Stable Gabor Phase Retrieval and Spectral Clustering

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Abstract

We consider the problem of reconstructing a signal f from its spectrogram, i.e., the magnitudes $|V_\varphi f|$ of its Gabor transform

$$V_\varphi f(x,y) := \int_{\mathbb{R}} f(t) e^{-\pi(t-x)^2} e^{-2\pi \mathrm{i} y t}\, dt, \quad x, y \in \mathbb{R}.$$

Such problems occur in a wide range of applications, from optical imaging of nanoscale structures to audio processing and classification.

While it is well-known that the solution of the above Gabor phase retrieval problem is unique up to natural identifications, the stability of the reconstruction has remained wide open. The present paper discovers a deep and surprising connection between phase retrieval, spectral clustering, and spectral geometry. We show that the stability of the Gabor phase reconstruction is bounded by the reciprocal of the *Cheeger constant* of the flat metric on $\mathbb{R}^2$, conformally multiplied with $|V_\varphi f|$. The Cheeger constant, in turn, plays a prominent role in the field of spectral clustering, and it precisely quantifies the "disconnectedness" of the measurements $V_\varphi f$.

It has long been known that a disconnected support of the measurements results in an instability—our result for the first time provides a converse in the sense that there are no other sources of instabilities.

Due to the fundamental importance of Gabor phase retrieval in coherent diffraction imaging, we also provide a new understanding of the stability properties of these imaging techniques: Contrary to most classical problems in imaging science whose regularization requires the promotion of smoothness or sparsity, the correct regularization of the phase retrieval problem promotes the "connectedness" of the measurements in terms of bounding the Cheeger constant from below. Our work thus, for the first time, opens the door to the development of efficient regularization strategies. © 2018 the Authors. *Communications on Pure and Applied Mathematics* is published by the Courant Institute of Mathematical Sciences and Wiley Periodicals, Inc.

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1 Introduction

1.1 Motivation

A signal is typically modeled as an element $f \in \mathcal{B}$ with $\mathcal{B}$ an ∞-dimensional Banach space. *Phase retrieval* refers to the reconstruction of a signal from phaseless linear measurements

$$(|\varphi_\omega(f)|)_{\omega\in\Omega}, \tag{1.1}$$

where $\Phi = (\varphi_\omega)_{\omega\in\Omega} \subset \mathcal{B}'$, the dual of $\mathcal{B}$. Since for any $\alpha \in \mathbb{R}$ the signal $e^{\mathbf{i}\alpha} f$ will yield the same phaseless linear measurements as f, a signal can only be reconstructed *up to global phase*, e.g., up to the identification $f \sim e^{\mathbf{i}\alpha} f$, where $\alpha \in \mathbb{R}$. If any $f \in \mathcal{B}$ can be uniquely reconstructed from its phaseless measurements (1.1), up to global phase, we say that Φ *does phase retrieval*.

Phase retrieval problems of the aforementioned type occur in a remarkably wide number of physical problems (often owing to the fact that the phase of a high-frequency wave cannot be measured), probably most prominently in coherent diffraction imaging [27, 30, 38, 42, 43] where Φ is either a Fourier or a Gabor dictionary. Other applications include quantum mechanics [40], audio processing [10, 11], or radar [36].

Given a concrete phase retrieval problem defined by a measurement system Φ, it is notoriously difficult to study whether Φ does phase retrieval, and there are only a few concrete instances where this is known. In the ∞-dimensional setting, examples of such instances include phase retrieval from Poisson wavelet measurements [48], from Gabor measurements [1], and from masked Fourier measurements [49], while it is known that the reconstruction of a compactly supported function from its Fourier magnitude is in general not uniquely possible [33].

From a computational standpoint, solving a given phase retrieval problem is even more challenging: Assuming that Φ does phase retrieval, an algorithmic reconstruction of a signal f would require additionally that the reconstruction be stable in the sense that

$$d_{\mathcal{B}}(f,g) \le c(f)\||\Phi(f)| - |\Phi(g)|\|_{\mathcal{D}} \quad \text{for all } g \in \mathcal{B} \tag{1.2}$$

holds true, where we have put

$$d_{\mathcal{B}}(f,g) := \inf_{\alpha\in\mathbb{R}} \|f - e^{\mathbf{i}\alpha} g\|_{\mathcal{B}}, \quad \Phi(f) := \begin{cases} \Omega \to \mathbb{C}, \\ \omega \mapsto \varphi_\omega(f), \end{cases}$$

and $\|\cdot\|_{\mathcal{D}}$ a suitable norm on the measurement space of functions $\Omega \to \mathbb{C}$.

1.2 Phase Retrieval Is Severely Ill-Posed

Despite its formidable relevance, the study of stability properties of phase retrieval problems has seen little progress until recently [1, 13]; a striking instability phenomenon has been identified by showing that $\sup_{f\in\mathcal{B}} c(f) = \infty$ whenever $\dim \mathcal{B} = \infty$ and some natural conditions on $\mathcal{B}$ and $\mathcal{D}$ are satisfied. Even worse, the stability of finite-dimensional approximations to such problems in general degenerates exponentially in a power of the dimension [2, 13]. This means that

> *every ∞-dimensional (and therefore every practically relevant) phase retrieval problem, as well as any fine-grained finite-dimensional approximation thereof, is unstable; phase retrieval is severely ill-posed.*

In view of this negative result, any phase retrieval problem needs to be regularized and any regularization strategy for a given phase retrieval problem requires a deeper understanding of the behavior of the local Lipschitz constant $c(f)$. This is a challenging problem requiring genuinely new methods: in [2] we show that all conventional regularization methods based on the promotion of smoothness or sparsity are unsuitable for the regularization of phase retrieval problems.

1.3 What Are the Sources for Instability?

We briefly summarize the current understanding of the situation.

A well-known source of instability (e.g., a very large constant $c(f)$), coined "multicomponent-type instability" in [1] arises whenever the measurements $\Phi(f)$ are separated in the sense that $f = u + v$ with $\Phi(u)$ and $\Phi(v)$ concentrated in disjoint subsets of Ω. Intiutively, in this case the function $g = u - v$ will produce measurements $|\Phi(g)|$ very close to the original measurements $|\Phi(f)|$, while the distance $d_{\mathcal{B}}(f, g)$ is not small at all, resulting in an instability (see also Figure 1.1 for an illustration). If $\mathcal{B}$ is a finite-dimensional Hilbert space over $\mathbb{R}$ (i.e., the real-valued case where only a sign and not the full phase needs to be determined), the correctness of this intuition has been proved in [8] and generalized in [3] to the setting of ∞-dimensional real or complex Banach spaces:

> *If the measurements $\Phi(f)$ are concentrated on a union of at least two disjoint domains, phase retrieval becomes unstable and correspondingly, the constant $c(f)$ becomes large.*

If $\mathcal{B}$ is a Banach space over $\mathbb{R}$ it is not very difficult to show that the "multicomponent-type instability" as just described is the only source of instability. More precisely, one can characterize $c(f)$, via the so-called σ-strong complement property (SCP), which indeed provides a measure for the disconnectedness of the measurements; see [3, 8]. While these results provide a complete characterization of the stability of phase retrieval problems over $\mathbb{R}$, we hasten to add that the verification of the σ-strong complement property is computationally intractable, which severely limits their applicability.

The (much more interesting) complex case is considerably more challenging and almost nothing is known. In this case the validity of the σ-SCP does not imply stability of the corresponding phase retrieval problem (it does not even imply uniqueness of the solution) [8].

Nevertheless, the results in the real-valued case suggest the following informal conjecture.

Conjecture 1.1. Phase retrieval is unstable if and only if the measurements are concentrated on at least two distinct domains. In other words: if $c(f)$ is large, then it is possible to partition the parameter set Ω into two disjoint domains $\Omega_1, \Omega_2 \subset \Omega$ such that the measurements $\Phi(f) : \Omega \to \mathbb{C}$ are "clustered" on Ω_1 and Ω_2, respectively.

While this conjecture seems to be folklore in the phase retrieval community [1] and ensuring connectedness of the essential support of the measurements is a common empirical regularization strategy [8, 26, 35, 47], we are not aware of any mathematical result that resolves Conjecture 1.1 for any concrete phase retrieval problem.

1.4 Phase Retrieval and Spectral Clustering

Looking at Conjecture 1.1, clustering problems in data analysis come to mind. We may, as a matter of fact, look into this field to formalize what it could possibly mean that "data is clustered on two disjoint sets." Let us suppose that $\Omega = \mathbb{R}^d$. We could interpret the measurements $|\Phi(f)| : \Omega \to \mathbb{R}_+$ as a density measure $d\mu = |\Phi(f)|dx$ (we shall also write μ^{d-1} for the induced surface measure) of data points and attempt to find two (or more) "clusters" (i.e., subsets of Ω) on which this measure is concentrated. In data analysis, the standard notion that describes the degree to which it is possible to divide data points into clusters is the Cheeger constant, which may be defined as

$$\inf_{C \subset \Omega} \frac{\mu^{d-1}(\partial C)}{\min(\mu(C), \mu(\Omega \setminus C))} = \inf_{\substack{C \subset \Omega, \\ \mu(C) \le \frac{1}{2}\mu(\Omega)}} \frac{\mu^{d-1}(\partial C)}{\mu(C)}; \tag{1.3}$$

see, for example, [17, 37, 44]. Looking at the above definition it becomes clear that the Cheeger constant indeed gives a measure of disconnectedness: if the constant above is small, there exists a partition of Ω into a set C and $\Omega \setminus C$ such that the volume of both C and $\Omega \setminus C$ is large, while the volume of the "interface" ∂C is small.

1.5 Contribution of This Paper

The present paper establishes a surprising connection between the mathematical analysis of clustering problems and phase retrieval: we show that for a Gabor dictionary

$$\Phi(f) = \left(V_\varphi f(x, y) := \int_{\mathbb{R}} f(t) e^{-\pi(t-x)^2} e^{-2\pi i t y} \, dt \right)_{(x,y) \in \mathbb{R}^2}$$

[1] For example, in [47, p. 1273], it is explicitly stated that "all instabilities ... we were able to observe in practice were of the form we described ...", meaning that they arise from measurements with disconnected components. Furthermore, [47] provides partial theoretical support for Conjecture 1.1 for phase retrieval problems based on wavelet measurements.

the Cheeger constant also characterizes the stability of the corresponding phase retrieval problem.

Given $f \in \mathcal{B}$, where $\mathcal{B}$ denotes a certain modulation space and $\|\cdot\|_{\mathcal{D}}$ a natural norm on the measurement space of functions on $\Omega = \mathbb{R}^2$, our main result, Theorem 2.9, shows that the stability constant $c(f)$ can be bounded from above (up to a fixed constant, independent of f) by $h(f)^{-1}$, where

$$h(f) = \inf_{\substack{C \subset \mathbb{R}^2 \text{ open, } \partial C \text{ is smooth,} \\ \int_C |V_\varphi f| \le \frac{1}{2}\int_{\mathbb{R}^2} |V_\varphi f|}} \frac{\|V_\varphi f\|_{L^1(\partial C)}}{\|V_\varphi f\|_{L^1(C)}}$$

denotes what we call the *Cheeger constant* of f. Note that the above definition is completely in line with (1.3) by setting $d\mu = |V_\varphi f(x,y)|dx\,dy$. The motivation for the term Cheeger constant stems from the fact that $h(f)$ is actually equal to the well-known Cheeger constant from Riemannian geometry [15] if we endow $\mathbb{R}^2$ with the Riemannian metric[2] induced by the metric tensor

$$\left(|V_\varphi f(x,y)| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)_{(x,y)\in\mathbb{R}^2}.$$

Such a metric is sometimes also called a *conformal multiplication of the flat metric* by $|V_\varphi f|$.

We would like to stress that our result can be regarded as a formalization and as a proof of Conjecture 1.1: The fact that $h(f)$ is small precisely describes the fact that the measurement space $\Omega = \mathbb{R}^2$ can be partitioned into two sets C and $\mathbb{R}^2 \setminus C$ such that both $\|V_\varphi f\|_{L^1(C)}$ and $\|V_\varphi f\|_{L^1(\mathbb{R}^2\setminus C)}$ are large, but on their separating boundary ∂C, the measurements are small. The quantity $h(f)$ is therefore a mathematical measure for the disconnectedness of the measurements. Indeed, as already mentioned, the Cheeger constant forms a crucial quantity in spectral clustering algorithms [45] and is, in the field of data science, a well-established quantity describing the degree of disconnectedness of data. Our results show that such a disconnectedness is the only possible source of instability of phase retrieval from Gabor measurements, and we find it quite remarkable that the notion of Cheeger constant, which is standard in clustering problems, occurs as a natural characterization of the stability of phase retrieval.

1.6 Implications

Aside from providing the first ever stability bounds for any realistic ∞-dimensional phase retrieval problem, our result has a number of important implications:

- Given measurements $V_\varphi f$, estimating the Cheeger constant $h(f)$ is a computationally tractable procedure [5, 39]. In this way one can decide from the measurements how noise stable the reconstruction is expected to be.

[2] Accepting the slight inaccuracy that $|V_\varphi f|$ may have zeros, one does not in general get a Riemannian metric.

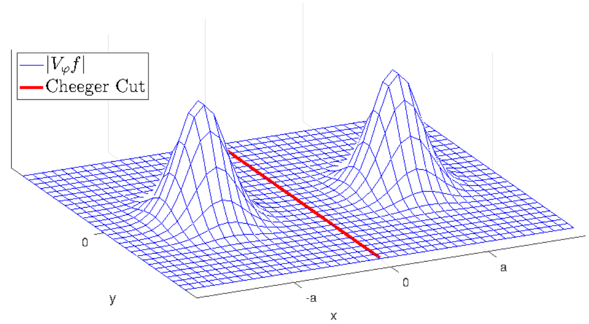

FIGURE 1.1. Standard examples of instabilities are constructed by adding functions whose measurements are essentially supported on sets that are far apart from each other. For the Gabor phase retrieval problem, such instabilities can be constructed as $f(\cdot) = \varphi(\cdot + a) + \varphi(\cdot - a)$, where $\varphi(\cdot) = e^{-\pi \cdot^2}$ denotes the Gaussian and $a > 0$ is a large real number. Since $V_\varphi f(x, y) = V_\varphi \varphi(x + a, y) + V_\varphi \varphi(x - a, y)$ holds true, Lemma A.5 yields that $|V_\varphi f| \approx |V_\varphi g|$, where $g(\cdot) = \varphi(\cdot + a) - \varphi(\cdot - a)$. Cutting the time-frequency plane along the line $x = 0$ results in two sets of equal measure w.r.t. $|V_\varphi f|(x, y) dx\, dy$. On the separating line (called a "Cheeger cut") the weight is small, therefore also the Cheeger constant will be very small. Our main result shows that all instabilities look like the above picture.

