^k_{e_{i_1}, \ldots, e_{i_k}} \xi, \nabla_t D^k_{e_{i_1}, \ldots, e_{i_k}} \eta \right\rangle$$

$$= \left\langle \nabla_t D^k \xi, D^k \eta \right\rangle + \left\langle D^k \xi, \nabla_t D^k \eta \right\rangle.$$

$\square$

Remark 3.2 In the above lemma, the special structure of the metric h is essential.

For a section $\xi \in \Gamma(V)$, we now define the kth energy density

$$e_k(\xi) = s^2 \left| D^k \nabla_t \xi \right|^2 + \left| D^{k+1} \xi \right|^2,$$

which is a nonnegative function on M and the kth energy as

$$E_k(\xi)(t) = \int_\Sigma e_k(\xi) \, dV_{g_t},$$

where dV_{g_t} is the volume element of the metric g_t and the integral has to be understood as an integral over $\{t\} \times \Sigma$.

Proposition 3.3 Let $\xi \in \Gamma(V)$ be spacelike compactly supported. Then its kth energy satisfies the evolution equation

$$\frac{d}{dt} E_k(\xi) = 2 \int_\Sigma \left\langle D^k \Box \xi, D^k \nabla_t \xi \right\rangle dV_{g_t} + \frac{1}{2} \int_\Sigma \left(\left| D^{k+1} \xi \right|^2 - s^2 \left| D^k \nabla_t \xi \right|^2 \right) \operatorname{tr}_g \dot{g} \, dV_{g_t}$$

$$+ 2s^2 \int_\Sigma \left\langle D^k \nabla_t \xi, \left[\nabla_t, D^k \right] \nabla_t \xi \right\rangle dV_{g_t} + 2 \int_\Sigma \left\langle D^k \nabla_t \xi, \left[D^* D, D^k \right] \xi \right\rangle dV_{g_t}$$

$$+ 2 \int_\Sigma \left\langle D^{k+1} \xi, \left[\nabla_t, D^{k+1} \right] \xi \right\rangle dV_{g_t}.$$

Proof We compute

$$\frac{d}{dt} E_k(\xi) = 2\dot{s} s \int_\Sigma \left| D^k \nabla_t \xi \right|^2 dV_{g_t} + \frac{1}{2} \int_\Sigma e_k(\xi) \operatorname{tr}_g \dot{g} \, dV_{g_t}$$

$$+ 2s^2 \int_\Sigma \left\langle \nabla_t D^k \nabla_t \xi, D^k \nabla_t \xi \right\rangle dV_{g_t} + 2 \int_\Sigma \left\langle \nabla_t D^{k+1} \xi, D^{k+1} \xi \right\rangle dV_{g_t}$$

$$= 2\dot{s}s \int_\Sigma \left| D^k \nabla_t \xi \right|^2 dV_{g_t} + \frac{1}{2} \int_\Sigma e_k(\xi) \operatorname{tr}_g \dot{g} \, dV_{g_t}$$

$$+ 2s^2 \int_\Sigma \left\langle D^k \nabla_t \nabla_t \xi, D^k \nabla_t \xi \right\rangle dV_{g_t} + 2s^2 \int_\Sigma \left\langle \left[\nabla_t, D^k \right] \nabla_t \xi, D^k \nabla_t \xi \right\rangle dV_{g_t}$$

$$+ 2 \int_\Sigma \left\langle D^{k+1} \nabla_t \xi, D^{k+1} \xi \right\rangle dV_{g_t} + 2 \int_\Sigma \left\langle \left[\nabla_t, D^{k+1} \right] \xi, D^{k+1} \xi \right\rangle dV_{g_t}.$$

Moreover, we have

$$2 \int_\Sigma \left\langle D^{k+1} \nabla_t \xi, D^{k+1} \xi \right\rangle dV_{g_t} = 2 \int_\Sigma \left\langle D^k \nabla_t \xi, D^* D^{k+1} \xi \right\rangle dV_{g_t}$$

$$= 2 \int_\Sigma \left\langle D^k \nabla_t \xi, D^k D^* D \xi \right\rangle dV_{g_t}$$

$$+ 2 \int_\Sigma \left\langle D^k \nabla_t \xi, \left[D^* D, D^k \right] \xi \right\rangle dV_{g_t}.$$

The statement follows by combining both equalities and using (3.1). □

In order to derive energy estimates for the vector spinors we have to take into account the special structure of the scalar product of the spinor bundle SM. The natural geometric scalar product which is invariant under the spin group is not positive definite and thus not a good candidate for analytic purposes. To obtain a positive definite scalar product one has to Clifford multiply the second factor with the timelike unit vector field e_0, see [3], which in our setup is given by $e_0 = s \partial_t$. Using our geometric setup we find

$$\nabla_t e_0 = 0, \qquad \nabla_i e_0 = \frac{1}{2} s g^{jk} \dot{g}_{ik} \partial_{x^j}.$$

Hence, we define the following kth energy density for the vector spinors ψ

$$e_k(\psi) = s^2 \left\langle D^k \nabla_t \psi, e_0 \cdot D^k \nabla_t \psi \right\rangle + \left\langle D^{k+1} \psi, e_0 \cdot D^{k+1} \psi \right\rangle,$$

which is a nonnegative function. In the following we will always employ the positive definite scalar product without mentioning it explicitly. However, since e_0 is not parallel with respect to the spatial coordinates the kth energy of the spinor satisfies

$$\frac{d}{dt} E_k(\psi) = 2 \int_\Sigma \left\langle D^k \Box \psi, D^k \nabla_t \psi \right\rangle dV_{g_t} + \frac{1}{2} \int_\Sigma \left(\left| D^{k+1} \psi \right|^2 - s^2 \left| D^k \nabla_t \psi \right|^2 \right) \operatorname{tr}_g \dot{g} \, dV_{g_t}$$

$$+ 2s^2 \int_\Sigma \left\langle D^k \nabla_t \psi, \left[\nabla_t, D^k \right] \nabla_t \psi \right\rangle dV_{g_t} + 2 \int_\Sigma \left\langle D^k \nabla_t \psi, \left[D^* D, D^k \right] \psi \right\rangle dV_{g_t}$$

$$+ 2 \int_\Sigma \left\langle D^{k+1} \psi, \left[\nabla_t, D^{k+1} \right] \psi \right\rangle dV_{g_t} + 2 \int_\Sigma \left\langle D^k \psi, e_0 \cdot \left(D^* e_0 \right) \cdot D^{k+1} \psi \right\rangle dV_{g_t}.$$

$$(3.2)$$

Lemma 3.4 *Let* $\xi \in \Gamma(T^* \Sigma^{\otimes k} \otimes V)$. *Then we have the identity*

$$([\nabla_t, D] \, \xi) \, (X, X_1, \ldots, X_k) = R^V_{\partial_t, X}(\xi(X_1, \ldots, X_k)) - \frac{1}{2} D_{\dot{g}(X)} \xi(X_1, \ldots, X_k)$$

$$+ \frac{1}{2} \sum_{i=1}^k \xi \, (X_1, \ldots, D_X \dot{g}(X_i), \ldots, X_k) \, .$$

Proof At first, we compute

$$
\begin{aligned}
&(\nabla_t D\xi)\,(X, X_1, \ldots, X_k)\\
&\quad = \nabla^V_{\partial_t}(D\xi(X, X_1, \ldots, X_k)) - (D\xi)(\nabla_{\partial_t} X, X_1, \ldots, X_k)\\
&\qquad - \sum_{i=1}^{k}(D\xi)(X, X_1, \ldots, \nabla_{\partial_t} X_i, \ldots, X_k)\\
&\quad = \nabla^V_{\partial_t}\left(\nabla^V_X(\xi(X_1, \ldots, X_k)) - \sum_{i=1}^{k}\xi(X_1, \ldots, D_X X_i, \ldots X_k)\right)\\
&\qquad - \left(D_{\nabla_{\partial_t} X}\xi\right)(X_1, \ldots, X_k) - \sum_{i=1}^{k}(D_X\xi)(X_1, \ldots, \nabla_{\partial_t} X_i, \ldots, X_k)\\
&\quad = \nabla^V_{\partial_t}\left(\nabla^V_X(\xi(X_1, \ldots, X_k))\right) - \sum_{i=1}^{k}(\nabla_t\xi)(X_1, \ldots, D_X X_i, \ldots X_k)\\
&\qquad - \sum_{i=1}^{k}\xi\left(X_1, \ldots, \nabla_{\partial_t} D_X X_i, \ldots X_k\right) - \sum_{\substack{i,j=1\\i\neq j}}^{n}\xi\left(X_1, \ldots, D_X X_i, \ldots, \nabla_{\partial_t} X_j, \ldots X_k\right)\\
&\qquad - \left(D_{\nabla_{\partial_t} X}\xi\right)(X_1, \ldots, X_k) - \sum_{i=1}^{k}(D_X\xi)\left(X_1, \ldots, \nabla_{\partial_t} X_i, \ldots, X_k\right).
\end{aligned}
$$