- Our results (in particular Corollary 2.10 below) for the first time open the door to the construction of regularization methods for the notoriously ill-posed phase retrieval problem from Gabor measurements. Any useful regularizer will have to promote the connectedness of the measurements in terms of keeping the value $h(f)$ above a certain threshold. To put it more pointedly:

  Contrary to most classical problems in imaging science whose regularization requires the promotion of smoothness or sparsity, the correct regularization of the phase retrieval problem promotes the "connectedness" of the measurements in terms of the Cheeger constant!

  We will explore algorithmic implications in future work.
- Often one has a priori knowledge on the data f to be measured in the sense that f belongs to a compact subset $\mathcal{C} \subset \mathcal{B}$ (such as, for example, piecewise smooth nonnegative functions). By studying the quantity $\inf_{f \in \mathcal{C}} h(f)$ we can for the first time decide what type of a priori knowledge is useful for the phase retrieval problem. We also expect our stability results to lead to insights on how to design masks ω such that the Gabor phase retrieval problem of the masked signal ωf becomes stable.

- In [1] it has been observed that for various applications, such as audio processing, the multicomponent-type instability is actually harmless because the assignment of different bulk phases to different connected components of the measurements is not recognizable by the human ear. Our results show that in fact no other instabilities occur which, for these applications, makes phase retrieval a stable problem! In particular, using our insights we expect to be able to make the concept of "multicomponent instability" of [1] rigorous. Furthermore, in Section 2.2 we outline how to algorithmically find multicomponent decompositions for unstable Gabor measurements using well-established spectral clustering algorithms that are precisely based on minimizing the Cheeger constant associated with the data [45].
- The quantity $h(f)^{-1}$ has another interpretation: it provides a bound for the *Poincaré constant* on the weighted $L^1(\mathbb{R}^2, \mu)$ space with measure $d\mu = |V_\varphi f|dx\,dy$. In fact, our results show that the stability of Gabor phase retrieval is controlled by the Poincaré constant. There exists a huge body of research providing bounds on such weighted Poincaré constants in terms of properties of $|V_\varphi f|$. By our results, every such result directly implies a stability result for phase retrieval from Gabor measurements.
- In Section 2.2 we outline an intimate connection between Gabor phase retrieval and the solution of the backward heat equation. Our results therefore also have implications for the latter problem, which we will study in detail in future work.

Our proof techniques are not restricted to the case of Gabor measurements but crucially assume that, up to multiplication with a smooth function, the measurements $\Phi(f)$ constitute a holomorphic function that is, for example, also satisfied if the measurements arise from a wavelet transform with a Poisson wavelet [48]. In terms of practical applications, the case of Gabor measurements is already of great relevance: Such measurements arise for instance in ptychography, a subfield of diffraction imaging where an extended object is scanned through a highly coherent X-ray beam, producing measurements that can be modeled as Gabor measurements [31, 42, 43]. Another application area is in audio processing, where phase retrieval from Gabor measurements arises in the so-called phase coherence problem for phase vocoders [7, 28, 41].

2 Summary of Our Main Result

2.1 Main Results of This Paper

This section summarizes our main results. We denote by $\mathcal{S}(\mathbb{R})$ the space of Schwartz test functions and with $\mathcal{S}'(\mathbb{R})$ its dual, the space of tempered distributions [46]. The short-time Fourier transform (STFT) is then defined as follows.

DEFINITION 2.1. Let $g \in \mathcal{S}(\mathbb{R})$. Then the *short-time Fourier tranform* (STFT) (with window function g) of a tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ is defined as[3]

$$V_g f(x,y) := \big(f, \overline{g(\cdot - x)}e^{-2\pi \mathrm{i} y \cdot}\big)_{\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})}.$$

If $g(t) = \varphi(t) := e^{-\pi t^2}$, we call the arising STFT the Gabor transform.

The functional analytic properties of the STFT are best studied within the framework of modulation spaces as defined below.

DEFINITION 2.2. Given $1 \le p \le \infty$, the *modulation space* $M^{p,p}(\mathbb{R})$ is defined as

$$M^{p,p}(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) : V_g f \in L^p(\mathbb{R}^2)\},$$

with induced norm

$$\|f\|_{M^{p,p}(\mathbb{R})} := \|V_g f\|_{L^p(\mathbb{R}^2)}.$$

Its definition is independent of $g \in \mathcal{S}(\mathbb{R})$; see [32].

Our goal will be to restore a signal f in a modulation space $M^{p,p}(\mathbb{R})$ from its phaseless Gabor measurements $|V_\varphi f| : \mathbb{R}^2 \to \mathbb{R}_+$, up to a global phase.

It is well-known that for any suitable window function the resulting phase retrieval problem is uniquely solvable:

THEOREM 2.3. *Suppose that* $g \in \mathcal{S}(\mathbb{R})$ *is such that its* ambiguity function

$$\mathcal{A}(g)(x,y) := \int_{\mathbb{R}} g(t)\overline{g(t-x)}e^{-2\pi \mathrm{i} t y}\,dt, \quad (x,y) \in \mathbb{R}^2,$$

is nonzero everywhere. Then, for any $f, h \in \mathcal{S}'(\mathbb{R})$ *with* $|V_g f| = |V_g h|$, *there exists* $\alpha \in \mathbb{R}$ *such that* $f = e^{\mathrm{i}\alpha} h$.

PROOF. This is essentially folklore. For the convenience of the reader we provide a proof in Appendix A. □

Since the Gabor window $\varphi(t) = e^{-\pi t^2}$ satisfies the assumptions of Theorem 2.3, we know that any f is uniquely, up to global phase, determined by its Gabor transform magnitudes $|V_\varphi f|$. For nice signals we even have an explicit reconstruction formula (see Theorem A.3 in Appendix A):

$$f(t) \cdot \overline{f(0)} = \mathcal{F}_2^{-1}\big(S\mathcal{F}|V_g f|^2/\mathcal{A}g\big)(t,t),$$

where $\mathcal{F}_2$ denotes the Fourier transform operator w.r.t. the second variable and S is defined by $SF(x,y) = F(y,x)$. We do not know, however, how to exploit this formula for the question of stability of our phase retrieval problem, and our methods do not make use of it.

What makes the Gabor transform special is that it possesses a lot of additional structure compared with an ordinary STFT. For instance, it turns out that the Gabor

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[3] If f is a regular tempered distribution, i.e., abusing notation,

$$(f,g)_{\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})} := \textstyle\int_{\mathbb{R}} f(t)g(t)dt \quad \text{for } g \in \mathcal{S}(\mathbb{R}),$$

we would get the usual formula $V_g f(x,y) = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi \mathrm{i} y t}\,dt$.

transform of a tempered distribution is, after simple modifications, a holomorphic function.

THEOREM 2.4. *Let* $z := x + \mathbf{i}y \in \mathbb{C}$. *Define* $\eta(z) := e^{\pi(|z|^2/2-\mathbf{i}xy)}$. *Then for every* $f \in \mathcal{S}'(\mathbb{R})$ *the function* $x+\mathbf{i}y \mapsto \eta(x,y)\cdot V_\varphi f(x,-y)$ *is an entire function.*

PROOF. This is again well-known, at least for $f \in L^2(\mathbb{R})$; see, for example, [6], where it is also shown that φ is essentially the only window function with this property. For the convenience of the reader, we present a proof in Appendix A. □

We are interested in stability estimates of the form (1.2). To this end we need to put a norm $\|\cdot\|_{\mathcal{D}}$ on the measurement space $\mathcal{S}'(\mathbb{R}^2)$. A suitable family of norms on the measurement space turns out to be the following:

DEFINITION 2.5. For $1 \le p, q < \infty$, $s > 0$, $r \in \mathbb{N}$, $D \subset \mathbb{R}^2$, and $F : D \to \mathbb{C}$ sufficiently smooth, we define the norms

$$\|F\|_{\mathcal{D}^{r,s}_{p,q}(D)} := \|F\|_{W^{r,p}(D)} + \|F\|_{L^q(D)} + \|(|x|+|y|)^s F(x,y)\|_{L^q(D)}$$

where $\|\cdot\|_{W^{r,p}(D)}$ denotes the Sobolev norm as defined in Section 2.5.

If $D = \mathbb{R}^2$ we simply write $\mathcal{D}^{r,s}_{p,q}$ instead of $\mathcal{D}^{r,s}_{p,q}(\mathbb{R}^2)$.

For $q = p$ and $s = 0$ the norm $\|.\|_{\mathcal{D}^{r,0}_{p,p}(D)}$ is equivalent to the Sobolev norm $\|.\|_{W^{r,p}(D)}$.

Remark 2.6. It may appear slightly irritating that the norms on measurement space include a polynomial weight. It turns out that without any polynomial weight (for example, putting $p = q = 2$ and $r = s = 0$), the stability constant $c(f)$ will in general be infinite (as a nontrivial exercise the reader may verify this for the function $f(t) = \frac{1}{1+t^2}$). In a sense the norms $\mathcal{D}^{r,s}_{p,q}(D)$ possess some symmetry between the space domain and the Fourier domain in the sense that they promote both spatial as well as Fourier-domain localization.

The norms as just introduced measure the time-frequency concentration of F in terms of both smoothness and spatial localization. Note that the last term in its definition, $\|(|x|+|y|)^s F(x,y)\|_{L^q(\mathbb{R}^2)}$, is not translation-invariant and therefore it will be convenient to apply the norm to what we call centered functions.

DEFINITION 2.7. A function $F : \mathbb{R}^2 \to \mathbb{C}$ is *centered* if $|F|$ possesses a maximum at the origin $(x,y) = (0,0)$.

Our setup is now complete; with $\Phi = (\varphi(\cdot - x)e^{2\pi \mathbf{i} y\cdot})_{(x,y)\in\mathbb{R}^2}$, $\mathcal{B} = M^{p,p}(\mathbb{R})$ a modulation space, and $\mathcal{D}$ the norm as defined above, we are interested in estimating the constant $c(f)$ as defined in (1.2).

The main insight of this paper is that the constant $c(f)$ behaves like the reciprocal of what we call the p-Cheeger constant of f. It is defined as follows:

DEFINITION 2.8. Let $p \in [1, \infty)$ and $D \subset \mathbb{R}^2$. For $f \in M^{p,p}(\mathbb{R})$, the *p-Cheeger constant* is defined by

$$h_{p,D}(f) := \inf_{\substack{C \subset D \text{ open: } \partial C \cap D \text{ is smooth,} \\ \int_C |V_\varphi f|^p \leq \frac{1}{2} \int_D |V_\varphi f|^p}} \frac{\|V_\varphi f\|_{L^p(\partial C)}^p}{\|V_\varphi f\|_{L^p(C)}^p}. \tag{2.1}$$

If $D = \mathbb{R}^2$ we simply write $h_p(f)$ instead of $h_{p,\mathbb{R}^2}(f)$.

As already mentioned in the introduction we borrowed here a term from spectral geometry. Indeed, our definition of $h_p(f)$ is equal to the usual Cheeger constant of the flat Riemannian manifold $\mathbb{R}^2$, conformally multiplied with $|V_\varphi f(x, y)|^p$; see [15].

We are ready to give an appetizer to our results by stating the following theorem, which confirms that disconnected measurements form the only source of instabilities for Gabor phase retrieval.

THEOREM 2.9. *Let $p \in [1, 2)$ and $q \in (\frac{2p}{2-p}, \infty)$. Suppose that $f \in M^{p,p}(\mathbb{R})$ is such that its Gabor transform $V_\varphi f$ is centered. Then there exists a constant $c > 0$* only depending on p, q, and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ *such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that*

$$d_{M^{p,p}(\mathbb{R})}(f, g) \leq c \cdot (1 + h_p(f)^{-1}) \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}}.$$

Theorem 2.9 is proved in Section 5.3, where the identical statement is given again in Theorem 5.11 for the reader's convenience. A local stability result is provided in Theorem 5.12.

The theorem above also establishes a noise stability result for reconstruction of a signal from noisy spectrogram measurements

$$\text{noisy measurements} = |V_\varphi f| + \eta.$$

COROLLARY 2.10. *Let $p \in [1, 2)$ and $q \in (\frac{2p}{2-p}, \infty)$. Suppose that $f \in M^{p,p}(\mathbb{R})$ is such that its Gabor transform $V_\varphi f$ is centered. Then there exists a constant $c > 0$* only depending on p, q, and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ *such that for any $\eta \in \mathcal{D}^{1,4}_{p,q}$ with $\|\eta\|_{\mathcal{D}^{1,4}_{p,q}} \leq \nu$ and any*

$$h \in \operatorname{argmin}_{g \in M^{p,p}(\mathbb{R})} \|(|V_\varphi f| + \eta) - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}},$$

it holds that

$$d_{M^{p,p}(\mathbb{R})}(f, h) \leq c \cdot (1 + h_p(f)^{-1}) \cdot \nu.$$

Due to its simplicity, we present the proof here.

PROOF. By Theorem 2.9, it holds that

$$\begin{aligned} d_{M^{p,p}(\mathbb{R})}(f, h) &\leq c \cdot (1 + h_p(f)^{-1}) \cdot \||V_\varphi f| - |V_\varphi h|\|_{\mathcal{D}^{1,4}_{p,q}} \\ &\leq c \cdot (1 + h_p(f)^{-1}) \cdot \big(\|(|V_\varphi f| + \eta) - |V_\varphi h|\|_{\mathcal{D}^{1,4}_{p,q}} + \nu\big). \end{aligned}$$

To finish the argument we note that, due to the definition of h, it holds that

$$\|(|V_\varphi f|+\eta)-|V_\varphi h|\|_{\mathcal{D}^{1,4}_{p,q}} \le \|(|V_\varphi f|+\eta)-|V_\varphi f|\|_{\mathcal{D}^{1,4}_{p,q}} \le \nu. \qquad \square$$

Typically, one is mainly interested in the reconstruction of a specific time-frequency region of f. To this end, we will also establish a local stability result, of which we here offer a special case in the following theorem.

THEOREM 2.11. *Let $p \in [1,2)$, $q \in (\frac{2p}{2-p},\infty)$ and $R>0$. Suppose that $f \in M^{p,p}(\mathbb{R})$ is such that its Gabor transform $V_\varphi f$ is centered. Suppose further that f is ε-concentrated on a ball $B_R(0) \subset \mathbb{R}^2$ in the sense that*

$$\int_{\mathbb{R}^2\setminus B_R(0)} |V_\varphi f(x,y)|^p\,dx\,dy \le \varepsilon^p.$$

Then there exists a constant $c>0$ only depending on p,q, and

$$\max\left\{\frac{\|V_\varphi f\|_{L^p(B_R(0))}}{\|V_\varphi f\|_{L^\infty(B_R(0))}}, \frac{\|V_{\varphi'} f\|_{L^\infty(B_R(0))}}{\|V_\varphi f\|_{L^\infty(B_R(0))}}\right\}$$

such that for any $g \in M^{p,p}(\mathbb{R})$ that is ε-concentrated in $B_R(0)$, it holds that

$$d_{M^{p,p}(\mathbb{R})}(f,g) \le c\cdot\big((1+h_{p,B_R(0)}(f)^{-1})\cdot\||V_\varphi f|-|V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}(B_R(0))}+\varepsilon\big).$$

Similarly to Corollary 2.10, a local noise stability result can also be deduced in an obvious way. We leave the details to the reader.

2.2 Putting Our Results in Perspective

In this subsection we briefly relate our results to the stable solution of the backwards heat equation and our previous work [1].

Connections with the Backwards Heat Equation

We would like to draw the reader's attention to an intricate connection between phase retrieval and the solution of the backwards heat equation.

Consider the heat equation in the plane:

$$\begin{aligned} u_t(t,x,y) &= \Delta u(t,x,y) = u_{xx}(t,x,y)+u_{yy}(t,x,y), \\ u(0,x,y) &= f(x,y), \quad x,y\in\mathbb{R},\ t>0. \end{aligned} \tag{2.2}$$

The backward heat equation problem, namely, (stably) reconstructing the initial value f given $u(t,\cdot,\cdot)$ for fixed t, is known to be severely ill-posed. Solving the heat equation in the frequency domain yields

$$\hat{u}(t,\xi,\eta) = \hat{f}(\xi,\eta)\cdot e^{-4\pi^2(\xi^2+\eta^2)t}.$$

Therefore solving the backward heat equation problem amounts to deconvolving $u(t,\cdot,\cdot)$ with a Gaussian kernel.

In Appendix A we show that

$$\mathcal{F}|V_g f|^2(\eta,\xi) = \mathcal{A}f(\xi,\eta)\cdot\mathcal{A}g(\xi,\eta)$$

(where $\mathcal{F}$ denotes the two-dimensional Fourier transform) as well as the fact that $\mathcal{A}g$ is a two-dimensional Gaussian for the Gaussian window $g = e^{-\pi \cdot^2}$. Thus, reconstructing the ambiguity function of f from the absolute values of its Gabor transform amounts to solving the backward heat equation problem. Consequently, the Gabor phase retrieval problem and the backward heat equation problem, as well as their stabilization, are closely related. We consider the investigation of the consequences of our results for the stabilization of the backwards heat equation an interesting problem for future work.

Comparison with the Results of [1]

Our result is very much inspired by stability results in recent work [1] by Rima Alaifari, Ingrid Daubechies, Rachel Yin, and one of the authors, and in fact grew out of this work.

In order to put our current results in perspective and to demonstrate the improvement of our present results over those in [1], we give a short comparison between the main stability results of [1] and the present paper.

In [1] it is shown that, for certain measurement scenarios (including Gabor and Poisson wavelet measurements), stable phase reconstruction is locally possible on subsets $\Omega' \subset \Omega$ on which the variation of the measurements, namely,

$$\frac{\sup_{\omega \in \Omega'} |\varphi_\omega(f)|}{\inf_{\omega \in \Omega'} |\varphi_\omega(f)|}$$

is bounded. However, in an ∞-dimensional problem, this quantity will not be bounded and therefore the results of [1] do not provide bounds for $c(f)$.

For concreteness we compare the sharpness of our result to the results of [1] at hand of a very simple example, namely a Gaussian signal $f = e^{-\pi t^2}$. A simple calculation (see Lemma A.5) reveals that

$$|V_\varphi f(x, y)| = r e^{-\pi/2(x^2+y^2)}$$

for some positive number r. Clearly, f is ε-concentrated on $B_R(0)$ with $\varepsilon \lesssim e^{-\pi/2R^2}$.

The results of [1] rely on the assumption that the measurements $V_\varphi f$ are of little variation on the domain of interest, which for our particular example is $B_R(0)$. The main parameter governing the stability in the results of [1] would be

$$\frac{\sup_{(x,y)\in B_R(0)} |V_\varphi f(x, y)|^2}{\inf_{(x,y)\in B_R(0)} |V_\varphi f(x, y)|^2} = e^{\pi R^2},$$

and the best stability bound that can be achieved using the results of [1] is thus of the form

$$\inf_{\alpha \in \mathbb{R}} \|f - e^{\mathrm{i}\alpha} g\|_{M^{2,2}(\mathbb{R})} \leq c \cdot \big(e^{\pi R^2} \cdot \| |V_\varphi f| - |V_\varphi g| \|_{W^{1,2}(B_R(0))} + e^{-\pi/2R^2}\big), \tag{2.3}$$

where g is an arbitrary function that is also ε-concentrated on $B_R(0)$.

We see that the stability bound obtainable from the results of [1] grows exponentially in R^2, which still suggests that the problem to reconstruct f from its spectrogram is severely ill-posed.

It turns out that this is not the case. In Appendix B (Theorem B.12) we see that $h_{p,B_R(0)}(f) \gtrsim 1$ with the implicit constant independent of R (in fact, this is well-known and follows from the Gaussian isoperimetric inequality and geometric arguments).

We can thus directly apply Theorem 2.11 and get the following:

THEOREM 2.12. *Let $f(t) = e^{-\pi t^2}$. Let $p \in [1,2)$, $q \in (\frac{2p}{2-p}, \infty)$, and $\varepsilon > 0$. Then there exists a constant $c > 0$* only depending on p, q, and ε *such that for any $R > 1$ and $g \in M^{p,p}(\mathbb{R})$ that is ε-concentrated in $B_R(0)$, it holds that*

$$\begin{aligned}\inf_{\alpha\in\mathbb{R}} &\|f - e^{\mathrm{i}\alpha} g\|_{M^{p,p}(\mathbb{R})} \\ &\leq c \cdot \big(\||V_\varphi f| - |V_\varphi g|\|_{W^{1,p}(B_R(0))} + R^4 \cdot \||V_\varphi f| \\ &\qquad - |V_\varphi g|\|_{L^q(B_R(0))} + e^{-\pi/2R^2}\big).\end{aligned}$$

We remark that a more careful analysis (which exploits the specific form of f) would yield an estimate of the form

$$\inf_{\alpha\in\mathbb{R}} \|f - e^{\mathrm{i}\alpha} g\|_{M^{p,p}(\mathbb{R})} \leq c \cdot \big(R \cdot \||V_\varphi f| - |V_\varphi g|\|_{W^{1,p}(B_R(0))} + e^{-\pi/2R^2}\big), \tag{2.4}$$

valid for every $p \in [1, \infty]$.

Comparing our result (2.4) with the bound (2.3) from [1], we see that our bound is much tighter. In particular,

> *our bound turns a superexponential growth of the stability constant into a low-order polynomial growth*!

A Partitioning Algorithm

Again we want to take up an idea from [1], where the concept of multicomponent phase retrieval was introduced: The multicomponent paradigm amounts to the following identification of measurements $F = V_\varphi f$, $G = V_\varphi g$:

$$F = \sum_{j=1}^{k} F_j \sim G = \sum_{j=1}^{k} e^{\mathrm{i}\alpha_j} F_j$$

for any $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ where the components $F_1, \dots, F_k$ are essentially supported on mutually disjoint domains $D_1, \dots, D_k$. This means we consider F and G to be close to each other whenever the quantity

$$\inf_{\alpha_1,\dots,\alpha_k} \sum_{j=1}^{k} \|F - e^{\mathrm{i}\alpha_j} G\|_{L^p(D_j)}$$

is small. Thus we no longer demand that there be a global phase factor but allow different phase factors that are constant on the distinct subdomains D_i. Since the human ear cannot recognize an identification $F \sim G$ whenever the measurements F_j are distant from each other, this notion of distance is sensible for the purpose of applications in audio.

Assume we are given a signal f such that the Cheeger constant $h_p(f)$ is small, meaning that we will expect the phase retrieval problem to be very unstable. A natural question to ask is whether it is possible to partition the time-frequency plane in subdomains $D_1, \dots, D_k$ such that Gabor phase retrieval is stable in the multicomponent sense, i.e.,

$$\inf_{\alpha_1,\dots,\alpha_k \in \mathbb{R}} \sum_{j=1}^{k} \|V_\varphi f - e^{\mathrm{i}\alpha_j} V_\varphi g\|_{L^p(D_j)} \le B \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}} \tag{2.5}$$

for moderately large $B > 0$ and all g.

Obviously the finer the partition, the smaller B will become. However, in view of the motivation from audio applications we will not want to choose a very fine partition, because then the corresponding multicomponent distance will not be naturally meaningful.

The challenge therefore is to find, given a signal f, a partition $D_1, \dots, D_k$ such that

(i) B is small and
(ii) the measurements $V_\varphi f \cdot \chi_{D_j}$ and $V_\varphi f \cdot \chi_{D_l}$ are distant for all $j \neq l$

simultaeously hold.

Corollary 5.14 tells us that B can essentially be bounded by the quantity

$$\min_{j=1,\dots,k} (1 + h_{p,D_j}(f)^{-1}) \cdot \left(1 + \frac{\kappa_j^p}{\delta_j^2}\right), \tag{2.6}$$

where

$$\delta_j = \min\left\{\sup\{r > 0 \colon B_r(z) \subset D, \inf_{\zeta \in B_r(z)} |V_\varphi f(\zeta)| \ge \frac{1}{2}\|V_\varphi f\|_{L^\infty(D_j)}\}, 1\right\}$$
$$\text{and} \quad \kappa_j = \frac{\|V_\varphi f\|_{L^p(D_j)}}{\|V_\varphi f\|_{L^\infty(D_j)}}.$$

Since in practice one only has finitely many samples of $|V_\varphi f|$ at hand, we consider a discrete version of this partitioning problem. Spectral clustering methods from graph theory provide algorithms that aim at finding partitions minimizing a discrete Cheeger ratio [12]. We now suggest an iterative approach. Once the domain D is partitioned into two components C and $D \setminus C$ (see Figure 2.1 top) we can again measure the disconnectedness of these two sets by estimating their respective Cheeger constants (see Figure 2.1 bottom left). If this estimate lies above a given threshold, we leave the set untouched in view of (ii). Otherwise we partition again. After carrying out this iterative procedure a few times, we expect to arrive at a

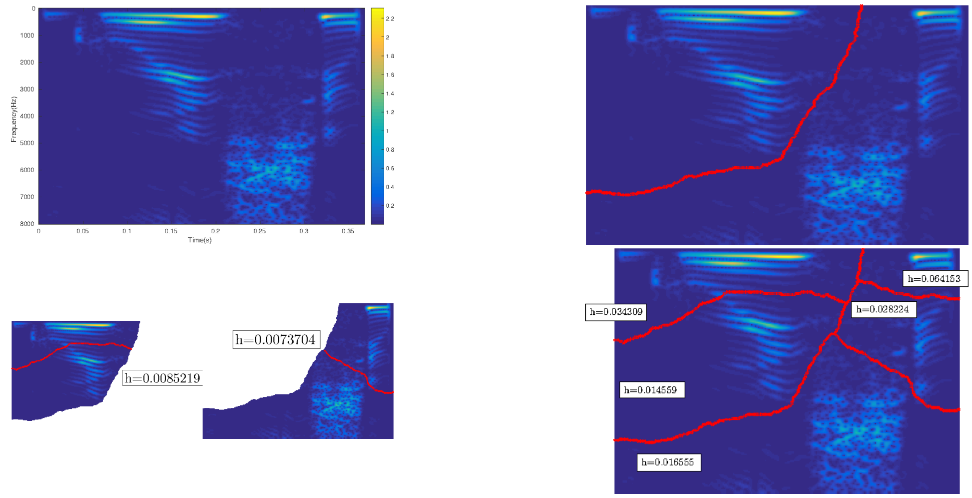

FIGURE 2.1. Top left: Magnitudes of the discrete Gabor transform of the signal "greasy" from the LTFAT toolbox (`http://ltfat.sourceforge.net/`). Top right: Partitioning of $|V_\varphi f|$ leading to a Cheeger constant $h_D(f) \approx 0.0019119$. Bottom left and middle: Further partitionings. Bottom right: The algorithm terminates as soon as the (estimated) Cheeger constants of all subdomains are above a given threshold.

partition $C_1, \dots, C_l$ of D such that each C_j is well connected (in terms of the Cheeger constant being large) and simultaneously for any $k \neq j$, the set $C_k \cup C_j$ is very disconnected (in terms of the Cheeger constant being small). We hence find a partition such that $h_{p,C_j}(f)$ is moderately large for all j. However, to use Theorem 5.14 we also need δ_j not to be too small and κ_j not to be too large, which can be verified a posteriori.

In Appendix C we describe the algorithm we used for the experiment illustrated in Figure 2.1 in detail.

2.3 Architecture of the Proof

The proof of our main result is quite convoluted and draws on techniques from different mathematical fields such as complex analysis, functional analysis, and spectral Riemannian geometry. For the benefit of the reader we provide a short sketch of our argument before we go into the details in the later sections.