Similarly, we find

$$
\begin{aligned}
&(D\,(\nabla_t\xi))\,(X, X_1, \ldots, X_k)\\
&\quad = \nabla^V_X((\nabla_t\xi)\,(X_1, \ldots, X_k)) - \sum_{i=1}^{k}(\nabla_t\xi)\,(X_1, \ldots, D_X X_i, \ldots, X_k)\\
&\quad = \nabla^V_X\left(\nabla^V_{\partial_t}(\xi\,(X_1, \ldots, X_k)) - \sum_{i=1}^{k}\xi\left(X_1, \ldots, \nabla_{\partial_t} X_i, \ldots, X_k\right)\right)\\
&\qquad - \sum_{i=1}^{k}(\nabla_t\xi)(X_1, \ldots, D_X X_i, \ldots, X_k)\\
&\quad = \nabla^V_X\left(\nabla^V_{\partial_t}(\xi(X_1, \ldots, X_k))\right) - \sum_{i=1}^{k}(D_X\xi)\left(X_1, \ldots, \nabla_{\partial_t} X_i, \ldots X_k\right)\\
&\qquad - \sum_{i=1}^{k}\xi\left(X_1, \ldots, D_X\nabla_{\partial_t} X_i, \ldots X_k\right) - \sum_{\substack{i,j=1\\i\neq j}}^{n}\xi\left(X_1, \ldots, D_X X_i, \ldots, \nabla_{\partial_t} X_j, \ldots X_k\right)\\
&\qquad - \sum_{i=1}^{k}(\nabla_t\xi)\,(X_1, \ldots, D_X X_i, \ldots, X_k).
\end{aligned}
$$

Summing up, we obtain

$$([\nabla_t, D]\xi)(X, X_1, \ldots, X_k) = \nabla^V_{\partial_t}\left(\nabla^V_X(\xi(X_1, \ldots, X_k))\right) - \nabla^V_X\left(\nabla^V_{\partial_t}(\xi(X_1, \ldots, X_k))\right)$$

$$- \sum_{i=1}^{k} \xi\left(X_1, \ldots, \nabla_{\partial_t} D_X X_i - D_X \nabla_{\partial_t} X_i, \ldots X_k\right)$$

$$- \left(D_{\nabla_{\partial_t} X}\xi\right)(X_1, \ldots, X_k).$$

As this expression is tensorial in X and the X_i, we may assume that the components of these vector fields with respect to a chart of Σ are independent of time. By raising an index with respect to g_t we can think of $\dot{g}_t$ as an endomorphism on $T^*\Sigma$. Then the formulas for the Christoffel symbols imply $\nabla_{\partial_t} X = \nabla_X \partial_t = \frac{1}{2}\dot{g}(X)$ and the same holds for X_i. Moreover, we have $[\partial_t, X] = [\partial_t, X_i] = 0$ and

$$\nabla_{\partial_t} D_X X_i - D_X \nabla_{\partial_t} X_i = \frac{1}{2}\dot{g}\left(D_X X_i\right) - D_X\left(\dot{g}\left(X_i\right)\right) = -\frac{1}{2}D_X\dot{g}\left(X_i\right).$$

By putting these facts together, we immediately get the statement of the Lemma. $\qquad\square$

In the following we will often make use of the so-called $\star$ notation. More precisely, we will use a $\star$ to denote various contractions between the objects involved.

We apply the general formula from above in the case where we have a map $\phi \in C^\infty(M, P)$, $[\nabla_t, D]\phi \in \Gamma(T^*\Sigma \otimes \phi^* TP)$ and we get

$$([\nabla_t, D]\phi)(X) = -\frac{1}{2}d\phi(\dot{g}(X)).$$

More generally, if $E = \phi^* TP$ in the Lemma above, we get

$$([\nabla_t, D]\xi)(X, X_1, \ldots, X_k) = R^P(d\phi(\partial_t), d\phi(X))(\xi(X_1, \ldots, X_k)) - \frac{1}{2}D_{\dot{g}(X)}\xi(X_1, \ldots, X_k)$$

$$+ \frac{1}{2}\sum_{i=1}^{k} \xi(X_1, \ldots, D_X\dot{g}(X_i), \ldots, X_k)$$

$$= R^P(\nabla_t\phi, D_X\phi)(\xi(X_1, \ldots, X_k)) - \frac{1}{2}D_{\dot{g}(X)}\xi(X_1, \ldots, X_k)$$

$$+ \frac{1}{2}\sum_{i=1}^{k} \xi(X_1, \ldots, D_X\dot{g}(X_i), \ldots, X_k).$$

By an iteration argument, we find

$$\left[\nabla_t, D^k\right]\nabla_t\phi = \sum_{\sum l_i + \sum m_j = k-1} {}^G\nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1-\text{times}} \star D^{l_2}\nabla_t\phi \star D^{l_3+1}\phi \star D^{l_4}\nabla_t\phi$$

$$+ \sum_{l=0}^{k-1} D^{k-l}\dot{g} \star D^l\nabla_t\phi,$$

$$\left[\nabla_t, D^{k+1}\right]\phi = \sum_{\sum l_i + \sum m_j = k-1} {}^G\nabla^{l_1} R^N \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1-\text{times}} \star D^{l_2}\nabla_t\phi \star D^{l_3+1}\phi \star D^{l_4+1}\phi$$

$$+ \sum_{l=0}^{k} D^{k-l}\dot{g} \star D^{l+1}\phi.$$

In the case of a vector spinor $\psi \in \Gamma(SM \otimes \phi^*TP)$, the formulas are

$$([\nabla_t, D]\,\psi)\,(X) = R^{SM}(\partial_t, X) \cdot \psi + R^P(\nabla_t\phi, D_X\phi)\,\psi + \frac{1}{2}D\phi\,(\dot{g}(X))$$

and

$$
\begin{aligned}
([\nabla_t, D]\,\xi)\,(X, X_1, \ldots, X_k) \\
= R^{SM}(\partial_t, X) \cdot \xi + R^P(\nabla_t\phi, D_X\phi)\,(\xi(X_1, \ldots, X_k)) \\
- \frac{1}{2}D_{\dot{g}(X)}\xi(X_1, \ldots, X_k) + \frac{1}{2}\sum_{i=1}^{k}\xi(X_1, \ldots, D_X\dot{g}(X_i), \ldots, X_k).
\end{aligned}
$$

By iterating the formula from above, we get

$$
\begin{aligned}
\left[\nabla_t, D^k\right]\nabla_t\psi = {} & \sum_{l=0}^{k-1} D^l\left(R^{SM}(\partial_t, \cdot)\right)\star D^{k-1-l}\nabla_t\psi + \sum_{l=0}^{k-1} D^{k-l}\dot{g}\star D^{l+1}\nabla_t\psi \\
& + \sum_{\sum l_i + \sum m_j = k-1} {}^G\nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi\star\cdots\star D^{m_{l_1}+1}\phi}_{l_1-\text{times}}\star D^{l_2}\nabla_t\phi\star D^{l_3+1}\phi\star D^{l_4}\nabla_t\psi
\end{aligned}
$$

and

$$
\begin{aligned}
\left[\nabla_t, D^{k+1}\right]\psi = {} & \sum_{l=0}^{k} D^l\left(R^{SM}(\partial_t, \cdot)\right)\star D^{k-l}\psi + \sum_{l=0}^{k} D^{k-l+1}\dot{g}\star D^{l+1}\psi \\
& + \sum_{\sum l_i + \sum m_j = k} {}^G\nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi\star\cdots\star D^{m_{l_1}+1}\phi}_{l_1-\text{times}}\star D^{l_2}\nabla_t\phi\star D^{l_3+1}\phi\star D^{l_4}\psi.
\end{aligned}
$$

Lemma 3.5 *Let $\xi \in \Gamma(T^*\Sigma^{\otimes k} \otimes V)$. Then we have the identity*

$$
\begin{aligned}
\left([D^*D, D]\,\xi\right)(X, X_1, \ldots, X_k) = {} & -D_{\mathrm{Ric}(X)}\xi(X_1, \ldots, X_k) \\
& - 2\sum_{j=1}^{k} D_{e_i}\xi(X_1, \ldots R_{X,e_i}X_j, \ldots, X_k) \\
& + D_{e_i}R^V_{X,e_i}(\xi(X_1, \ldots, X_k)) + 2R^V_{X,e_i}\left(D_{e_i}\xi(X_1, \ldots, X_k)\right) \\
& - \sum_{j=1}^{k}\xi\left(X_1, \ldots D_{e_i}R_{X,e_i}X_j, \ldots, X_k\right),
\end{aligned}
$$

where $\{e_i\}_{1 \le i \le n-1}$ is a local orthonormal frame. Here and throughout the proof, we sum over i.

Proof A direct computation yields

$$
\begin{aligned}
\left([D^*D, D]\,\xi\right)(X, X_1, \ldots, X_k) \\
= D^3_{X,e_i,e_i}\xi(X_1, \ldots, X_k) - D^3_{e_i,e_i,X}\xi(X_1, \ldots, X_k) \\
= D^3_{X,e_i,e_i}\xi(X_1, \ldots, X_k) - D^3_{e_i,X,e_i}\xi(X_1, \ldots, X_k) \\
+ D^3_{e_i,X,e_i}\xi(X_1, \ldots, X_k) - D^3_{e_i,e_i,X}\xi(X_1, \ldots, X_k) \\
= R_{X,e_i}D\xi(e_i, X_1, \ldots, X_k) + D_{e_i}R_{X,e_i}\xi(X_1, \ldots, X_k)
\end{aligned}
$$

$$= -D_{\mathrm{Ric}(X)}\xi(X_1, \ldots, X_k) - \sum_{j=1}^{k} D_{e_i}\xi\left(X_1, \ldots R_{X,e_i}X_j, \ldots, X_k\right)$$

$$+ R_{X,e_i}\left(D_{e_i}\xi(X_1, \ldots, X_k)\right) + D_{e_i}\left(R^V \circ \xi\right)(X, e_i, X_1, \ldots, X_k)$$

$$- D_{e_i}\sum_{j=1}^{k}\xi\left(X_1, \ldots R_{X,e_i}X_j, \ldots, X_k\right)$$

$$= -D_{\mathrm{Ric}(X)}\xi(X_1, \ldots, X_k) - 2\sum_{j=1}^{k} D_{e_i}\xi\left(X_1, \ldots R_{X,e_i}X_j, \ldots, X_k\right)$$

$$+ D_{e_i}R^V_{X,e_i}(\xi(X_1, \ldots, X_k)) + 2R_{X,e_i}\left(D_{e_i}\xi(X_1, \ldots, X_k)\right)$$

$$- \sum_{j=1}^{k}\xi\left(X_1, \ldots D_{e_i}R_{X,e_i}X_j, \ldots, X_k\right).$$

Here, $R^V \circ \xi \in \Gamma(T^*\Sigma^{\otimes k+2} \otimes V)$ is defined as

$$\left(R^V \circ \xi\right)(X, Y, X_1, \ldots, X_k) = R^V_{X,Y}(\xi(X_1, \ldots, X_k))$$

and DR^V is the covariant derivative of the curvature endomorphism on V restricted to vectors tangential to Σ. In other words,

$$D_X R^V_{Y,Z}\xi := D_X\left(R^V_{Y,Z}\xi\right) - R^V_{D_X Y,Z}\xi - R^V_{Y,D_X Z}\xi - R^V_{Y,Z}\left(\nabla_X\xi\right).$$

$\square$

In the case $\phi \in C^\infty(M, P)$, we obtain

$$\left([D^*D, D]\phi\right)(X) = -D\phi\left(\mathrm{Ric}^M(X)\right) + R^P\left(D\phi(X), D\phi(e_i)\right)(D\phi(e_i))$$

and for $\xi \in \Gamma(T^*\Sigma^{\otimes k} \otimes \phi^*TP)$, we have

$$\left([D^*D, D]\xi\right)(X, X_1, \ldots X_k) = 2\sum_{l=1}^{k} D_{e_i}\xi\left(X_1, \ldots, R^M(e_i, X)X_l, \ldots X_k\right)$$

$$+ \sum_{l=1}^{k}\xi\left(X_1, \ldots, D_{e_i}R^M(e_i, X)X_l, \ldots X_k\right) - D_{\mathrm{Ric}(X)}\xi(X_1, \ldots, X_k)$$

$$+ D_{D\phi(e_i)}R^P(D\phi(X), D\phi(e_i))(\xi(X_1, \ldots X_k))$$

$$+ R^P\left(D^2\phi(e_i, X), D\phi(e_i)\right)(\xi(X_1, \ldots X_k))$$

$$+ R^P\left(D\phi(X), D^2\phi(e_i, e_i)\right)(\xi(X_1, \ldots X_k))$$

$$+ 2R^P\left(D\phi(X), D\phi(e_i)\right)\left(D_{e_i}\xi(X_1, \ldots, X_k)\right).$$

Thus by iteration we get

$$\left[D^*D, D^k\right]\phi = \sum_{l=0}^{k-1} D^l R^M \star D^{k-l}\phi$$

$$+ \sum_{\sum l_i + \sum m_j = k-1} {}^G\nabla^{l_1}R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1-\text{times}} \star D^{l_2+1}\phi \star D^{l_3+1}\phi \star D^{l_4+1}\phi.$$

In the case of vector spinors, that is $\psi \in \Gamma(SM \otimes \phi^* TP)$, we obtain

$$
\begin{aligned}
\left([D^*D, D]\psi\right)(X) = &-D_{\mathrm{Ric}(X)}\psi + 2R^P(D\phi(X), D\phi(e_i))(D\psi(e_i)) \\
&+ 2R^{SM}_{X,e_i} \cdot D_{e_i}\psi + D_{e_i} R^{SM}_{X,e_i} \cdot \psi \\
&+ \nabla_{D\phi(e_i)} R^P(D\phi(X), D\phi(e_i))\psi + R^P\left(D^2\phi(e_i, X), D\phi(e_i)\right)\psi \\
&+ R^P\left(D\phi(X), D^2\phi(e_i, e_i)\right)\psi
\end{aligned}
$$

and for $\xi \in \Gamma(T^*\Sigma^{\otimes k} \otimes SM \otimes \phi^* TP)$, we have

$$
\begin{aligned}
\left([D^*D, D]\xi\right)(X, X_1, \ldots, X_k) = &-D_{\mathrm{Ric}(X)}\xi(X_1, \ldots, X_k) \\
&+ 2R^P(D\phi(X), D\phi(e_i))\left(D_{e_i}\psi(X_1, \ldots, X_k)\right) \\
&+ 2R^{SM}_{X,e_i} \cdot D_{e_i}\psi(X_1, \ldots, X_k) + D_{e_i} R^{SM}_{X,e_i} \cdot \psi(X_1, \ldots, X_k) \\
&+ \nabla_{D\phi(e_i)} R^P(D\phi(X), D\phi(e_i))(\psi(X_1, \ldots, X_k)) \\
&+ R^P\left(D^2\phi(e_i, X), D\phi(e_i)\right)(\psi(X_1, \ldots, X_k)) \\
&+ R^P\left(D\phi(X), D^2\phi(e_i, e_i)\right)(\psi(X_1, \ldots, X_k)) \\
&+ 2\sum_{l=1}^{k} D_{e_i}\xi\left(X_1, \ldots, R^M(e_i, X)X_l, \ldots X_k\right) \\
&+ \sum_{l=1}^{k}\xi\left(X_1, \ldots, D_{e_i} R^M(e_i, X)X_l, \ldots X_k\right).
\end{aligned}
$$

By iteration, we then obtain

$$
\begin{aligned}
\left[D^*D, D^k\right]\psi = &\sum_{l=0}^{k-1} D^l R^M \star D^{k-l}\psi + \sum_{l=0}^{k} D^l R^{SM} \star D^{k-l}\psi \\
&+ \sum_{\sum l_i + \sum m_j = k} {}^G\nabla^{l_1} R^N \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1-\text{times}} \star D^{l_2+1}\phi \star D^{l_3+1}\phi \star D^{l_4}\psi.
\end{aligned}
$$

We conclude this section with a very important lemma, which we will frequently make use of when deriving energy estimates.

Lemma 3.6 *Let (M^m, g) be an m-dimensional Riemannian manifold, r a natural number satisfying $r > \frac{m}{2}$, $E \to M$ a Riemannian vector bundle with a connection, $k \in \mathbb{N}$, $\xi_1, \ldots \xi_k \in H^r(E)$ and l_i $i = 1, \ldots k$ natural numbers satisfying $\sum_i l_i \le r$. Then the following inequality holds*

$$
\int_M \mathrm{II}_{i=1}^k \left|\nabla^{l_i}\xi_i\right|^2 dV \le C_{Sob} \cdot \mathrm{II}_{i=1}^k \|\xi_i\|_{H^r}.
$$

If in addition, we have $\xi_{k+1} \in H^{r-1}(E)$ and $\sum_{i=1}^{k+1} l_i \le r - 1$, then

$$
\int_M \mathrm{II}_{i=1}^{k+1} \left|\nabla^{l_i}\xi_i\right|^2 dV \le C_{Sob} \cdot \mathrm{II}_{i=1}^k \|\xi_i\|_{H^r} \cdot \|\xi_{k+1}\|_{H^{r-1}}.
$$

Here, $C_{sob} = C_{sob}(g, l_1, \ldots, l_k, r)$ depends on the constant from the Sobolev embedding on M and the numbers $l_1, \ldots, l_k, r$.

Proof We prove the first inequality, the second one is shown very similarly. Choose $p_i \in [1, \infty]$, $i = 1, \ldots, l_k$ such that $\frac{1}{p_i} > \frac{1}{2} - \frac{r - l_i}{n}$ so that $\left\| \nabla^{l_i} \xi_i \right\|_{L^{p_i}} \leq C \left\| \xi_i \right\|_{H^r}$ by Sobolev embedding. Furthermore,

$$
\sum_{i=1}^{k} \left(\frac{1}{2} - \frac{r - l_i}{m} \right) = \frac{k}{2} - \frac{kr}{m} + \frac{1}{m} \left(\sum_{i=1}^{k} l_i \right)
$$

$$
= \frac{1}{2} + (k - 1) \left(\frac{1}{2} - \frac{r}{m} \right) + \frac{1}{m} \left(\sum_{i=1}^{k} l_i - r \right) < \frac{1}{2}
$$

due to our assumptions. Therefore, we can choose the p_i such that $\sum_{i=1}^{k} \frac{1}{p_i} = \frac{1}{2}$. An application of the Hölder inequality finishes the proof of the inequality. $\qquad\square$

3.2 Energy of the map

We define the kth energy density of the map part as

$$
e_k(\phi) = s^2 \left| D^k \partial_t \phi \right|^2 + \left| D^k D\phi \right|^2 \tag{3.3}
$$

and the kth energy as

$$
E_k(\phi) = \int_\Sigma e_k(\phi) \, dV_{g_t}. \tag{3.4}
$$

The kth total energy is given by

$$
F_k(\phi) = \sum_{l=0}^{k} E_l(\phi) = \int_\Sigma \sum_{l=0}^{k} e_l(\phi) \, dV_{g_t} = s^2 \|\nabla_t \phi\|_{H^k}^2 + \|D\phi\|_{H^k}^2, \tag{3.5}
$$

where the Sobolev norms are taken with respect to the metric g_t. In (3.9) below, we also define the total energy $F_k(\psi)$ of the spinor part, which we already need in the following proposition:

Proposition 3.7 *Let $T > 0$, $r \in \mathbb{N}$, $r > (n - 1)/2$, $k \in \{1, \ldots, r\}$ and (ϕ, ψ) be a solution of (2.4), (2.5) such that*

$$
\phi \in C^0 \left([0, T), H^{r+1}(\Sigma, P) \right) \cap C^1 \left([0, T), H^r(\Sigma, P) \right),
$$
$$
\psi \in C^0 \left([0, T), H^r \left(M, SM \otimes \phi^* TP \right) \right) \cap C^1 \left([0, T), H^{r-1} \left(M, SM \otimes \phi^* TP \right) \right).
$$

Then the kth energy of the map satisfies the following inequality

$$
\frac{d}{dt} E_k(\phi) \leq C_1(k) \left(\|\dot{g}\|_{C^{k-1}} + \|\partial_t \log N\|_{C^k} + s^{-1} \|D \log N\|_{C^k} + s^{-1} \|R^\Sigma\|_{C^{k-1}} \right) F_k(\phi)
$$

$$
- (n - 2) s \dot{s} \int_\Sigma \left| D^k \nabla_t \phi \right|^2 \, dV_{g_t}
$$

$$
+ C_2(k, n, g) s^{-1} \left\| R^P \right\|_{C^{k-1}} \sum_{l=1}^{k-1} F_r(\phi)^{2+l/2}
$$

$$
+ C_3(k, n, g) s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{1+l/2} F_{r-1}(\psi)
$$

$$+ C_4(k, n, g)s^{1-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{1+l/2} F_{r-1}(\psi)$$

$$+ C_5(k, n, g)s^{3-2n} \left\| N^{4-2n} \right\|_{C^k} \left\| R^P \right\|_{C^{k+1}} \sum_{l=0}^{k} F_r(\phi)^{1/2+l/2} F_{r-1}(\psi)^2,$$

$$(3.6)$$

where the positive constants C_i, $i \in \{1, \dots, 5\}$ *depend on* n, k *and the Sobolev constant of the metric* g_t.

Proof Assume for the moment that the initial data is compactly supported such that the solution is spacelike compactly supported. Using the general formula (3.3) we find

$$\frac{d}{dt} E_k(\phi) = 2 \int_\Sigma \left\langle D^k \Box_h \phi, D^k \nabla_t \phi \right\rangle dV_{g_t} + \frac{1}{2} \int_\Sigma \left(\left| D^k D\phi \right| - s^2 \left| D^k \nabla_t \phi \right|^2 \right) \operatorname{tr}_g \dot{g} \, dV_{g_t}$$

$$+ 2s^2 \int_\Sigma \left\langle \left[\nabla_t, D^k \right] \nabla_t \phi, D^k \nabla_t \phi \right\rangle dV_{g_t} + 2 \int_\Sigma \left\langle \left[D^* D, D^k \right] \phi, D^k \nabla_t \phi \right\rangle dV_{g_t}$$

$$+ 2 \int_\Sigma \left\langle \left[\nabla_t, D^{k+1} \right] \phi, D^{k+1} \phi \right\rangle dV_{g_t}.$$

Due to finite speed of propagation and an exhaustion procedure, this equality also holds generally for solutions in the above space. Note also that we will use Lemma 3.6 frequently in the proof without mentioning it explicitly. We have to estimate all terms on the right hand side and start by estimating the commutator terms

$$s^2 \int_\Sigma \left\langle D^k \nabla_t \phi, \left[\nabla_t, D^k \right] \nabla_t \phi \right\rangle dV_{g_t}$$

$$= s^2 \sum_{\sum l_i + \sum m_j = k-1} \int_\Sigma \nabla^{l_1} R^P \star \underbrace{D^{m_1+1} \phi \star \cdots \star D^{m_{l_1}+1} \phi}_{l_1\text{-times}} \star D^{l_2} \partial_t \phi \star D^{l_3+1} \phi \star D^{l_4} \partial_t \phi \star D^k \partial_t \phi \, dV_{g_t}$$

$$+ s^2 \sum_{l=0}^{k-1} \int_\Sigma D^{k-l} \dot{g} \star D^{l+1} \partial_t \phi \star D^k \partial_t \phi \, dV_{g_t}$$

$$\leq C(k)s^2 \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} \| D\phi \|_{H^r}^{l+1} \| \partial_t \phi \|_{H^r}^3 + C(k)s^2 \| \dot{g} \|_{C^{k-1}} \| \partial_t \phi \|_{H^k}^2$$

$$\leq C(k)s^{-1} \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} (F_r(\phi))^{2+l/2} + C(k) \| \dot{g} \|_{C^{k-1}} F_k(\phi).$$

The second commutator can be controlled as follows

$$\int_\Sigma \left\langle D^{k+1} \phi, \left[\nabla_t, D^{k+1} \phi \right] \right\rangle dV_{g_t}$$

$$= \sum_{\sum l_i + \sum m_j = k-1} \int_\Sigma \nabla^{l_1} R^P \star \underbrace{D^{m_1+1} \phi \star \cdots \star D^{m_{l_1}+1} \phi}_{l_1\text{-times}} \star D^{l_2} \partial_t \phi \star D^{l_3} D\phi \star D^{l_4} D\phi \star D^k D\phi \, dV_{g_t}$$

$$+ \sum_{l=0}^{k} \int_\Sigma D^{k-l} \dot{g} \star D^l D\phi \star D^k D\phi \, dV_{g_t}$$

$$\leq C(k) \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} \| D\phi \|_{H^r}^{l+3} \| \partial_t \phi \|_{H^r} + C(k) \| \dot{g} \|_{C^{k-1}} \| D\phi \|_{H^k}^2$$

$$\leq C(k)s^{-1} \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} (F_r(\phi))^{2+l/2} + C(k)\|\dot{g}\|_{C^{k-1}} F_k(\phi).$$

The third commutator can be estimated as follows

$$\int_\Sigma \left\langle D^k \partial_t \phi, \left[D^* D, D^k \right] \phi \right\rangle dV_{g_t}$$

$$= \sum_{l=0}^{k-1} \int_\Sigma D^l R^\Sigma \star D^{k-l} \phi \star D^k \partial_t \phi \, dV_{g_t}$$

$$+ \sum_{\sum l_i + \sum m_j = k-1} \int_\Sigma \nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}} \star D^{l_2} D\phi \star D^{l_3} D\phi \star D^{l_4} D\phi \star D^k \nabla_t \phi \, dV_{g_t}$$

$$\leq C(k) \left\| R^\Sigma \right\|_{C^{k-1}} \|D\phi\|_{H^{k-1}} \|\partial_t \phi\|_{H^k} + C(k) \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} \|D\phi\|_{H^r}^{l+3} \|\partial_t \phi\|_{H^r}$$

$$\leq C(k)s^{-1} \left\| R^\Sigma \right\|_{C^{k-1}} F_k(\phi) + C(k)s^{-1} \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} (F_r(\phi))^{2+l/2}.$$

As a second step we estimate the terms that arise when inserting the equation (2.4) for ϕ into (3.3). In order to estimate the first term we calculate

$$(Ns)^{-1} \nabla_{\nabla(Ns)} \phi = -s\dot{s}\nabla_t \phi - s^2 N^{-1} \partial_t N \nabla_t \phi + N^{-1} \langle DN, D\phi \rangle_g.$$

This allows us to derive the following estimate

$$\int_\Sigma \left\langle D^k \left((Ns)^{-1} \nabla_{\nabla(Ns)} \phi \right), D^k \nabla_t \phi \right\rangle dV_{g_t}$$

$$= -s\dot{s} \int_\Sigma \left| D^k \nabla_t \phi \right|^2 dV_{g_t} + \sum_{l=0}^{k} \int_\Sigma D^l \left(\frac{DN}{N} \right) \star D^{k-l} D\phi \star D^k \nabla_t \phi \, dV_{g_t}$$

$$+ s^2 \sum_{l_i}^{k} \int_\Sigma D^{l_1} \left(\frac{\partial_t N}{N} \right) \star D^{l_2} \nabla_t \phi \star D^k \nabla_t \phi \, dV_{g_t}$$

$$\leq -s\dot{s} \int_\Sigma \left| D^k \nabla_t \phi \right|^2 dV_{g_t} + C(k)s^{-1} \|D\log N\|_{C^k} F_k(\phi) + C(k)\|\partial_t \log N\|_{C^k} F_k(\phi).$$

In order to treat the second term on the right hand side of (2.4) we have to consider spatial and time derivatives separately

$$D^k \left(R^P(\psi, i\partial_t \cdot \psi) d\phi(\partial_t) \right)$$

$$= \sum_{\sum l_i + \sum m_j = k} \nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}} \star D^{l_2} \partial_t \star D^{l_3} \psi \star D^{l_4} \psi \star D^{l_5} \nabla_t \phi,$$

$$D^k \left(R^P(\psi, i\partial_i \cdot \psi) d\phi(\partial_i) \right)$$

$$= \sum_{\sum l_i + \sum m_j = k} \nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}} \star D^{l_2} \psi \star D^{l_3} \psi \star D^{l_4} D\phi,$$

such that we find

$$D^k h^{\alpha\beta}\left(N^{2-n}R^P(\psi, i\partial_\alpha \cdot \psi)d\phi(\partial_\beta)\right)$$

$$= \sum_{\sum l_i + \sum m_j = k} D^{l_1}N^{2-n}\star\left(s^2\nabla^{l_2}R^P\star\underbrace{D^{m_1+1}\phi\star\cdots\star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}\star D^{l_3}\partial_t\star D^{l_4}\psi\star D^{l_5}\psi\star D^{l_6}\nabla_t\phi\right.$$

$$\left.+\nabla^{l_2}R^P\star\underbrace{D^{m_1+1}\phi\star\cdots\star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}\star D^{l_3}\psi\star D^{l_4}\psi\star D^{l_5}D\phi\right).$$