Let us start with the following observation: Given two functions $F_1, F_2 : D \to \mathbb{C}$, we have

$$\inf_{\alpha\in\mathbb{R}} \|F_1 - e^{\mathrm{i}\alpha} F_2\|_{L^p(D)}^p = \inf_{a\in\mathbb{C},\ |a|=1} \int_D \left| \frac{F_2(z)}{F_1(z)} - a \right|^p w(z)\,dz, \tag{2.7}$$

where $w(z)dz$ is the Lebesgue measure with density $w(z) = |F_1(z)|^p$.

Now suppose that we could just disregard the constraint $|a| = 1$ in the above formula (2.7) (in Section 4 we develop tools which effectively amount to an equivalent result). Then, using the notation $L^p(D, w)$ for the L^p space with respect to the measure $w\,dz$, we would need to estimate a term of the form

$$\inf_{a\in\mathbb{C}} \left\| \frac{F_2}{F_1} - a \right\|_{L^p(D,w)}. \tag{2.8}$$

The Poincaré inequality tells us that (provided w and D are "nice") there exists a constant $C_{\mathrm{poinc}}(p, D, w) < \infty$, depending only on the domain and the weight, such that (2.7) can be bounded by

$$C_{\mathrm{poinc}}(p, D, w) \cdot \left\| \nabla \frac{F_2}{F_1} \right\|_{L^p(D,w)}. \tag{2.9}$$

Now spectral geometry enters the picture. Cheeger's inequality [16] says that the Poincaré constant on a Riemannian manifold can be controlled by the reciprocal of the Cheeger constant. We would like to apply this result to the metric induced by the metric tensor

$$\left(w(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)_{z\in D}$$

in order to get a bound on $C_{\mathrm{poinc}}(p, D, w)$. However, since w in our case arises from Gabor measurements, it generally has zeros and therefore does not qualify as a Riemannian manifold. In Appendix B.1 we will show that for $F_1 = V_\varphi f$

$$C_{\mathrm{poinc}}(p, D, w) \leq \frac{4p}{h_{p,D}(f)},$$

where $h_{p,D}(f)$ as defined in Definition 2.8 holds true, nevertheless.

Assuming that all heuristics up to this point were correct, we get a bound of the form

$$\inf_{\alpha\in\mathbb{R}} \|F_1 - e^{\mathbf{i}\alpha} F_2\|_{L^p(D)} \leq c \cdot h_{p,D}(f)^{-1} \cdot \left\| \nabla \frac{F_2}{F_1} \right\|_{L^p(D,w)},$$

where here and in the following c denotes an unspecified constant.

We are faced with the problem of converting $\|\nabla(F_2/F_1)\|_{L^p(D,w)}$ into a useful estimate in the difference $|F_1| - |F_2|$.

Now complex analysis enters the picture. If $F_1(z) = V_\varphi f(x, -y)$ and $F_2(z) = V_\varphi g(x, -y)$ it is known that the quotient F_2/F_1 is a meromorphic function (Theorem 2.4), which, almost everywhere, satisfies the Cauchy-Riemann equations. It is a simple exercise (Lemma 3.4) to verify that for any meromorphic function $F = F_2/F_1$ it holds that

$$|\nabla F| = \sqrt{2}\big|\nabla|F|\big| \tag{2.10}$$

almost everywhere. This is great, since we now can get a bound that only depends on the absolute values $|F_1|$ and $|F_2|$!

To summarize, if all our heuristics were correct, we would get a bound of the form

$$\inf_{\alpha\in\mathbb{R}} \|F_1 - e^{\mathbf{i}\alpha} F_2\|_{L^p(D)} \leq c \cdot h_{p,D}(f)^{-1} \cdot \left\| \nabla \frac{|F_2|}{|F_1|} \right\|_{L^p(D,w)}.$$

If we now apply the quotient rule to the estimate above and utilize the fact that $w = |F_1|^p$, we would get a bound of the form

$$\begin{aligned}
&\inf_{\alpha\in\mathbb{R}} \|F_1 - e^{\mathbf{i}\alpha} F_2\|_{L^p(D)} \\
&\quad \leq c \cdot h_{p,D}(f)^{-1} \\
&\qquad \cdot \left(\left\| \left(\frac{\nabla|F_1|}{|F_1|} \right) \cdot (|F_1| - |F_2|) \right\|_{L^p(D)} + \|\nabla|F_1| - \nabla|F_2|\|_{L^p(D)} \right).
\end{aligned} \tag{2.11}$$

This is precisely Theorem 3.3 and Theorem 5.3 (although the details of these results and their proofs are significantly more delicate than this informal discussion may suggest; see Section 4).

The estimate (2.11) is already close to what one would like to have, were it not for the term $\nabla|F_1|/|F_1|$ in the first summand of the right-hand side of (2.11). Indeed, since F_1 will in general have zeroes, this term will not be bounded.

Here again complex analysis will come to our rescue: The function $F_1(z) = V_\varphi f(x,-y)$ is, after multiplication with a suitable function η, an entire function of order 2. Jensen's formula [20] provides bounds for the distribution of zeros of F_1 and this allows us to show that, for $1 \leq p < 2$ the norms $\|\nabla|F_1|/|F_1|\|_{L^p(B_R(0))}$ grow at most like a low-order polynomial in R, which is, remarkably, independent of f! These arguments are carried out in Section 5.2.

Finally, we can put all our estimates together and arrive at our main stability theorems, which are summarized in Section 5.3.

2.4 Outline

The outline of this article is as follows. In Section 3 we start by proving a general stability result, Theorem 3.3, for phase retrieval problems. This result, which depends on some at-this-point-unspecified constants, namely an analytic Poincaré constant and a sampling constant, is inspired by and generalizes the main result of [1]. In Section 4 we gain control of the two unspecified constants of the main result in Section 3 and show that they can be controlled in terms of the global variation of the measurements as defined in Definition 4.5; see Theorem 5.3. In Section 5 we specialize to the case of Gabor phase retrieval. We first show that the global variation of Gabor measurements is independent of the signal to be analyzed, which will yield an estimate of the type (2.11); see Theorem 5.3. Finally, in Section 5.2 we remove the logarithmic derivative in the estimate of Theorem 5.3 at the expense of introducing weighted norms in the error estimate; see Proposition 5.7 whose proof requires deep function-theoretic properties of the Gabor transform. In Section 5.3 we formulate and prove our main stability result.

Finally, Appendix A is concerned with auxiliary properties of the Gabor phase retrieval problem, and in Appendix B we state and prove several auxiliary facts related to Cheeger and Poincaré constants. In Appendix C we provide some details on spectral clustering algorithms that aim at estimating Cheeger constants of graphs.

2.5 Notation

We pause here to collect some notation that will be used throughout this article. Since some proofs will turn out to be quite technical, we hope that this will prevent the reader from getting lost in his or her reading.

- For $1 \leq p < \infty$, $D \subset \mathbb{R}^2$, and a weight function $w : D \to \mathbb{R}_+$, we write
$$L^p(D, w) := \{F : D \to \mathbb{C} :: \|F\|_{L^p(D,w)} < \infty\},$$
where
$$\|F\|^p_{L^p(D,w)} := \int_D |F(u)|^p w(u) du.$$
If $w \equiv 1$ we simply write $L^p(D)$ instead of $L^p(D, 1)$.
- For $w : D \to \mathbb{R}_+$ we shall write
$$w(D) := \int_D w(u) du.$$
- For $F : D \to \mathbb{C}$ and $w : D \to \mathbb{R}_+$ we shall write
$$F^w_D := \frac{1}{w(D)} \int_D F(u) w(u) du.$$
- For $1 \leq p < \infty$, $k \in \mathbb{N}$, and a weight function $w : D \to \mathbb{R}_+$, we write (somewhat informally)
$$W^{k,p}(D, w) := \{F : D \to \mathbb{C} :: \|F\|_{W^{k,p}(D,w)} < \infty\}$$
for the Sobolev space, where
$$\|F\|^p_{W^{k,p}(D,w)} := \sum_{\alpha+\beta \leq k} \left\| \frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} F \right\|^p_{L^p(D,w)}.$$
If $w \equiv 1$ we simply write $W^{k,p}(D)$ instead of $W^{k,p}(D, 1)$; see [24].
- We shall often identify $\mathbb{R}^2$ with $\mathbb{C}$ via the isomorphism $(x, y) \in \mathbb{R}^2 \leftrightarrow z := x + \mathbf{i}y \in \mathbb{C}$. Using this identification we may also interpret a subset $D \subset \mathbb{R}^2$ as a subset of $\mathbb{C}$.
- For $z \in \mathbb{C}$ we shall write $B_r(z)$ for the ball of radius r around z.
- For a set $C \subset \mathbb{R}^2$ let $|C|$ denote the two-dimensional Lebesgue measure of C. For a smooth curve $A \subset \mathbb{R}^2$, let $\ell(A)$ denote the euclidean length of A.
- For $D \subset \mathbb{C}$ we denote $\mathcal{O}(D)$ the ring of holomorphic functions on D and $\mathcal{M}(D)$ the field of meromorphic functions on D.

- For $F : D \to \mathbb{C}$ we may write
$$F(z) = u(x, y) + \mathbf{i}v(x, y)$$
where u and v denote the real and imaginary part of F, respectively. We shall also write
$$F'(z) := \frac{\partial}{\partial x}u(x, y) + \mathbf{i}\frac{\partial}{\partial x}v(x, y),$$
whenever defined.
- For $D \subset \mathbb{C}$ we denote χ_D the characteristic function of D.

3 A First Stability Result for Phase Reconstruction from Holomorphic Measurements

The starting point of our work will be a general stability result, Theorem 3.3, which we prove in the present section. The estimate will essentially depend on two quantities: an analytic Poincaré constant and a sampling constant. We will see later on how these two constants can be controlled, but for the time being we simply present their definitions.

DEFINITION 3.1. Given a domain $D \subset \mathbb{C}$, $1 \le p < \infty$, a number $\delta > 0$, a point $z_0 \in D$ such that $B_\delta(z_0) \subset D$, and a weight $w : D \to \mathbb{R}_+$, we define $C^a_{\text{poinc}}(p, D, z_0, \delta, w) > 0$ as the smallest constant such that

$$\|F - F(z_0)\|_{L^p(D,w)} \le C^a_{\text{poinc}}(p, D, z_0, \delta, w)\|F'\|_{L^p(D,w)} \tag{3.1}$$

for all $F \in \mathcal{M}(D) \cap \mathcal{O}(B_\delta(z_0)) \cap W^{1,p}(D, w)$.

We will refer to $C^a_{\text{poinc}}(p, D, z_0, \delta, w)$ as an "analytic Poincaré constant." We will see later on, in Section 4.1, how one can control this quantity.

Next we define what we call a "sampling constant."

DEFINITION 3.2. Let D be a domain, $w : D \to \mathbb{R}_+$, and $G \in L^p(D, w)$. Then we define, for $z_0 \in D$, $1 \le p < \infty$, the *sampling constant*

$$C_{\text{samp}}(p, D, z_0, G, w) := \frac{\|G(z_0)\|_{L^p(D,w)}}{\|G\|_{L^p(D,w)}} = \frac{|G(z_0)| \cdot w(D)^{1/p}}{\|G\|_{L^p(D,w)}}.$$

Later on, in Section 4.2 we will see how to control this quantity.

Having defined the notion of analytic Poincaré constant and sampling constant, we can now state and prove the following general stability result.

THEOREM 3.3. *Let $D \subset \mathbb{C}$ and $1 \le p < \infty$. Suppose that $F_1, F_2 \in L^p(D)$ are smooth functions such that there exists a continuous, nowhere-vanishing function $\eta : D \to \mathbb{C}$ for which both functions $\eta \cdot F_1, \eta \cdot F_2 \in \mathcal{O}(D)$.*

Suppose that $z_0 \in D$ and $\delta > 0$ with $B_\delta(z_0) \subset D$ and

$$|F_1(z)| > 0 \quad \textit{for all } z \in B_\delta(z_0).$$

Then the following estimate holds:

$$
\begin{aligned}
&\inf_{\alpha\in\mathbb{R}} \|F_1 - e^{\mathbf{i}\alpha}F_2\|_{L^p(D)} \\
&\quad\le C_{\text{samp}}(p, D, z_0, |F_2/F_1| - 1, |F_1|^p)\||F_2| - |F_1|\|_{L^p(D)} \\
&\qquad + C^a_{\text{poinc}}(p, D, z_0, \delta, |F_1|^p) \\
&\qquad\quad \cdot \big(\|\nabla|F_1| - \nabla|F_2|\|_{L^p(D)} + \|\nabla \log|F_1|(|F_1| - |F_2|)\|_{L^p(D)}\big).
\end{aligned}
\tag{3.2}
$$

We remark that this result draws its inspiration from, and generalizes, the main result of [1]. At its heart lies the following elementary lemma, which is proved in [1] and which follows directly from the Cauchy-Riemann equations.

LEMMA 3.4. *Suppose that $F \in \mathcal{M}(D)$. Then for any $z = x + \mathbf{i}y \in D$ that is not a pole of F we have the equality*

$$
|F'(z)| = |\nabla|F|(x, y)| = \sqrt{(|F|_x(x, y))^2 + (|F|_y(x, y))^2}.
$$

Having Lemma 3.4 at hand, we can now proceed to the proof of the main result of this section.

PROOF OF THEOREM 3.3. We need to bound the quantity

$$
\|F_2(z) - e^{\mathbf{i}\alpha}F_1(z)\|_{L^p(D)} \tag{3.3}
$$

for suitable $\alpha \in \mathbb{R}$.

Step 1. As a first step we start by developing a basic estimate. Consider

$$
F := F_2/F_1.
$$

By assumption it holds that $F \in \mathcal{M}(D)$.