This allows us to derive the following estimate:

$$\int_\Sigma \left\langle D^k\left(-\frac{1}{2}h^{\alpha\beta}(Ns)^{2-n}R^P(\psi, i\partial_\alpha \cdot \psi)d\phi(\partial_\beta)\right), D^k\nabla_t\phi\right\rangle dV_{g_t}$$

$$= -\frac{s^{2-n}}{2}\sum_{\sum l_i + \sum m_j = k}\int_\Sigma D^{l_1}N^{2-n}$$

$$\star s^2\left(\nabla^{l_2}R^P\star\underbrace{D^{m_1+1}\phi\star\cdots\star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}\star D^{l_3}\partial_t\star D^{l_4}\psi\star D^{l_5}\psi\star D^{l_6}\nabla_t\phi\star D^k\nabla_t\phi\right.$$

$$\left.+D^{l_2}R^P\star\underbrace{D^{m_1+1}\phi\star\cdots\star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}\star D^{l_3}\psi\star D^{l_4}\psi\star D^{l_5}D\phi\star D^k\nabla_t\phi\right)dV_{g_t}$$

$$\leq C(k,n)s^{4-n}\left\|N^{2-n}\right\|_{C^k}\left\|R^P\right\|_{C^k}\|\partial_t\|_{C^k}\sum_{l=0}^k\|D\phi\|_{H^r}^l\|\psi\|_{H^r}^2\|\nabla_t\phi\|_{H^r}^2$$

$$+ C(k,n)s^{2-n}\left\|N^{2-n}\right\|_{C^k}\left\|R^P\right\|_{C^k}\sum_{l=0}^k\|D\phi\|_{H^r}^{l+1}\|\psi\|_{H^r}^2\|\nabla_t\phi\|_{H^r}$$

$$\leq C(k,n)s^{2-n}\left\|N^{2-n}\right\|_{C^k}\left\|R^P\right\|_{C^k}\|\partial_t\|_{C^k}\sum_{l=0}^k F_r(\phi)^{1+l/2}F_{r-1}(\psi)$$

$$+ C(k,n)s^{1-n}\left\|N^{2-n}\right\|_{C^k}\left\|R^P\right\|_{C^k}\sum_{l=0}^k F_r(\phi)^{1+l/2}F_{r-1}(\psi).$$

The third term on the right hand side of (2.4) can be computed as

$$D^k\left(N^{4-2n}\left\langle\left(\nabla R^P\right)^\sharp(\psi,\psi)\psi,\psi\right\rangle\right)$$

$$= \sum_{\sum l_i + \sum m_j = k}D^{l_1}N^{4-2n}\star\nabla^{l_2+1}R^P\star\underbrace{D^{m_1+1}\phi\star\cdots\star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}\star D^{l_3}\psi\star D^{l_4}\psi\star D^{l_5}\psi\star D^{l_6}\psi.$$

This allows us to derive the following estimate

$$\int_\Sigma \left\langle D^k \left((Ns)^{4-2n} \left\langle \left(\nabla R^P \right)^\sharp (\psi, \psi) \psi, \psi \right\rangle \right), D^k \partial_t \phi \right\rangle dV_{g_t}$$

$$= s^{4-2n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} N^{4-2n} \star \nabla^{l_2+1} R^P$$

$$\star \underbrace{D^{m_1+1} \phi \star \cdots \star D^{m_{l_2}+1} \phi}_{l_2\text{-times}} \star D^{l_3} \psi \star D^{l_4} \psi \star D^{l_5} \psi \star D^{l_6} \psi \star D^k \partial_t \phi \, dV_{g_t}$$

$$\leq s^{4-2n} C(k) \left\| N^{4-2n} \right\|_{C^k} \left\| R^P \right\|_{C^{k+1}} \sum_{l=0}^{k} \| D\phi \|_{H^r}^l \| \psi \|_{H^r}^4 \| \partial_t \phi \|_{H^r}$$

$$\leq s^{3-2n} C(k) \left\| N^{4-2n} \right\|_{C^k} \left\| R^P \right\|_{C^{k+1}} \sum_{l=0}^{k} F_r(\phi)^{1/2+l/2} F_{r-1}(\psi)^2.$$

Adding up the different contributions concludes the proof. $\qquad\square$

3.3 Energy of the spinor

We define the kth energy density of the spinor part as

$$e_k(\psi) = s^2 \left| D^k \nabla_t \psi \right|^2 + \left| D^k D\psi \right|^2 \tag{3.7}$$

and the kth energy as

$$E_k(\psi) = \int_\Sigma e_k(\psi) \, dV_{g_t}. \tag{3.8}$$

The kth total energy is given by

$$F_k(\psi) = \sum_{l=0}^{k} E_l(\psi) + \| \psi \|_{L^2}^2 = \int_\Sigma \sum_{l=0}^{k} e_l(\psi) \, dV_{g_t} + \| \psi \|_{L^2}^2 = s^2 \| \nabla_t \psi \|_{H^k}^2 + \| \psi \|_{H^{k+1}}^2. \tag{3.9}$$

Note that

$$\frac{d}{dt} \| \psi \|_{L^2}^2 \leq 2 \| \nabla_t \psi \|_{L^2} \| \psi \|_{L^2} + \frac{1}{2} \| \operatorname{tr} \dot{g} \|_{L^\infty} \| \psi \|_{L^2}^2 \leq \left(s^{-1} + \frac{1}{2} \| \operatorname{tr} \dot{g} \|_{L^\infty} \right) F_1(\psi). \tag{3.10}$$

Proposition 3.8 *Let $T > 0$, $r \in \mathbb{N}$, $r > (n-1)/2$, $k \in \{1, \ldots, r-1\}$ and (ϕ, ψ) be a solution of (2.4), (2.5) such that*

$$\phi \in C^0 \left([0, T), H^{r+1}(\Sigma, P) \right) \cap C^1 \left([0, T), H^r(\Sigma, P) \right),$$
$$\psi \in C^0 \left([0, T), H^r \left(M, SM \otimes \phi^* TP \right) \right) \cap C^1 \left([0, T), H^{r-1} \left(M, SM \otimes \phi^* TP \right) \right).$$

Then the kth energy of the spinor satisfies the following inequality

$$\frac{d}{dt} E_k(\psi) \le C_1(k,n) \left(\left\| R^{SM}(\partial_t, \cdot) \right\|_{C^{k-1}} + \|\dot{g}\|_{C^k} + \|D^* e_0\|_{L^\infty} \right) F_k(\psi)$$

$$+ C_2(n,k) \cdot s^{-1} \left(\left\| R^\Sigma \right\|_{C^{k-1}} + \left\| R^{SM} \right\|_{C^k} + \left\| \mathrm{scal}^M \right\|_{C^k} \right) F_k(\psi)$$

$$+ C_3(n,k,g) \left(s^{-1} + \|\partial_t\|_{C^k} \right) \left\| R^P \right\|_{C^{k+1}} \sum_{l=0}^{k} F_r(\phi)^{l/2+1} F_{r-1}(\psi)$$

$$+ C_4(n,k,g) s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^{k+1}} \|\partial_t\|_{C^k} \sum_{l=0}^{k+1} F_r(\phi)^{l/2} F_{r-1}(\psi)^2$$

$$+ (n-2) \cdot C_5(n,k,g) s^{3-n} \left\| N^{2-n} \partial_t \log N \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{l/2} F_{r-1}(\psi)^2$$

$$+ (n-2) \cdot C_6(n,k,g) \dot{s} s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{l/2} F_{r-1}(\psi)^2$$

$$+ C_7(n,k,g) s^{1-n} \left\| N^{2-n} \right\|_{C^{k+1}} \left\| R^P \right\|_{C^{k+1}} \sum_{l=0}^{k+1} F_r(\phi)^{l/2} F_{r-1}(\psi)^2$$

$$+ C_8(n,k,g) s^{1-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k}^2 \sum_{l=0}^{2k} F_r(\phi)^{l/2} F_{r-1}(\psi)^3,$$

$$(3.11)$$

where the positive constants C_i, $i \in \{3, \dots, 8\}$ depend on n, k and the Sobolev constant of the metric g_t.

Remark 3.9 The curvature quantities coming from the spinor bundle above have to be understood as the following sections: $R^{SM}(\partial_t, \cdot) \in \Gamma(T^*\Sigma \otimes \mathrm{End}(SM))$, $R^{SM} = R^{SM}(.,.) \in \Gamma(\Lambda^2 T\Sigma \otimes \mathrm{End}(SM))$, both by restricting to vectors tangent to Σ. Moreover $\partial_t \in \Gamma(\mathrm{End}(SM))$ acts by Clifford multiplication. The C^k norms with respect to g_t are then defined in a canonical way.

Proof In order to derive an energy estimate for the spinor we make use of (3.2). As before, we may at first assume that the solution is spacelike compactly supported before we obtain this equality for the general case by an exhaustion procedure. We will also frequently use Lemma 3.6 in this proof. Again, we have to estimate all terms on the right hand side and start by estimating the commutator terms:

$$s^2 \int_\Sigma \left\langle D^k \nabla_t \psi, \left[\nabla_t, D^k \right] \psi \right\rangle dV_{g_t}$$

$$= s^2 \sum_{l=0}^{k-1} \int_\Sigma D^l \left(R^{SM}(\partial_t, \cdot) \right) \star D^{k-l-1} \nabla_t \psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$+ s^2 \sum_{l=0}^{k-1} \int_\Sigma D^{k-l} \dot{g} \star D^{l+1} \nabla_t \psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$+ s^2 \sum_{\sum l_i + \sum m_j = k-1} \int_\Sigma \nabla^{l_1} R^P$$

$$\star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}} \star D^{l_2} \nabla_t \phi \star D^{l_3} D\phi \star D^{l_4} \nabla_t \psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$\leq s^2 C(k) \left\| R^{SM}(\partial_t, \cdot) \right\|_{C^{k-1}} \|\nabla_t \psi\|_{H^k}^2 + s^2 C(k) \|\dot{g}\|_{C^k} \|\nabla_t \psi\|_{H^k}^2$$

$$+ s^2 C(k) \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} \|D\phi\|_{H^r}^{l+1} \|\partial_t \phi\|_{H^r} \|\nabla_t \psi\|_{H^{r-1}}^2$$

$$\leq C(k) \left\| R^{SM}(\partial_t, \cdot) \right\|_{C^{k-1}} F_k(\psi) + C(k) \|\dot{g}\|_{C^k} F_k(\psi)$$

$$+ s^{-1} C(k) \left\| R^P \right\|_{C^{k-1}} \sum_{l=0}^{k-1} F_r(\phi)^{l/2+1} F_{r-1}(\psi).$$

The second commutator term can be estimated as follows:

$$\int_\Sigma \left\langle \left[D^*D, D^k \right] \psi, D^k \nabla_t \psi \right\rangle \, dV_{g_t}$$

$$= \sum_{l=0}^{k-1} \int_\Sigma D^l R^M \star D^{k-l} \psi \star D^k \nabla_t \psi \, dV_{g_t} + \sum_{l=0}^{k} \int_\Sigma D^l R^{SM} \star D^{k-l} \psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$+ \sum_{\sum l_i + \sum m_j = k} \int_\Sigma \nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}} \star D^{l_2} D\phi \star D^{l_3} D\phi \star D^{l_4}\psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$\leq C(k) \left\| R^M \right\|_{C^{k-1}} \|\psi\|_{H^k} \|\nabla_t \psi\|_{H^k} + C(k) \left\| R^{SM} \right\|_{C^k} \|\psi\|_{H^k} \|\nabla_t \psi\|_{H^k}$$

$$+ C(k) \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} \|D\phi\|_{H^r}^{l+2} \|\psi\|_{H^r} \|\nabla_t \psi\|_{H^{r-1}}$$

$$\leq s^{-1} C(k) \left(\left\| R^M \right\|_{C^{k-1}} + \left\| R^{SM} \right\|_{C^k} \right) F_k(\psi)$$

$$+ s^{-1} C(k) \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{l/2+1} F_{r-1}(\psi).$$

The third commutator can be controlled as follows:

$$\int_\Sigma \left\langle \left[\nabla_t, D^{k+1} \right] \psi, D^{k+1}\psi \right\rangle \, dV_{g_t}$$

$$= \sum_{l=0}^{k} \int_\Sigma D^l \left(R^{SM}(\partial_t, \cdot) \right) \star D^{k-l}\psi \star D^{k+1}\psi \, dV_{g_t}$$

$$+ \sum_{l=0}^{k} \int_\Sigma D^{k-l-1}\dot{g} \star D^{l+1}\psi \star D^{k+1}\psi \, dV_{g_t}$$

$$+ \sum_{\sum l_i + \sum m_j = k-1} \int_\Sigma \nabla^{l_1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}}$$

$$\star D^{l_2}\partial_t \phi \star D^{l_3} D\phi \star D^{l_4}\psi \star D^{k+1}\psi \, dV_{g_t}$$

$$\leq C(k) \left\| R^{SM}(\partial_t, \cdot) \right\|_{C^k} \|\psi\|_{H^{k+1}}^2 + C(k) \|\dot{g}\|_{C^k} \|\psi\|_{H^{k+1}}^2$$

$$+ C(k) \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} \|D\phi\|_{H^r}^{l+1} \|\nabla_t \phi\|_{H^r} \|\psi\|_{H^r}^2$$

$$\leq C(k) \left\| R^{SM}(\partial_t, \cdot) \right\|_{C^k} F_k(\psi) + C(k) \|\dot{g}\|_{C^k} F_k(\psi)$$

$$+ s^{-1} C(k) \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{l/2+1} F_{r-1}(\psi).$$

As a next step we estimate the terms that arise when inserting (2.5) into (3.3). The first term can easily be controlled as

$$\int_{\Sigma} \left\langle D^k \left(\frac{\mathrm{scal}^M}{4} \psi \right), D^k \nabla_t \psi \right\rangle dV_{g_t} = \frac{1}{4} \sum_{l=0}^{k} \int_{\Sigma} D^l \mathrm{scal}^M \star D^{k-l} \psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$\leq C(k) \left\| \mathrm{scal}^M \right\|_{C^k} s^{-1} F_k(\psi).$$

Regarding the second term on the right hand side of (2.5) we again expand space and time contributions

$$h^{\alpha\gamma} h^{\beta\delta} \partial_\alpha \cdot \partial_\beta \cdot R^P \left(d\phi \left(\partial_\gamma \right), d\phi(\partial_\delta) \right) \psi = -2s^2 g^{ij} \partial_t \cdot \partial_i \cdot R^P \left(d\phi(\partial_t), d\phi(\partial_j) \right) \psi$$
$$+ g^{ij} g^{kl} \partial_i \cdot \partial_k \cdot R^P \left(d\phi(\partial_j), d\phi(\partial_l) \right) \psi.$$

Note that we do not get a term proportional to $|d\phi(\partial_t)|^2$ due to symmetry reasons. We then find

$$D^k \left(h^{\alpha\gamma} h^{\beta\delta} \partial_\alpha \cdot \partial_\beta \cdot R^P \left(d\phi(\partial_\gamma), d\phi(\partial_\delta) \right) \psi \right)$$

$$= \sum_{\sum l_i + \sum m_j = k} \left(D^{l_1} R^P \star \underbrace{D^{m_1+1} \phi \star \cdots \star D^{m_{l_1}+1} \phi}_{l_1\text{-times}} \star D^{l_2} D\phi \star D^{l_3} D\phi \star D^{l_4} \psi \right)$$

$$+ s^2 \sum_{\sum l_i + \sum m_j = k} \left(D^{l_1} \partial_t \star D^{l_2} R^P \star \underbrace{D^{m_1+1} \phi \star \cdots \star D^{m_{l_2}+1} \phi}_{l_2\text{-times}} \star D^{l_3} d\phi(\partial_t) \star D^{l_4} d\phi \star D^{l_5} \psi \right).$$

This allows us to derive the following estimate

$$\int_{\Sigma} \left\langle D^k \left(h^{\alpha\gamma} h^{\beta\delta} \partial_\alpha \cdot \partial_\beta \cdot R^P \left(d\phi(\partial_\gamma), d\phi(\partial_\delta) \right) \psi \right), D^k \nabla_t \psi \right\rangle dV_{g_t}$$

$$= \sum_{\sum l_i + \sum m_j = k} \int_{\Sigma} D^{l_1} R^P \star \underbrace{D^{m_1+1} \phi \star \cdots \star D^{m_{l_1}+1} \phi}_{l_1\text{-times}} \star D^{l_2} d\phi \star D^{l_3} d\phi \star D^{l_4} \psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$+ s^2 \sum_{\sum l_i + \sum m_j = k} \int_{\Sigma} D^{l_1} \partial_t \star D^{l_2} R^P \star \underbrace{D^{m_1+1} \phi \star \cdots \star D^{m_{l_2}+1} \phi}_{l_2\text{-times}}$$

$$\star D^{l_3} d\phi(\partial_t) \star D^{l_4} d\phi \star D^{l_5} \psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$\leq C(k) \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} \| D\phi \|_{H^r}^{l+2} \| \psi \|_{H^r} \| \nabla_t \psi \|_{H^k}$$

$$+ s^2 C(k) \| \partial_t \|_{C^k} \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} \| D\phi \|_{H^r}^{l+1} \| \nabla_t \phi \|_{H^r} \| \psi \|_{H^r} \| \nabla_t \psi \|_{H^k}$$

$$\leq s^{-1} C(k) \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{l/2+1} F_{r-1}(\psi) + C(k) \| \partial_t \|_{C^k} \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} F_r(\phi)^{l/2+1} F_{r-1}(\psi).$$

To manipulate the third term on the right hand side of (2.5) we first consider the derivative with respect to t. We find

$$\nabla_t \left(R^P(\psi, \psi)(Ns)^{2-n} \right) \partial_t \cdot \psi = \left(\nabla_{d\phi(\partial_t)} R^P \right)(\psi, \psi)(Ns)^{2-n}\partial_t \cdot \psi$$
$$+ 2R^P \left(\nabla_t \psi, \psi \right)(Ns)^{2-n}\partial_t \cdot \psi$$
$$+ (2-n)R^P(\psi, \psi)(Ns)^{2-n}\partial_t \log N \partial_t \cdot \psi$$
$$+ (2-n)\dot{s}s^{1-n}N^{2-n}R^P(\psi, \psi)\partial_t \cdot \psi.$$