Pick α such that

$$
|F(z_0) - e^{\mathbf{i}\alpha}| = \big||F(z_0)| - 1\big|. \tag{3.4}
$$

Now consider for $z \in D$ arbitrary

$$
\begin{aligned}
|F_2(z) - e^{\mathbf{i}\alpha}F_1(z)| &= |F_1(z)||F(z) - e^{\mathbf{i}\alpha}| \\
&\le |F_1(z)|(|F(z) - F(z_0)| + |F(z_0) - e^{\mathbf{i}\alpha}|) \\
&= |F_1(z)| \cdot |F(z) - F(z_0)| + |F_1(z)| \cdot ||F(z_0)| - 1|.
\end{aligned}
\tag{3.5}
$$

It follows that

$$
\begin{aligned}
\|F_2(z) - e^{\mathbf{i}\alpha}F_1(z)\|_{L^p(D)} &\le \|F(z) - F(z_0)\|_{L^p(D,|F_1|^p)} \\
&\quad + \||F(z_0)| - 1\|_{L^p(D,|F_1|^p)} =: \text{(I)} + \text{(II)}.
\end{aligned}
$$

Step 2. Estimating (II). By Definition 3.2 with $w = |F_1|^p$, we see that

$$
\text{(II)} = C_{\text{samp}}(p, D, z_0, |F| - 1, w)\||F_1| - |F_2|\|_{L^p(D)}.
$$

Step 3. Estimating (I). By Definition 3.1 with $w = |F_1|^p$ and $F \in \mathcal{O}(B_\delta(z_0))$ (which follows from the fact that F_1 is nonzero on $B_\delta(z_0)$), we get that

$$
\text{(I)} \le C^a_{\text{poinc}}(p, D, z_0, \delta, w) \cdot \|F'\|_{L^p(D,w)}. \tag{3.6}
$$

We now need to get a bound on $\|F'\|_{L^p(D,w)}$ in terms of $\||F_1| - |F_2|\|_{W^{1,p}(D)}$ to finish the proof. This is where our key lemma, Lemma 3.4, comes into play, stating that

$$\|F'\|_{L^p(D,w)} = \|\nabla|F|\|_{L^p(D,w)};$$

see also (2.10). It thus remains to achieve a bound for $\|\nabla|F|\|_{L^p(D,w)}$. To this end we calculate

$$\begin{aligned}\nabla|F| &= \frac{|F_1|\nabla|F_2| - |F_2|\nabla|F_1|}{|F_1|^2} \\ &= |F_1|^{-2}(\nabla|F_1|(|F_1| - |F_2|) + |F_1|(\nabla|F_2| - \nabla|F_1|)),\end{aligned}$$

which holds at least for all points where neither F_1 nor F_2 vanishes, hence almost everywhere. We get that

$$\|\nabla|F|\|_{L^p(D,w)} \le \|\nabla \log|F_1|(|F_1|-|F_2|)\|_{L^p(D)} + \|\nabla|F_2|-\nabla|F_1|\|_{L^p(D)}. \quad \square$$

As it stands, Theorem 3.3 is not yet satisfactory for at least two reasons. First, it is not yet clear how the analytic Poincaré constant and the sampling constant can be (simultaeously) controlled. Second, the term $\|\nabla \log|F_1|(|F_1|-|F_2|)\|_{L^p(D)}$ in the estimate (3.2) is difficult to interpret since the logarithmic derivative $\nabla \log|F_1|$ will in general be unbounded. The purpose of the remainder of this article is to show that all these dependencies can be absorbed into a natural quantity that describes the degree of disconnectedness of the measurements.

4 Balancing the Constants

Having the technical result in Theorem 3.3 at hand, the next task is to get a grip on the error term on the right-hand side of (3.2). Indeed, we will show that both the analytic Poincaré constant, as well as the sampling constant, can be simultaneously controlled.

4.1 Weighted Analytic Poincaré Inequalities

While the concept of the analytic Poincaré constant does not seem to be very widely studied, the classical Poincaré constant as defined next is certainly much better known.

DEFINITION 4.1. For $1 \le p < \infty$ denote by $C_{\mathrm{poinc}}(p, D, w)$ the *Poincaré constant* of the domain D w.r.t. the weight w, i.e., the optimal constant C such that for all $F \in W^{1,p}(D, w) \cap \mathcal{M}(D)$ we have

$$\|F - F_D^w\|_{L^p(D,w)} \le C\|\nabla F\|_{L^p(D,w)}.$$

There exists a huge body of work devoted to the study of weighted Poincaré inequalities as just described. In Appendix B we present a collection of results that are especially relevant for the present paper.

Remark 4.2. Observe that in Definition 4.1, the defining inequality only needs to be satisfied for meromorphic functions. This is certainly nonstandard but sufficient for our purposes, where F will always be the quotient of two (up to normalization) holomorphic functions. The reason for this somewhat odd definition is that we will ultimately estimate the Poincaré constant in terms of the Cheeger constant related to the measurements. The proof of this estimate is carried out in Appendix B, but it does not necessarily apply to all functions $F \in W^{1,p}(D,w)$, the reason being the famous *Lavrentiev phenomenon*, which states that smooth functions need not necessarily be dense in $W^{1,p}(D,w)$ [50].

The next result shows that analytic Poincaré constants as defined in Definition 3.1 can be, to some extent, controlled by the usual Poincaré constant as defined in Definition 4.1.

LEMMA 4.3. *With the notation of Definition* 3.1*, we have the estimate*

$$\begin{aligned} &C^a_{\text{poinc}}(p,D,z_0,\delta,w) \\ &\quad\le C_{\text{poinc}}(p,D,w) \\ &\qquad\cdot\left(1+w(D)^{1/p}\cdot\inf_{0<a\le\delta}\frac{w(B_a(z_0))^{1-1/p}\|w^{-1}\|_{L^\infty(B_a(z_0))}}{|B_a(z_0)|}\right). \end{aligned}\tag{4.1}$$

PROOF. The analytic Poincaré inequality as defined in Definition 3.1 applies to functions F that are holomorphic in $B_\delta(z_0)$, so, for any disc $B := B_a(z_0) \subset D$ with $0 < a < \delta$, it holds that $F(z_0) = F_B := \frac{1}{|B|}\int_B F(z)dz$; thus we need to estimate

$$\begin{aligned} \|F-F(z_0)\|_{L^p(D,w)} &= \|F-F_B\|_{L^p(D,w)} \\ &\le \|F-F^w_D\|_{L^p(D,w)} + \|F_B-F^w_D\|_{L^p(D,w)}. \end{aligned}\tag{4.2}$$

The first summand above is bounded by $C_{\text{poinc}}(p,D,w)\|\nabla F\|_{L^p(D,w)}$, by the definition of the Poincaré constant.

For the second summand we estimate

$$\begin{aligned} |F_B-F^w_D| &\le \frac{1}{|B|}\int_B |F(z)-F^w_D|dz \\ &\le \frac{\|w^{-1}\|_{L^\infty(B)}}{|B|}\int_B |F(z)-F^w_D|w(z)dz, \end{aligned}$$

which, by Hölder's inequality, can be bounded by

$$\frac{\|w^{-1}\|_{L^\infty(B)}}{|B|}\|F-F^w_D\|_{L^p(D,w)}w(B)^{1-1/p}.$$

Thus, it holds that

$$\|F_B-F^w_D\|_{L^p(D,w)} \le \frac{\|w^{-1}\|_{L^\infty(B)}}{|B|}\|F-F^w_D\|_{L^p(D,w)}w(B)^{1-1/p}w(D)^{1/p}.$$

Applying the Poincaré inequality again yields that the second summand in (4.2) can be bounded by

$$C_{\text{poinc}}(p, D, w) \cdot \frac{\|w^{-1}\|_{L^\infty(B)}}{|B|} w(B)^{1-1/p} w(D)^{1/p} \cdot \|\nabla F\|_{L^p(D,w)}.$$

Since the expression above continuously depends on $a > 0$, we can also admit $a = \delta$. This proves the claim. □

Taking a close look at the statement of Lemma 4.3, we see that the analytic Poincaré constant at z_0 can be controlled by the classical Poincaré constant whenever there exists a not-too-small neighborhood around z_0 such that the weight function w is lower-bounded on this neighborhood. Since we will later on apply this result to very specific weight functions, we will see that such z_0 can always be found.

4.2 Weighted Stable Point Evaluations

Having obtained an estimate for the analytic Poincaré constant in the previous subsection, we go on to develop bounds for the sampling constant that occurs in the right-hand side of (3.2). We start with the following lemma, which shows that there exist "many" points with a given sampling constant.

LEMMA 4.4. *Suppose that $D \subset \mathbb{C}$ is a domain and $w : D \to \mathbb{R}_+$ a weight function, and let $G \in L^p(D, w)$ for $1 \leq p < \infty$. For $C > 0$ we denote*

$$\begin{aligned} D_C(G) &:= \{z \in D\colon \|G(z)\|_{L^p(D,w)} \leq C\|G\|_{L^p(D,w)}\} \\ &= \{z \in D\colon C_{\text{samp}}(p, D, z, G, w) \leq C\}. \end{aligned}$$

Then

$$w(D_C(G)) \geq w(D) \cdot \left(1 - \frac{1}{C^p}\right).$$

PROOF. We compute

$$\int_{D \setminus D_C(G)} |G(x)|^p w(x) dx + \int_{D_C(G)} |G(x)|^p w(x) dx = \|G\|_{L^p(D,w)}^p.$$

By the definition of $D_C(G)$ we have that

$$|G(x)|^p > \frac{C^p}{w(D)} \|G\|_{L^p(D,w)}^p \quad \text{for all } x \in D \setminus D_C(G),$$

and this implies that

$$w(D \setminus D_C(G)) \frac{C^p}{w(D)} \|G\|_{L^p(D,w)}^p + \int_{D_C(G)} |G(x)|^p w(x) dx \leq \|G\|_{L^p(D)}^p.$$

Consequently,

$$\big(w(D) - w(D_C(G))\big) \frac{C^p}{w(D)} \leq 1,$$

and this yields the statement. □

4.3 Simultaneously Balancing Poincaré and Sampling Constants

Since Theorem 5.3 requires simultaneous control of the Poincaré and the sampling constant, we now show how the results of the previous two subsections may be combined to achieve this. We consider, for simplicity, the case that $D \subset \mathbb{C}$ is convex such that the boundary of D has bounded curvature—the more general case would be more technical and is therefore omitted (see, however, Remark 4.8).

DEFINITION 4.5. Let $D \subset \mathbb{C}$ and $F_1\colon \bar{D} \to \mathbb{C}$ be differentiable. We define the *global variation* of F_1 as

$$\delta_D(F_1) := \min\left\{\frac{1}{2}\cdot\frac{\|F_1\|_{L^\infty(D)}}{\|\nabla|F_1|\|_{L^\infty(D)}},\ 1\right\}. \tag{4.3}$$

The following elementary result will be used later on.

LEMMA 4.6. *Let $D \subset \mathbb{C}$ be convex and $F_1\colon \bar{D} \to \mathbb{C}$ be a differentiable function. Suppose that z_0 is a maximum of $|F_1|$ in D, e.g., $|F_1(z_0)| = \|F_1\|_{L^\infty(D)}$. Then it holds that*

$$\inf_{z\in B_{\delta_D(F_1)}(z_0)\cap D} |F_1(z)| \geq \frac{1}{2}\|F_1\|_{L^\infty(D)}. \tag{4.4}$$

PROOF. This is a simple consequence of the fact that for all $z \in D$

$$\big||F_1|(z_0) - |F_1|(z)\big| \leq |z - z_0|\cdot\|\nabla|F_1|\|_{L^\infty(D)} \tag{4.5}$$

and

$$|F_1(z_0)| = \|F_1\|_{L^\infty(D)}. \tag{4.6}$$

Suppose that $z \in B_{\delta_D(F_1)}(z_0)\cap D$. Then $|z-z_0|\cdot\|\nabla|F_1|\|_{L^\infty(D)} \leq \frac{1}{2}|F_1(z_0)|$. By (4.5) and (4.6) it follows that $|F_1(z)| \geq \frac{1}{2}\|F_1\|_{L^\infty(D)}$. □

The following proposition shows that the analytic Poincaré and sampling constants can always be balanced, provided that the quantity $\delta_D(F_1)$ is not too small.

PROPOSITION 4.7. *Let $1 \leq p < \infty$. Suppose that $D \subset \mathbb{C}$ is convex and that the curvature of the boundary ∂D is everywhere bounded by 1. Suppose $F_1 : \bar{D} \to \mathbb{C}$ is differentiable, $\delta_D(F_1)$ as defined in Definition 4.5 is positive, and $G \in L^p(D, |F_1|^p)$. Then there exists $z \in D$ with*

$$\begin{aligned} &C^a_{\mathrm{poinc}}(p, D, z, \delta_D(F_1)/4, |F_1|^p)\\ &\quad\leq C_{\mathrm{poinc}}(p, D, |F_1|^p)\\ &\qquad\cdot\left(1 + \frac{2^p}{\pi\delta_D(F_1)^2/16}\cdot\frac{\|F_1\|^p_{L^p(D)}}{\|F_1\|^p_{L^\infty(D)}}\right)\end{aligned} \tag{4.7}$$

and

$$C_{\mathrm{samp}}(p, D, z, G, |F_1|^p) \leq \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}}\cdot(\delta_D(F_1)^2\pi)^{-1/p}\cdot 2\cdot 16^{-1/p}. \tag{4.8}$$

Furthermore, it holds that

$$\inf_{u \in B_{\delta_D(F_1)/4}(z)} |F_1(u)| > 0. \tag{4.9}$$

PROOF. Suppose that $z_0 \in \overline{D}$ is a maximum of $|F_1|$ and put $\delta := \delta_D(F_1) \leq 1$ as defined in (4.3). First we note that by our assumptions on D and by the definition (4.3) it holds that the set $B_\delta(z_0) \cap D$ contains a ball of radius $\delta/2$; i.e., there exists $\widetilde{z}_0$ such that

$$B_{\delta/2}(\widetilde{z}_0) \subset D \quad \text{and} \quad \inf_{z \in B_{\delta/2}(\widetilde{z}_0)} |F_1(z)| \geq \frac{1}{2} \|F_1\|_{L^\infty(D)}.$$

But this implies that for all $z \in B_{\delta/4}(\widetilde{z}_0)$ it holds that

$$B_{\delta/4}(z) \subset D \quad \text{and} \quad \inf_{u \in B_{\delta/4}(z)} |F_1(u)| \geq \frac{1}{2} \|F_1\|_{L^\infty(D)}.$$

Using this fact, we start by estimating the analytic Poincaré constant for such a z, with the estimate from Lemma 4.3. More precisely, we will use the estimate (4.1) with $a = \delta/4$ and $B_a := B_{\delta/4}(z)$, which yields that

$$\begin{aligned}
&C^a_{\text{poinc}}(p, D, z, \delta/4, |F_1|^p) \\
&\quad \leq C_{\text{poinc}}(p, D, |F_1|^p) \\
&\qquad \cdot \left(1 + \|F_1\|_{L^p(D)} \cdot \frac{\|F_1\|^{p-1}_{L^p(B_{\delta/4})} 2^p \|F_1\|^{-p}_{L^\infty(D)}}{\pi\delta^2/16}\right) \\
&\quad \leq C_{\text{poinc}}(p, D, |F_1|^p) \cdot \left(1 + \frac{2^p}{\pi\delta^2/16} \cdot \frac{\|F_1\|^p_{L^p(D)}}{\|F_1\|^p_{L^\infty(D)}}\right).
\end{aligned}$$

Recall that the above estimate holds for any $z \in B_{\delta/4}(\widetilde{z}_0)$.