As a next step we calculate the k-th spatial derivative of this expression

$$D^k \left(\left(\nabla_{d\phi(\partial_t)} R^P \right)(\psi, \psi)N^{2-n}\partial_t \cdot \psi \right) = \sum_{\sum l_i + \sum m_j = k}^{k} D^{l_1} N^{2-n} \star D^{l_2+1} R^P$$
$$\star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1+1}\phi}_{l_2+1\text{-times}} \star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi \star D^{l_7} d\phi(\partial_t),$$

$$D^k \left(R^P(\nabla_t \psi, \psi)N^{2-n}\partial_t \cdot \psi \right) = \sum_{\sum l_i + \sum m_j = k}^{k} D^{l_1} N^{2-n} \star D^{l_2} R^P$$
$$\star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}} \star D^{l_3}\nabla_t \psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi,$$

$$D^k \left(R^P(\psi, \psi)N^{2-n}\partial_t \log N \partial_t \cdot \psi \right) = \sum_{\sum l_i + \sum m_j = k}^{k} D^{l_1} N^{2-n}\partial_t \log N \star D^{l_2} R^P$$
$$\star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}} \star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi,$$

$$D^k \left(N^{2-n}R^P(\psi, \psi)\partial_t \cdot \psi \right) = \sum_{\sum l_i + \sum m_j = k}^{k} D^{l_1} N^{2-n} \star D^{l_2} R^P$$
$$\star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}} \star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi.$$

These manipulations allow us to derive the following estimates:

$$s^2 \int_\Sigma \left\langle D^k \left(\nabla_t \left(R^P(\psi, \psi)(Ns)^{2-n} \right) \partial_t \cdot \psi \right), D^k \nabla_t \psi \right\rangle dV_{g_t}$$
$$= s^{4-n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} N^{2-n} \star D^{l_2+1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1+1}\phi}_{l_2+1\text{-times}}$$
$$\star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi \star D^{l_7} d\phi(\partial_t) \star D^k \nabla_t \psi \, dV_{g_t}$$
$$+ s^{4-n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} N^{2-n} \star D^{l_2} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}$$
$$\star D^{l_3}\nabla_t \psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi \star D^k \nabla_t \psi \, dV_{g_t}$$
$$+ (2-n)\dot{s}s^{3-n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} N^{2-n} \star D^{l_2} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}$$

$$\star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$+ (2-n)s^{4-n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} N^{2-n}\partial_t \log N \star D^{l_2} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}$$

$$\star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\partial_t \star D^{l_6}\psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$\leq C(k)s^{4-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^{k+1}} \|\partial_t\|_{C^k} \sum_{l=0}^k \|D\phi\|_{H^r}^l \|\psi\|_{H^r}^3 \|\nabla_t \phi\|_{H^r} \|\nabla_t \psi\|_{H^k}$$

$$+ C(k)s^{4-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^k \|D\phi\|_{H^r}^l \|\psi\|_{H^r}^2 \|\nabla_t \psi\|_{H^k}^2$$

$$+ C(k,n)(n-2)\dot{s}s^{3-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^k \|D\phi\|_{H^r}^l \|\psi\|_{H^r}^3 \|\nabla_t \psi\|_{H^k}$$

$$+ C(k,n)(n-2)s^{4-n} \left\| N^{2-n}\partial_t \log N \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^k \|D\phi\|_{H^r}^l \|\psi\|_{H^r}^3 \|\nabla_t \psi\|_{H^k}$$

$$\leq C(k)s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^{k+1}} \|\partial_t\|_{C^k} \sum_{l=0}^k F_r(\phi)^{l/2+1/2} F_{r-1}(\psi)^2$$

$$+ C(k)s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^k F_r(\phi)^{l/2} F_{r-1}(\psi)^2$$

$$+ C(k,n)(n-2)\dot{s}s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^k F_r(\phi)^{l/2} F_{r-1}(\psi)^2$$

$$+ C(k,n)(n-2)s^{3-n} \left\| N^{2-n}\partial_t \log N \right\|_{C^k} \left\| R^P \right\|_{C^k} \|\partial_t\|_{C^k} \sum_{l=0}^k F_r(\phi)^{l/2} F_{r-1}(\psi)^2.$$

As a next step we take care of the spatial derivatives in the third term on the right hand side of (2.5). These can be computed as

$$D\left(R^P(\psi,\psi)(Ns)^{2-n}\right)\cdot\psi = \left(D_{d\phi}R^P\right)(\psi,\psi)(Ns)^{2-n}\cdot\psi + 2R^P(D\psi,\psi)(Ns)^{2-n}\cdot\psi$$
$$+ s^{2-n}R^P(\psi,\psi)D(N)^{2-n}\cdot\psi.$$

The kth spatial derivative of these terms acquire the forms

$$D^k\left(\left(D_{d\phi}R^P\right)(\psi,\psi)(Ns)^{2-n}\cdot\psi\right)$$
$$= s^{2-n} \sum_{\sum l_i + \sum m_j = k} D^{l_1+1}R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1+1}\phi}_{(l_1+1)\text{-times}} \star D^{l_2}\psi \star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}N^{2-n},$$

$$D^k\left(R^P(D\psi,\psi)(Ns)^{2-n}\cdot\psi\right)$$
$$= s^{2-n} \sum_{\sum l_i + \sum m_j = k} D^{l_1}R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}} \star D^{l_2}D\psi \star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}N^{2-n},$$

$$D^k\left((s)^{2-n}R^P(\psi,\psi)D(N)^{2-n}\cdot\psi\right)$$

$$= s^{2-n} \sum_{\sum l_i + \sum m_j = k} D^{l_1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_1}+1}\phi}_{l_1\text{-times}} \star D^{l_2}\psi \star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5} D N^{2-n}.$$

These manipulations allow us to derive the following estimates:

$$\int_\Sigma \left\langle D^k \left(D\left(R^P(\psi,\psi)(Ns)^{2-n}\right) \partial_i \cdot \psi \right), D^k \nabla_t \psi \right\rangle dV_{g_t}$$

$$= s^{2-n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} N^{2-n} \star D^{l_2+1} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1+1}\phi}_{(l_2+1)\text{-times}}$$

$$\star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$+ s^{2-n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} N^{2-n} \star D^{l_2} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}$$

$$\star D^{l_3} D\psi \star D^{l_4}\psi \star D^{l_5}\psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$+ s^{2-n} \sum_{\sum l_i + \sum m_j = k} \int_\Sigma D^{l_1} D\left(N^{2-n}\right) \star D^{l_2} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1}\phi}_{l_2\text{-times}}$$

$$\star D^{l_3}\psi \star D^{l_4}\psi \star D^{l_5}\psi \star D^k \nabla_t \psi \, dV_{g_t}$$

$$\leq C(k) s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^{k+1}} \sum_{l=0}^{k} \| D\phi\|_{H^r}^{l+1} \|\psi\|_{H^r}^3 \|\nabla_t \psi\|_{H^k}$$

$$+ C(k) s^{2-n} \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} \| D\phi\|_{H^r}^{l} \|\psi\|_{H^r}^3 \|\nabla_t \psi\|_{H^k}$$

$$+ C(k) s^{2-n} \left\| N^{2-n} \right\|_{C^{k+1}} \left\| R^P \right\|_{C^k} \sum_{l=0}^{k} \| D\phi\|_{H^r}^{l} \|\psi\|_{H^r}^3 \|\nabla_t \psi\|_{H^k}$$

$$\leq C(k) s^{1-n} \left\| N^{2-n} \right\|_{C^{k+1}} \left\| R^P \right\|_{C^{k+1}} \sum_{l=0}^{k+1} F_r(\phi)^{l/2} F_{r-1}(\psi)^2.$$

To control the last term from the right hand side of (2.5) we calculate

$$D^k \left((Ns)^{2-n} R^P(\psi,\psi) R^P(\psi,\psi)\psi \right)$$

$$= s^{2-n} \sum_{\sum l_{i_1} + \sum m_{i_2} + \sum q_{i_3} = k} D^{l_1} N^{2-n} \star D^{l_2} R^P \star \underbrace{D^{m_1+1}\phi \star \cdots \star D^{m_{l_2}+1+1}\phi}_{l_2\text{-times}}$$

$$\star D^{l_3} R^P \star \underbrace{D^{q_1+1}\phi \star \cdots \star D^{q_{l_3}+1+1}\phi}_{l_3\text{-times}} \star D^{l_4}\psi \star D^{l_5}\psi \star D^{l_6}\psi \star D^{l_7}\psi \star D^{l_8}\psi.$$

Consequently, we obtain the following estimate

$$\int_\Sigma \left\langle D^k \left((Ns)^{2-n} R^P(\psi,\psi) R^P(\psi,\psi)\psi \right), D^k \nabla_t \psi \right\rangle dV_{g_t}$$

$$\leq s^{2-n} C(k) \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k}^2 \sum_{l=0}^{2k} \| D\phi\|_{H^r}^{l} \|\psi\|_{H^r}^5 \|\nabla_t \psi\|_{H^k}$$

$$\leq s^{1-n} C(k) \left\| N^{2-n} \right\|_{C^k} \left\| R^P \right\|_{C^k}^2 \sum_{l=0}^{2k} F_r(\phi)^{l/2} F_{r-1}(\psi)^3.$$

The last term of (3.2) can be estimated as

$$\int_{\Sigma} \left\langle D^k \psi, e_0 \cdot \left(D^* e_0 \right) \cdot D^{k+1} \psi \right\rangle dV_{g_t} \leq \| D^* e_0 \|_{L^\infty} \| \psi \|_{H^r}^2.$$

Adding up the different contributions yields the claim. $\qquad\square$

Remark 3.10 Note that all terms on the right hand side of (3.11) have a similar analytic structure except the terms proportional to C_3 and C_8, which contain higher powers of the H^r-norms of ϕ and ψ.