We now abbreviate $B_{\delta/4} := B_{\delta/4}(\widetilde{z}_0)$ and show that there exists such a $z \in B_{\delta/4}$ that also generates good sampling constants. Let $w = |F_1|^p$. By (4.4), we have that

$$w(B_{\delta/4}) = \int_{B_{\delta/4}} |F_1(z)|^p \, dz \geq \|F_1\|^p_{L^\infty(D)} \frac{\pi\delta^2}{2^p \cdot 16}. \tag{4.10}$$

The measure of "good" sampling points

$$D_C(G) := \{z \in D \colon C_{\text{samp}}(p, D, z, G, |F_1|^p) \leq C\}$$

by Lemma 4.4 satisfies $w(D_C(G)) \geq w(D) \cdot (1 - 1/C^p)$. Therefore, if

$$C > \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}} \cdot (\delta^2\pi)^{-1/p} \cdot 2 \cdot 16^{1/p}$$

by (4.10), it holds that $w(D_C(G)) > w(D) - w(B_{\delta/4})$, implying that $D_C(G) \cap B_{\delta/4} \neq \varnothing$. Any z in this intersection will satisfy the desired estimates. $\square$

The result of Proposition 4.7 may still seem very technical. However, we have succeeded in providing bounds for both the analytic Poincaré constant as well as the sampling constant that appear in the right-hand side of (3.2).

Indeed, from Proposition 4.7 we can infer that these constants essentially depend only on the Poincaré constant of the measurements $|F_1|^p$ and the quantity $\delta_D(F_1)$.

Remark 4.8. As an alternative to the global variation as defined in (4.5), we may look at the quantity

$$\begin{aligned}\widetilde{\delta}_D(F_1) := \min\Big\{ \sup\Big\{ & r > 0: \\ & B_r(z) \subset D,\ \inf_{\zeta \in B_r(z)} |F_1(\zeta)| \geq \frac{1}{2}\|F_1\|_{L^\infty(D)} \Big\}, 1 \Big\}\end{aligned} \tag{4.11}$$

Replicating the proof of Proposition 4.7 reveals that for any $G \in L^p(D, |F_1|^p)$ there is a $z \in D$ such that

$$\begin{aligned}C^a_{\mathrm{poinc}}(p, D, z, \widetilde{\delta}_D(F_1)/2, |F_1|^p) \leq {} & C_{\mathrm{poinc}}(p, D, |F_1|^p) \\ & \cdot \left(1 + \frac{2^p}{\pi \widetilde{\delta}_D(F_1)^2/4} \cdot \kappa_D^p\right)\end{aligned} \tag{4.12}$$

and

$$C_{\mathrm{samp}}(p, D, z, G, |F_1|^p) \leq \kappa_D \cdot (\widetilde{\delta}_D(F_1)^2 \pi)^{-1/p} \cdot 2 \cdot 4^{-1/p}, \tag{4.13}$$

where we denote $\kappa_D := \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}}$. Additionally, it holds that

$$\inf_{u \in B_{\widetilde{\delta}_D(F_1)/2}(z)} |F_1(u)| > 0. \tag{4.14}$$

Note that in contrast to Proposition 4.7 we do not need the domain D to be convex, and its boundary does not have to meet any curvature assumptions.

In the next section we shall see that the quantity $\delta_D(F_1)$ can always be uniformly bounded if F_1 arises as the Gabor transform of any $f \in \mathcal{S}'(\mathbb{R})$, e.g., $F_1(z) = V_\varphi f(x, -y)$.

5 Gabor Phase Retrieval

Up to now all results have applied to general functions F_1 and F_2, which map from a domain $D \subset \mathbb{C}$ to $\mathbb{C}$ and which are holomorphic after multiplication with a function η. Indeed, by combining Theorem 3.3 with Proposition 4.7, we obtain a stability result that essentially depends only on the Poincaré constant $C_{\mathrm{poinc}}(p, D, |F_1|^p)$ and the quantity $\delta_D(F_1)$.

We will, from now on, specialize to the case that F_1 is, up to a reflection, the Gabor transform of a function $f \in \mathcal{S}'(\mathbb{R})$, e.g., $F_1(z) = V_\varphi f(x, -y)$, where $\varphi(t) = e^{-\pi t^2}$ and $V_\varphi f$ is defined as in Definition 2.1. The Gabor transform enjoys a lot of structure that allows us to obtain major improvements in the general stability bound (3.2). In order to estimate $\inf_{\alpha\in\mathbb{R}} \|V_\varphi f - e^{\mathrm{i}\alpha} V_\varphi g\|_{L^p(D)}$ we will apply the results of Chapters 3 and 4 to F_1, $F_2(z) = V_\varphi g(x,-y)$, and the reflected domain $\{\overline{z}\colon z \in D\}$.

First, in Section 5.1 we shall see that the quantity $\delta_D(V_\varphi f)$ can essentially be bounded independently of f, which will finally give us complete control over the implicit constants that appear in the estimate (3.2). Then, in Section 5.2 we will show that, in the case of Gabor measurements, the term involving a logarithmic derivative in (3.2) can be absorbed into an error term with respect to a norm $\mathcal{D}^{r,s}_{p,q}$ for suitable parameters. This latter result will exploit deep function-theoretic properties of the Gabor transform. Finally, in Section 5.3 we will put all these results together and present our final stability estimates for Gabor phase retrieval.

5.1 Balancing the Constants

The goal of the present section is to establish the following result.

PROPOSITION 5.1. *Let $D \subset \mathbb{C}$ and suppose $f \in M^{\infty,\infty}(\mathbb{R})$. Then there exists $\delta > 0$, only depending on $\|V_\varphi f\|_{L^\infty(D)}/\|V_{\varphi'} f\|_{L^\infty(D)}$, such that*

$$\delta_D(V_\varphi f) \geq \delta. \tag{5.1}$$

For $D = \mathbb{C}$ we get a stronger statement: There exists a universal constant $\delta \geq (2^{5/4}\pi)^{-1}$ with

$$\inf_{f\in M^{\infty,\infty}(\mathbb{R})} \delta_{\mathbb{C}}(V_\varphi f) \geq \delta. \tag{5.2}$$

PROOF. The proof proceeds by showing that the L^∞ norm of the gradient of $|V_\varphi f|$ cannot be much larger than the L^∞ norm of $|V_\varphi f|$. Indeed, a simple calculation (or a look at equations (3) and (5) in [7]) reveals that $\big|\nabla|V_\varphi f|\big| = |V_{\varphi'} f|$, which directly implies $\|\nabla|V_\varphi f|\|_{L^\infty(D)} = \|V_{\varphi'} f\|_{L^\infty(D)}$. Looking at (4.3) implies (5.1).

When $D = \mathbb{C}$ we can use the norm equivalence $\|V_\varphi\cdot\|_{L^p(\mathbb{C})} \sim \|V_{\varphi'}\cdot\|_{L^p(\mathbb{C})}$ on $M^{p,p}(\mathbb{R})$ for any $p \in [1,\infty]$ (see [32, prop. 11.3.2(c)]). To get a positive lower bound of δ, we denote V_φ^*, the adjoint of the Gabor transform, by

$$V_\varphi^* F := \int_{\mathbb{R}}\int_{\mathbb{R}} F(x,y) M_y T_x \varphi \, dx\, dy,$$

where T and M denote the translation and the modulation operator, respectively, and rewrite

$$V_{\varphi'} f = V_{\varphi'}\left(\frac{1}{\|\varphi\|_{L^2(\mathbb{R})}} V_\varphi^* V_\varphi f\right) = 2^{1/4} \cdot V_{\varphi'} V_\varphi^* V_\varphi f.$$

Applying equation (11.29) from the proof of proposition 11.3.2 in [32] allows us to estimate pointwise

$$|V_{\varphi'} f| = 2^{1/4} \cdot |V_{\varphi'} V_\varphi^* V_\varphi f| \leq 2^{1/4} \cdot (|V_\varphi f| * |V_{\varphi'}\varphi|),$$

and therefore

$$\|V_{\varphi'} f\|_{L^\infty(\mathbb{C})} \le 2^{1/4} \cdot \|V_\varphi f\|_{L^\infty(\mathbb{C})} \|V_{\varphi'}\varphi\|_{L^1(\mathbb{C})}. \tag{5.3}$$

As an elementary integration exercise, the reader verifies that

$$\|V_{\varphi'}\varphi\|_{L^1(\mathbb{C})} = \pi. \tag{5.4}$$

From the definition of $\delta_{\mathbb{C}}(V_\varphi f)$ and from equations (5.3) and (5.4), it follows that

$$\delta_{\mathbb{C}}(V_\varphi f) = \min\left\{\frac{1}{2} \cdot \frac{\|V_\varphi f\|_{L^\infty(\mathbb{C})}}{\|V_{\varphi'} f\|_{L^\infty(\mathbb{C})}}, 1\right\} \ge \frac{1}{2^{5/4}\pi}. \qquad \square$$

As a corollary we get the following result for $D = \mathbb{C}$.

COROLLARY 5.2. *Let $1 \le p < \infty$. Suppose that $f \in M^{\infty,\infty}(\mathbb{R})$ and let $F_1(z) = V_\varphi f(x, -y)$. Furthermore, suppose that $G \in L^p(\mathbb{C}, |F_1|^p)$. Then there exist constants $c, \delta > 0$ (independent of f and G!) such that there exists $z \in \mathbb{C}$ with*

$$C^a_{\mathrm{poinc}}(p, \mathbb{C}, z, \delta, |F_1|^p) \le c \cdot C_{\mathrm{poinc}}(p, \mathbb{C}, |F_1|^p) \cdot \left(1 + \frac{\|F_1\|^p_{L^p(\mathbb{C})}}{\|F_1\|^p_{L^\infty(\mathbb{C})}}\right), \tag{5.5}$$

$$C_{\mathrm{samp}}(p, \mathbb{C}, z, G, |F_1|^p) \le c \cdot \frac{\|F_1\|_{L^p(\mathbb{C})}}{\|F_1\|_{L^\infty(\mathbb{C})}}, \tag{5.6}$$

and

$$\inf_{u \in B_\delta(z)} |F_1(u)| > 0. \tag{5.7}$$

PROOF. This is a direct consequence of Propositions 4.7 and 5.1. $\square$

Observe that for $f, g \in M^{p,p}(\mathbb{R})$ the function $G := |F_2/F_1| - 1$ is an element of $L^p(\mathbb{C}, |F_1|^p)$, where we put $F_1(z) = V_\varphi f(x, -y)$ and $F_2(z) = V_\varphi g(x, -y)$. Furthermore, let us point out that $M^{p,p}(\mathbb{R})$ is contained in $M^{\infty,\infty}(\mathbb{R})$ for all $p \in [1, \infty]$. Applying Corollary 5.2, together with the well-known fact that $\eta \cdot F_1$ and $\eta \cdot F_2$ are holomorphic for suitable η (Theorem 2.4) to Theorem 3.3, we get the following stability result:

THEOREM 5.3. *Suppose that $f \in M^{p,p}(\mathbb{R})$ and $g \in M^{p,p}(\mathbb{R})$. Then there exists a constant $c > 0$ only depending on $\|V_\varphi f\|_{L^p(\mathbb{C})}/\|V_\varphi f\|_{L^\infty(\mathbb{C})}$ such that*

$$\begin{aligned} &\inf_{\alpha \in \mathbb{R}} \|f - e^{\mathbf{i}\alpha} g\|_{M^{p,p}(\mathbb{R})} \\ &\quad \le c \cdot (1 + C_{\mathrm{poinc}}(p, \mathbb{C}, |V_\varphi f|^p)) \\ &\qquad \cdot \big(\||V_\varphi f| - |V_\varphi g|\|_{W^{1,p}(\mathbb{C})} + \|\nabla \log|V_\varphi f|(|V_\varphi f| - |V_\varphi g|)\|_{L^p(\mathbb{C})}\big). \end{aligned} \tag{5.8}$$

For general $D \subset \mathbb{C}$ we get the following result.

COROLLARY 5.4. *Suppose that $f \in M^{\infty,\infty}(\mathbb{R})$ and let $F_1(z) = V_\varphi f(x, -y)$. Suppose that $D \subset \mathbb{C}$ satisfies the assumptions of Proposition 4.7 and that $G \in L^p(D, |F_1|^p)$. Then there exists a constant c that only depends monotonically increasingly on $\|V_{\varphi'} f\|_{L^\infty(D)}/\|V_\varphi f\|_{L^\infty(D)}$ (and that is otherwise independent* *of f and D!), a constant $\delta > 0$ that depends monotonically decreasingly on $\|V_\varphi f\|_{L^\infty(D)}/\|V_{\varphi'} f\|_{L^\infty(D)}$ (and that is otherwise independent of f and D!), and $z \in D$ with $B_\delta(z) \subset D$ such that*

$$C^a_{\mathrm{poinc}}(p, D, z, \delta, |F_1|^p) \le c \cdot C_{\mathrm{poinc}}(p, D, |F_1|^p) \cdot \left(1 + \frac{\|F_1\|^p_{L^p(D)}}{\|F_1\|^p_{L^\infty(D)}}\right), \tag{5.9}$$

$$C_{\mathrm{samp}}(p, D, z, G, |F_1|^p) \le c \cdot \frac{\|F_1\|_{L^p(D)}}{\|F_1\|_{L^\infty(D)}}. \tag{5.10}$$

and

$$\inf_{u \in B_\delta(z)} |F_1(u)| > 0. \tag{5.11}$$

PROOF. This is a direct consequence of Theorem 2.4, Proposition 4.7, and Proposition 5.1. □

As before, Corollary 5.4 directly leads to a stability result for Gabor phase retrieval.