3.4 Energy of the coupled pair

Before we prove an energy estimate for the coupled pair, we give more geometric interpretations of the terms appearing in the above estimates. At first, recall that the second fundamental form $\mathrm{II} \in \Gamma(T^* \Sigma^{\odot 2})$ of a hypersurface $(\{t\} \times \Sigma, g_t) \subset (M, h)$ is given by

$$\mathrm{II}(X, Y) = \langle \nabla_X Y - D_X Y, \nu \rangle_h = \frac{1}{2} \left\langle s^2 \dot{g}(X, Y) \cdot \partial_t, \nu \right\rangle_h = -\frac{s}{2} \dot{g}(X, Y),$$

where ν is the future-directed unit normal of the hypersurface.

Lemma 3.11 *We have the estimates*

$$\| \dot{g} \|_{C^k} \leq 2s^{-1} \| \mathrm{II} \|_{C^k}, \qquad \| D e_0 \|_{L^\infty} \leq \| \mathrm{II} \|_{L^\infty}, \qquad \| \partial_t \|_{C^k} \leq s^{-1} \left(1 + \| \mathrm{II} \|_{C^{k-1}} \right),$$

$$\left\| R^\Sigma \right\|_{C^k} \leq C(k) \sum_{l=0}^{k} \| \mathrm{II} \|_{C^k}^l \| R^M \|_{C^k} + C(k) \| \mathrm{II} \|_{C^k}^2,$$

$$\left\| R^{SM} \right\|_{C^k} \leq C(k, n) \sum_{l=0}^{k} \| \mathrm{II} \|_{C^k}^l \left\| R^M \right\|_{C^k},$$

$$\left\| R^{SM}(\partial_t, \cdot) \right\|_{C^k} \leq C(k, n) s^{-1} \sum_{l=0}^{k+1} \| \mathrm{II} \|_{C^k}^l \left\| R^M \right\|_{C^k}.$$

Here, we defined the C^k-norm of R^M by taking the kth covariant derivative with respect to h but taking the norm with respect to the Riemannian reference metric $s^{-2} dt^2 + g_t$ in order to get a nonnegative quantity.

Proof The first estimate follows from the definition of II, the second and third from the facts that $\nabla_X \partial_t = \frac{1}{2} \dot{g}(X)$ and $e_0 = s \partial_t$. Now we recall that for every tensor $T \in \Gamma(T^* \Sigma^{\otimes k})$ on the manifold M, the difference between the covariant derivatives ∇ and D can be expressed as

$$DT = \nabla T + \mathrm{II} \star T,$$

which by induction yields

$$D^k T = \sum_{l_1 + l_2 + \sum m_i = k} \underbrace{D^{m_1} \mathrm{II} \star \cdots \star D^{m_{l_1}} \mathrm{II}}_{l_1 - \text{times}} \star \nabla^{l_2} T.$$

This formula in combination with the Gauß equation

$$R^\Sigma = R^M + \mathbb{II} \star \mathbb{II}$$

yields the fourth inequality. To prove the last two formulas, we recall that

$$R^{SM}(X, Y)\psi = \frac{1}{4} \sum_{\alpha,\beta=0}^{n-1} R^M\left(X, Y, \partial_\alpha, \partial_\beta\right) \partial_\alpha \cdot \partial_\beta \cdot \psi$$

for $X, Y \in \Gamma(TM)$, where $\{\partial_\alpha\}$ is a local pseudo-orthonormal frame. Therefore, we get in the above situation

$$\left(D^k R^{SM}\right)\psi = \sum_{l_1+l_2+\sum m_i=k} \underbrace{D^{m_1}\mathbb{II} \star \cdots \star D^{m_{l_1}}\mathbb{II}}_{l_1-\text{times}} \star \nabla^{l_2} R^M \star \psi,$$

which yields the fifth inequality. Similarly, by using the product rule, we obtain

$$\left(D^k R^{SM}(\partial_t, \cdot)\right)\psi = \sum_{\sum l_i+\sum m_j=k} D^{l_1} \partial_t \star \underbrace{D^{m_1}\mathbb{II} \star \cdots \star D^{m_{l_2}}\mathbb{II}}_{l_2-\text{times}} \star \nabla^{l_3} R^M \star \psi$$

and by using the third inequality we obtain the last one. $\square$

At this point we are ready to control the total energy of (ϕ, ψ), which we define by

$$F_r(\phi, \psi) = F_r(\phi) + F_{r-1}(\psi) = \sum_{k=0}^{r} E_k(\phi) + \sum_{k=0}^{r-1} E_k(\psi) + \|\psi\|_{L^2}^2$$
$$= s^2\|\partial_t\phi\|_{H^r}^2 + \|D\phi\|_{H^r}^2 + s^2\|\nabla_t\psi\|_{H^{r-1}}^2 + \|D\psi\|_{H^{r-1}}^2 + \|\psi\|_{L^2}^2.$$

Proposition 3.12 *Let $T > 0$, $r \in \mathbb{N}$, $r > (n-1)/2$ and (ϕ, ψ) be a solution of* (2.4), (2.5) *such that*

$$\phi \in C^0\left([0, T), H^{r+1}(\Sigma, P)\right) \cap C^1\left([0, T), H^r(\Sigma, P)\right),$$
$$\psi \in C^0\left([0, T), H^r\left(M, SM \otimes \phi^*TP\right)\right) \cap C^1\left([0, T), H^{r-1}\left(M, SM \otimes \phi^*TP\right)\right).$$

Suppose that $\dot{s} \geq 0$ and that the following uniform bounds

$$0 < C_2 \leq N \leq C_3, \quad \|N\|_{C^{r+1}} < C_4, \quad \|\nabla_\nu N\|_{C^{r+1}} < C_5,$$
$$\|\mathbb{II}\|_{C^r} < C_6, \quad \left\|R^M\right\|_{C^r} < C_7, \quad \left\|R^P\right\|_{C^{r+1}} < C_8, \quad \|\mathbb{II}\|_{L^\infty} < C_9 s^{-1}$$

hold for some positive constants C_i, $i = 1, \ldots, 9$. Suppose finally that there is a uniform bound on all the Sobolev constants of g_t. Then the total energy satisfies the following inequality

$$\frac{d}{dt} F_r(\phi, \psi) \leq C \cdot s^{-1} \sum_{l=0}^{2r+4} F_r(\phi, \psi)^{l/2+1} + C(n-2)\dot{s}s^{1-n} \sum_{l=0}^{r} F_r(\phi, \psi)^{l/2+2}, \quad (3.12)$$

where $C > 0$ depends on the above bounds.

Proof The estimate is a direct consequence of the evolution inequalities (3.6), (3.10), (3.11), Lemma 3.11 and elementary estimates. $\square$

Proof of Theorem 1.1 The existence of a local solution to the system (1.2), (1.3) can be obtained by standard methods, see for example [21, Proposition 9.12]. In order to prove long-time existence, it suffices to prove longtime existence of the solution (ϕ, ψ) of the equivalent system (2.4), (2.5) on the conformal manifold (M, h).

Moreover, by the continuation criterion for hyperbolic partial differential equations [21, Lemma 9.14] if suffices to obtain a uniform bound on $F_r(\phi, \psi)$ for all times $t \in [0, \infty)$.

We set $f(t) = (n-2)s^{1-n}\dot{s}$. Note that $f(t)$ is integrable with respect to t. For $n = 2$, this is trivial. For $n > 2$, we get due to the assumptions on s that

$$(n-2)\int_0^\infty \dot{s}s^{1-n}dt = -\int_0^\infty \frac{d}{dt}\left(s^{2-n}\right)dt = -\left(s^{2-n}|_{t=\infty} - s^{2-n}|_{t=0}\right) = s(0)^{2-n} < \infty.$$

As long as $F_r(\phi, \psi) \leq 1$ we have the differential inequality

$$\frac{d}{dt}F_r(\phi, \psi) \leq C\left(s^{-1}(t) + f(t)\right)F_r(\phi, \psi),$$

which can easily be integrated as

$$F_r(\phi, \psi)|_{t=T} \leq F_r(\phi, \psi)|_{t=0}\exp\left(\int_0^T (s^{-1}(t) + f(t))dt\right).$$

Now, we set $\Phi := \int_0^\infty (s^{-1}(t) + f(t))dt < \infty$ and choose $\varepsilon > 0$ small enough such that $F_r(\phi, \psi)|_{t=0} \leq (2\Phi)^{-1}$. Suppose that T_0 is the first time for which $F_r(\phi, \psi)|_{T_0} = 1$. However, the energy inequality from above gives

$$F_r(\phi, \psi)|_{t=T_0} \leq \Phi F_r(\phi, \psi)|_{t=0} = \frac{1}{2} < 1,$$

which yields a contradiction. Therefore we can conclude that

$$F_r(\phi, \psi) < 1 < \infty$$

for all times $t \in [0, \infty)$ completing the proof. $\square$

Proof of Theorem 1.3 The proof is as above but we additionally assume that $\psi \equiv 0$. In this case, we just need the assumptions in Proposition 3.7 and not the slightly stronger ones in Proposition 3.8. As the spinor is not involved, we obviously can also remove the spin condition. $\square$

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