THEOREM 5.5. *Suppose that $f, g \in M^{p,p}(\mathbb{R})$ and let $D \subset \mathbb{C}$ satisfy the assumptions of Proposition 4.7. Then there exists a constant $c > 0$ only depending on*

$$\max\left\{\frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}, \frac{\|V_{\varphi'} f\|_{L^\infty(D)}}{\|V_\varphi f\|_{L^\infty(D)}}\right\}$$

such that

$$\begin{aligned} &\inf_{\alpha \in \mathbb{R}} \|V_\varphi f - e^{\mathbf{i}\alpha} V_\varphi g\|_{L^p(D)} \\ &\quad \le c \cdot (1 + C_{\mathrm{poinc}}(p, D, |V_\varphi f|^p)) \\ &\qquad \cdot \big(\||V_\varphi f| - |V_\varphi g|\|_{W^{1,p}(D)} + \|\nabla \log|V_\varphi f|(|V_\varphi f| - |V_\varphi g|)\|_{L^p(D)}\big). \end{aligned} \tag{5.12}$$

5.2 Controlling the Logarithmic Derivative

Compared to Theorem 2.9, Theorems 5.3 and 5.5 are now independent of any choice of z_0, which is very nice. However, the term involving the logarithmic derivative of $|V_\varphi f|$ in (5.8) and (5.12) is still bothersome, in particular because, in general, $|\nabla \log|V_\varphi f||$ will certainly be unbounded. It turns out that in the case of Gabor measurements this quantity can be absorbed into an error term with respect to a norm as defined in Definition 2.5. The proof of this fact is, however, quite difficult and involves deep function-theoretic properties of the Gabor transform.

We begin by estimating the norms of the logarithmic derivative of the modules of a Gabor transform $V_\varphi f$ on discs with growing radii. It turns out that these can be estimated independently of the original signal $f \in M^{\infty,\infty}(\mathbb{R})$. The reason for this perhaps surprising fact is that the holomorphic function $\eta \cdot V_\varphi f$, with suitable η, satisfies certain restricted growth properties. Jensen's formula relates the distribution of zeros of a holomorphic function with its growth rate, which allows us to bound the number of zeros of $V_\varphi f$ in a given disc. This in turn will yield a bound on the norm of the logarithmic derivative of $|V_\varphi f|$ as follows.

PROPOSITION 5.6. *Let $1 \leq r < 2$. There exists a constant C that only depends on r such that for all $f \in M^{\infty,\infty}(\mathbb{R})$ and all $R > 1$ we have an estimate*

$$\|\nabla \log|V_\varphi f|\|_{L^r(B_R(z_0))} \leq CR^3,$$

where z_0 is a maximum of $|V_\varphi f|$.

PROOF. Let $R > 1$ be fixed. We assume w.l.o.g. that $z_0 = 0$ (otherwise f can be translated and modulated such that the origin 0 is a maximum of $|V_\varphi f|$).

Step 1. PREPARATIONS. Let $F_1(z) = V_\varphi f(x, -y)$ and $\eta(z) := e^{\pi(\frac{|z|^2}{2} - \mathrm{i}xy)}$. Then, by Theorem 2.4, the function $G = \eta \cdot F_1$ is an entire function. Applying the gradient yields

$$\nabla|G| = \nabla|\eta| \cdot |F_1| + |\eta| \cdot \nabla|F_1|,$$

and thus for any z such that $F_1(z) \neq 0$, it holds that

$$\frac{\nabla|F_1|(z)}{|F_1|(z)} = \frac{\nabla|G|(z)}{|G|(z)} - \frac{\nabla|\eta|(z)}{|\eta|(z)} = \frac{\nabla|G|(z)}{|G|(z)} - \pi\begin{pmatrix} x \\ y \end{pmatrix}.$$

By the triangle inequality and equation (2.10) we can estimate

$$\left|\frac{\nabla|F_1|(z)}{|F_1|(z)}\right| \leq \left|\frac{\nabla|G|(z)}{|G|(z)}\right| + \pi\left|\begin{pmatrix} x \\ y \end{pmatrix}\right| = \left|\frac{\nabla|G|(z)}{|G|(z)}\right| + \pi|z|. \tag{5.13}$$

In order to estimate the norm of the logarithmic derivative of $|G|$, we will exploit the *Poisson-Jensen formula* [20], which states that for any entire function G with zeros z_j (repeated according to multiplicity) and $R > 0$ it holds that

$$\begin{aligned} \log|G(z)| = &- \sum_{|z_j|<R} \log\left|\frac{R^2 - \bar{z}_j z}{R(z - z_j)}\right| \\ &+ \frac{1}{2\pi}\int_0^{2\pi} \Re\left(\frac{Re^{\mathrm{i}\theta} + z}{Re^{\mathrm{i}\theta} - z}\right) \log|G(Re^{\mathrm{i}\theta})| d\theta, \quad |z| < R. \end{aligned} \tag{5.14}$$

To calculate the logarithmic derivative of $|G|$ we set

$$\hbar(z, u) := \frac{R^2 - \bar{u}z}{R(z - u)}$$

and

$$\Bbbk(z, \theta) := \frac{R^2 - x^2 - y^2}{(R\cos\theta - x)^2 + (R\sin\theta - y)^2}$$

and rewrite (5.14) as follows:

$$\log|G(z)| = -\sum_{|z_j|<R} \log|\hbar(z, z_j)| + \frac{1}{2\pi}\int_0^{2\pi} \Bbbk(z, \theta) \cdot \log|G(Re^{\mathrm{i}\theta})| d\theta. \tag{5.15}$$

By (5.13) and by differentiation of (5.15) we obtain for $|z| < R$ and $F_1(z) \neq 0$ that

$$\left|\frac{\nabla|F_1|(z)}{|F_1|(z)}\right| \leq (|\mathrm{I(z)}| + |\mathrm{II(z)}| + |\mathrm{III(z)}|), \tag{5.16}$$

where we put

$$\mathrm{I(z)} := \sum_{|z_j|<R} \nabla \log|\hbar(z, z_j)|,$$

$$\mathrm{II(z)} := \frac{1}{2\pi}\int_0^{2\pi} \nabla k(z,\theta)\cdot \log|G(Re^{\mathbf{i}\theta})|d\theta,$$

$$\mathrm{III(z)} := z,$$

where in the definition of II the integral of a vector-valued function is to be understood as the vector that contains the integrals of the individual entries. We will derive estimates for $\|\mathrm{I}\|_{\mathrm{L^r(B_{R/2}(0))}}$, $\|\mathrm{II}\|_{\mathrm{L^r(B_{R/2}(0))}}$, and $\|\mathrm{III}\|_{\mathrm{L^r(B_{R/2}(0))}}$ separately. Before that we need to bound the number of zeros of $V_\varphi f$ on discs.

Step 2. BOUNDING THE NUMBER OF ZEROS OF $V_\varphi f$. We will require the well-known *Jensen's formula*, which is a special case ($z = 0$) of equation (5.14):

$$\log|G(0)| = -\sum_{|z_j|<R}\log\left|\frac{R}{z_j}\right| + \frac{1}{2\pi}\int_0^{2\pi}\log|G(Re^{\mathbf{i}\theta})|d\theta.$$

By our assumption that $|V_\varphi f(0)| = 1$ this implies

$$\sum_{|z_j|<R}\log\left|\frac{R}{z_j}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log|G(Re^{\mathbf{i}\theta})|d\theta. \tag{5.17}$$

Jensen's formula allows us to bound the number of zeros in a ball of radius R:

$$\begin{aligned}
|\{j : |z_j| < R\}| &= \sum_{|z_j|<R} 1 \\
&\leq \frac{1}{\log 2}\sum_{|z_j|<2R}\log\left|\frac{2R}{z_j}\right| \\
&= \frac{1}{\log 2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\log|G(2Re^{\mathbf{i}\theta})|d\theta \\
&= \frac{1}{\log 2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\log|F_1(2Re^{\mathbf{i}\theta})| + \log|\eta(2Re^{\mathbf{i}\theta})|d\theta \\
&= \frac{1}{\log 2}\left(\frac{1}{2\pi}\int_0^{2\pi}\log|F_1(2Re^{\mathbf{i}\theta})|d\theta + 2\pi R^2\right) \leq \frac{2\pi}{\log 2}R^2,
\end{aligned}$$

where we used that $|F_1| \leq 1$.

Step 3. ESTIMATING THE NORM OF I. First we calculate $|\nabla \log|\hbar|(z,u)|$. Recall (Lemma 3.4) that for meromorphic functions the length of the gradient of the modulus coincides with the absolute value of the complex derivative (almost everywhere). Thus we calculate

$$\begin{aligned}
|\nabla \log|\hbar|(z,u)| &= |\nabla|h|(z,u)/h(z,u)| \\
&= |h'(z,u)/h(z,u)| \\
&= \left|\frac{-\bar{u}R(z-u)-(R^2-\bar{u}z)R}{R^2(z-u)} \Big/ \frac{R^2-\bar{u}z}{R(z-u)}\right| \\
&= \left|\frac{\bar{u}}{R^2-\bar{u}z}+\frac{1}{z-u}\right|.
\end{aligned}$$

Therefore we can estimate for $|z| < R/2$

$$\begin{aligned}
|\mathrm{I(z)}| \le \sum_{|z_j|<R} \big|\nabla|\hbar|(z, z_j)\big| &= \sum_{|z_j|<R}\left|\frac{\bar{z}_j}{R^2-\bar{z}_j z}+\frac{1}{z-z_j}\right| \\
&\le \sum_{|z_j|<R}\left|\frac{\bar{z}_j}{R^2-\bar{z}_j z}\right| + \sum_{|z_j|<R}|z-z_j|^{-1} \\
&\le \frac{2\pi}{\log 2}R^2\cdot\frac{2}{R}+\sum_{|z_j|<R}|z-z_j|^{-1}.
\end{aligned}$$

One can calculate that for $s > 0$ and $r \in [1,2)$

$$\|z \mapsto z^{-1}\|_{L^r(B_s(0))} = c \cdot s^{2/r-1},$$

where c is a finite constant that only depends on r. Therefore

$$\left\| z \mapsto \frac{1}{z-z_j}\right\|_{L^r(B_{R/2}(0))} \le \|z \mapsto z^{-1}\|_{L^r(B_{R/2}(0))} = c' \cdot R^{2/r-1},$$

where c' again only depends on r. Since the number of zeros in $B_R(0)$ is at most of the order of R^2, there is a constant c'' that only depends on r such that

$$\begin{aligned}
\|\mathrm{I}\|_{\mathrm{L^r(B_{R/2}(0))}} &\le c''(R\|1\|_{L^r(B_{R/2}(0))} + R^2 R^{2/r-1}) \\
&= c''((\pi/4)^{1/r}+1)R^{2/r+1} \le 2c''R^3.
\end{aligned} \tag{5.18}$$

Step 4. ESTIMATING THE NORM OF II(z). First we compute the derivative of $\Bbbk$:

$$\nabla \Bbbk(z,\theta) = \frac{\begin{pmatrix}-2x\\-2y\end{pmatrix}|Re^{\mathrm{i}\theta}-z|^2 - |R-z|^2\begin{pmatrix}-2(R\cos\theta - x)\\-2(R\sin\theta - y)\end{pmatrix}}{|Re^{\mathrm{i}\theta}-z|^4}.$$

Therefore we obtain for $|z| < R/2$ that

$$|\nabla k(z,\theta)| \le \frac{2|z|}{(R/2)^2} + \frac{2R^2}{(R/2)^3} = 24R^{-1}.$$

Before we continue to estimate II we introduce $M_G(R) := \max_{|z|\le R}|G(z)|$ for $R > 0$. Since $|F_1(0)| = 1$ and $|F_1| \le 1$ we can conclude that

$$1 \le M_G(R) \le e^{\frac{\pi}{2}R^2}. \tag{5.19}$$

Hence we obtain

$$\begin{aligned}
|\mathrm{II}(\mathrm{z})| &\le \frac{1}{2\pi}\int_0^{2\pi} |\nabla k(z,\theta)||\log|G(Re^{i\theta})||d\theta \\
&\le \frac{12}{\pi}R^{-1}\int_0^{2\pi} |\log|G(re^{\mathrm{i}\theta})||d\theta \\
&\le \frac{12}{\pi}R^{-1}\left(\int_0^{2\pi} \big|\log|G(re^{\mathrm{i}\theta})| - \log M_G(R)\big|d\theta + \int_0^{2\pi} \log M_G(R)d\theta\right) \\
&= \frac{12}{\pi}R^{-1}\int_0^{2\pi} 2\log M_G(R) - \log|G(Re^{\mathrm{i}\theta})|d\theta.
\end{aligned}$$

By (5.17) we know that $\int_0^{2\pi} \log|G(Re^{\mathrm{i}\theta})|d\theta$ is nonnegative. Using (5.19) we further estimate II and arrive at

$$|\mathrm{II}(z)| \le \frac{24}{\pi}R^{-1}\int_0^{2\pi} \log M_G(R)d\theta \le 48R^{-1}\cdot\frac{\pi}{2}R^2 = 24\pi R,$$

and consequently there is a constant c''', which only depends on r, such that

$$\|\mathrm{II}\|_{L^r(B_{R/2}(0))} \le 24\pi R\cdot((R/2)^2\pi)^{1/r} \le c'''R^3. \tag{5.20}$$

Step 5. PUTTING THE ESTIMATES TOGETHER. It only remains to bound the norm of $\mathrm{III}(z) = z$. However, it is an easy exercise to calculate

$$\|z \mapsto z\|_{L^r(B_{R/2}(0))} = c''''R^{1+2/r}$$

for a constant c'''' depending on r only. Together with (5.18), (5.20), and (5.16), this yields that there exists a $C > 0$, which only depends on r, such that

$$\|\nabla\log|F_1|\|_{L^r(B_{R/2}(0))} \le CR^3 \quad \text{for all } R > 0.$$

Substitution of R by $2R$ concludes the proof. $\square$

Proposition 5.6 yields important information on how fast the L^r-norm of the logarithmic derivative of a Gabor magnitude can possibly grow as the size of the integration domain increases. It is remarkable that this quantity can be bounded independently of the original signal.

Moreover, with Proposition 5.6 in hand we can go on to control the bothersome logarithm term in the estimates of Theorems 5.3 and 5.5:

PROPOSITION 5.7. *Let $p \in [1,2)$ and $q \in (\frac{2p}{2-p}, \infty)$. Then there exists a polynomial σ of maximal order 4 such that for any $f \in M^{\infty,\infty}(\mathbb{R})$ with $V_\varphi f$ centered (see Definition 2.7), all domains $D \subset \mathbb{C}$ with $0 \in \mathbb{C}$ ($D = \mathbb{C}$ is allowed!), and all measurable functions $\Delta : D \to \mathbb{C}$, it holds that*

$$\|\nabla \log|V_\varphi f| \cdot \Delta\|_{L^p(D)} \le \|\Delta \cdot \sigma(|\cdot|)\|_{L^q(D)}.$$

In particular, it holds that

$$\|\nabla \log|V_\varphi f| \cdot \big(|V_\varphi f| - |V_\varphi g|\big)\|_{L^p(D)} \le \|\big(|V_\varphi f| - |V_\varphi g|\big) \cdot \sigma(|\cdot|)\|_{L^q(D)}$$

for any $g \in \mathcal{S}'(\mathbb{R})$.

PROOF. We only consider $D = \mathbb{C}$ since the general case can be proven in the exact same way.

Let $D_0 := B_1(0)$ and $D_j := B_{2^j}(0) \setminus B_{2^{j-1}}(0)$ for $j \ge 1$; then

$$\|\nabla \log|V_\varphi f| \cdot \Delta\|_{L^p(\mathbb{C})}^p = \sum_{j\ge 0} \int_{D_j} |\nabla \log|V_\varphi f||^p \cdot |\Delta|^p. \tag{5.21}$$

The numbers $s = \frac{q}{q-p}$ and $s' = \frac{q}{p}$ are Hölder-conjugated. Denoting $r := ps = \frac{pq}{q-p}$ we have

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q} > \frac{1}{p} - \frac{2-p}{2p} = \frac{1}{2},$$

and therefore $r < 2$. Applying Hölder's inequality we obtain

$$\begin{aligned}\int_{D_j} |\nabla \log|V_\varphi f||^p \cdot |\Delta|^p &\le \|\nabla \log|V_\varphi f||\|_{L^r(D_j)}^p \cdot \|\Delta\|_{L^q(D_j)}^p \\ &\le (C \cdot 2^{3j})^p \cdot \|\Delta\|_{L^q(D_j)}^p\end{aligned}$$

where C is the constant from Propositon 5.6 and only depends on r. Using Hölder's inequality for sums yields

$$\begin{aligned}\|\nabla \log|V_\varphi f| \cdot \Delta\|_{L^p(\mathbb{C})}^p &\le C^p \cdot \sum_{j\ge 0} 2^{-\frac{j}{s}} \cdot 2^{3pj+\frac{j}{s}} \left(\int_{D_j} |\Delta|^q\right)^{1/s'} \\ &\le C^p \cdot \left(\sum_{j\ge 0} 2^{-j}\right)^{1/s} \cdot \left(\sum_{j\ge 0} 2^{(3ps'+\frac{s'}{s})j} \int_{D_j} |\Delta|^q\right)^{1/s'} \\ &= C^p \cdot 2^{1/s} \cdot \left(\int_{\mathbb{C}} |\Delta|^q \cdot \sum_{j\ge 0} 2^{(3ps'+\frac{s'}{s})j} \chi_{D_j}\right)^{1/s'}.\end{aligned}$$

For $z \in D_j$ it holds that

$$2^{(3ps'+\frac{s'}{s})j} \le 2 \cdot (|z|^{3ps'+\frac{s'}{s}} + 1) = 2 \cdot (|z|^{(3+1/r)q} + 1) \le 2 \cdot (|z|^4 + 1)^q.$$

Therefore we arrive at

$$\begin{aligned}\|\nabla \log|V_\varphi f| \cdot \Delta\|_{L^p(\mathbb{C})}^p &\le C^p \cdot 2^{1/s} \cdot \left(\int_{\mathbb{C}} |\Delta|^q \cdot 2(|z|^4+1)^q\right)^{1/s'} \\ &= \left(\int_{\mathbb{C}} |\Delta|^q \cdot (C 2^{1/r}(|z|^4+1))^q\right)^{1/s'} \\ &= \|\Delta \cdot \sigma(|.|)\|_{L^q(\mathbb{C})}^p,\end{aligned}$$

where we set $\sigma(z) := C 2^{1/r}(z^4+1)$. The second claim follows immediately by considering $\Delta = |V_\varphi f| - |V_\varphi g|$. □

5.3 Putting Everything Together

We can now apply Proposition 5.7 to control the logarithmic derivative in Theorem 5.3 and immediately get the following result.

THEOREM 5.8. *Let $p \in [1,2)$ and $q \in (\frac{2p}{2-p}, \infty)$. Suppose that $f \in M^{p,p}(\mathbb{R})$ is such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f). Then there exists a constant $c > 0$* only depending on p, q, and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ *such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that*

$$d_{M^{p,p}(\mathbb{R})}(f,g) \le c \cdot (1 + C_{\mathrm{poinc}}(p, \mathbb{C}, |V_\varphi f|^p)) \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}}.$$

We can also establish the following local version, which follows by combining Proposition 5.7 and Theorem 5.5.

THEOREM 5.9. *Let $p \in [1,2)$ and $q \in (\frac{2p}{2-p}, \infty)$. Suppose that $D \subset \mathbb{C}$ satisfies the assumptions of Proposition* 4.7. *Suppose that $f \in M^{p,p}(\mathbb{R})$ is such that its Gabor transform $V_\varphi f$ is centered (otherwise we could translate and modulate f). Then there exists a constant $c > 0$* only depending on p, q, and

$$\max\left\{\frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}, \frac{\|V_{\varphi'} f\|_{L^\infty(D)}}{\|V_\varphi f\|_{L^\infty(D)}}\right\}$$

such that for any $g \in M^{p,p}(\mathbb{R})$ it holds that

$$\inf_{\alpha\in\mathbb{R}} \|V_\varphi f - e^{\mathbf{i}\alpha} V_\varphi g\|_{L^p(D)} \le \\ c \cdot (1 + C_{\mathrm{poinc}}(p, D, |V_\varphi f|^p)) \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}(D)}.$$

It remains to interpret the weighted Poincaré constants $C_{\mathrm{poinc}}(p, D, |V_\varphi f|^p)$ and $C_{\mathrm{poinc}}(p, \mathbb{C}, |V_\varphi f|^p)$. In Appendix B we prove the following result.

THEOREM 5.10. *Let $p \in [1,2]$. For every connected domain $D \subset \mathbb{R}^2$ ($D = \mathbb{C}$ is allowed!) and every $f \in \mathcal{S}'(\mathbb{R})$, it holds that*

$$C_{\mathrm{poinc}}(p, D, |V_\varphi f|^p) \le \frac{4p}{h_{p,D}(f)},$$

where $h_{p,D}(f)$ is defined as in Definition 2.8.

Combining Theorem 5.10 with Theorem 5.8 we obtain the following fundamental stability result.

THEOREM 5.11. *Let* $p \in [1,2)$ *and* $q \in (\frac{2p}{2-p}, \infty)$. *Suppose that* $f \in M^{p,p}(\mathbb{R})$ *is such that its Gabor transform* $V_\varphi f$ *is centered* (*otherwise we could translate and modulate* f). *Then there exists a constant* $c > 0$ only depending on p, q, and the quotient $\|f\|_{M^{p,p}(\mathbb{R})}/\|f\|_{M^{\infty,\infty}(\mathbb{R})}$ *such that for any* $g \in M^{p,p}(\mathbb{R})$ *it holds that*

$$d_{M^{p,p}(\mathbb{R})}(f,g) \leq c \cdot (1 + h_p(f)^{-1}) \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}}.$$

Combining Theorem 5.10 with Theorem 5.9, we obtain the following fundamental local stability result.

THEOREM 5.12. *Let* $p \in [1,2)$ *and* $q \in (\frac{2p}{2-p}, \infty)$. *Suppose that* $D \subset \mathbb{C}$ *satisfies the assumptions of Proposition* 4.7. *Suppose that* $f \in M^{p,p}(\mathbb{R})$ *is such that its Gabor transform* $V_\varphi f$ *is centered* (*otherwise we could translate and modulate* f). *Then there exists a constant* $c > 0$, only depending on p, q, and

$$\max\left\{\frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}, \frac{\|V_{\varphi'} f\|_{L^\infty(D)}}{\|V_\varphi f\|_{L^\infty(D)}}\right\}$$

such that for any $g \in M^{p,p}(\mathbb{R})$ *it holds that*

$$\inf_{\alpha\in\mathbb{R}} \|V_\varphi f - e^{\mathrm{i}\alpha} V_\varphi g\|_{L^p(D)} \leq c \cdot (1 + h_{p,D}(f)^{-1}) \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}(D)}.$$

Remark 4.8, together with Proposition 5.7 and Theorem 5.10, gives us a slightly different version of the stability result in Theorem 5.12, where we can drop the assumptions on the domain D altogether.

THEOREM 5.13. *Let* $p \in [1,2)$ *and* $q \in (\frac{2p}{2-p}, \infty)$. *Suppose that* $f \in M^{p,p}(\mathbb{R})$ *is such that its Gabor transform* $V_\varphi f$ *is centered* (*otherwise we could translate and modulate* f), *and let* $\delta := \widetilde{\delta}_D(V_\varphi f)$ *as defined in* (4.11) *and* $\kappa := \frac{\|V_\varphi f\|_{L^p(D)}}{\|V_\varphi f\|_{L^\infty(D)}}$. *Then there exists a constant* $c > 0$ only depending on p and q *such that for any* $g \in M^{p,p}(\mathbb{R})$ *it holds that*

$$\inf_{\alpha\in\mathbb{R}} \|V_\varphi f - e^{\mathrm{i}\alpha} V_\varphi g\|_{L^p(D)} \leq c \cdot (1 + h_{p,D}(f)^{-1}) \cdot \left(1 + \frac{\kappa^p}{\delta^2}\right) \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}(D)}.$$

As a consequence, we get the following multicomponent-type stability result.

COROLLARY 5.14. *Let* $p \in [1,2)$ *and* $q \in (\frac{2p}{2-p}, \infty)$, *and let* D *be partitioned in subdomains* $D_1, \dots, D_s$, *i.e.,*

$$D_i \subset D \text{ open}, \quad D_i \cap D_j = \varnothing \text{ for } i \neq j, \quad \text{and} \quad \bigcup_{i=1}^s \overline{D_i} = \overline{D}.$$

Suppose that $f \in M^{p,p}(\mathbb{R})$ *is such that its Gabor transform* $V_\varphi f$ *is centered (otherwise we could translate and modulate* f*). Let*

$$B := \max_{i=1,\dots,s} (1 + h_{p,D_i}(f)^{-1}) \cdot \left(1 + \frac{\kappa_i^p}{\delta_i^2}\right)$$

where we set $\delta_i := \widetilde{\delta}_{D_i}(V_\varphi f)$ *as defined in* (4.11) *and*

$$\kappa_i := \frac{\|V_\varphi f\|_{L^p(D_i)}}{\|V_\varphi f\|_{L^\infty(D_i)}}. \tag{5.22}$$

Then there exists a constant $c > 0$ only depending on p and q *such that for any* $g \in M^{p,p}(\mathbb{R})$ *it holds that*

$$\sum_{i=1}^{s} \inf_{\alpha_i \in \mathbb{R}} \|V_\varphi f - e^{\mathbf{i}\alpha} V_\varphi g\|_{L^p(D)} \leq c \cdot B \cdot \||V_\varphi f| - |V_\varphi g|\|_{\mathcal{D}^{1,4}_{p,q}(D)}.$$

Appendix A The STFT Does Phase Retrieval

In this section we present two remarkable properties of the Gabor transform. First we show that by multiplication with a function η (which is independent of the signal f) the Gabor transform $V_\varphi f$ becomes an entire function (compare to [32, prop. 3.4.1]).

THEOREM A.1. *Let* $z := x + iy \in \mathbb{C}$ *and let* $\eta(z) := e^{\pi(\frac{|z|^2}{2} - \mathbf{i}xy)}$. *Then for every* $f \in \mathcal{S}'(\mathbb{R})$ *the function* $z \mapsto \eta(z) \cdot V_\varphi f(x, -y)$ *is an entire function.*

PROOF. For fixed f define $F(z) := \eta(z) \cdot V_\varphi f(x, -y)$. Since any tempered distribution is a derivative of finite order of a continuous function of polynomial growth [29, theorem 8.3.1.], we can find a function h with these properties such that

$$\begin{aligned} F(z) &= \eta(z) \cdot \left(\frac{d^k}{dt^k} h(\cdot), e^{-\pi(\cdot - x)^2} e^{2\pi \mathbf{i} y \cdot} \right)_{\mathcal{S}'(\mathbb{R}) \times \mathcal{S}(\mathbb{R})} \\ &= (-1)^k \eta(z) \int_{\mathbb{R}} h(t) \frac{d^k}{dt^k} \left(e^{-\pi(t-x)^2} e^{2\pi \mathbf{i} y t} \right) dt \end{aligned}$$

for some $k \in \mathbb{N}$. With $g(t, z) := e^{-\pi(t-x)^2} e^{2\pi \mathbf{i} y t}$ we get

$$\frac{\partial}{\partial t} g(t, z) = -2\pi(t - z) g(t, z).$$

A simple induction argument yields that any higher derivative of g w.r.t. t is of the form $p(t - z) \cdot g(t, z)$ where p is a polynomial. Since for any t the function $z \mapsto \eta(z) g(t, z)$ is holomorphic, so is the integrand of

$$F(z) = \int_{\mathbb{R}} (-1)^k h(t) \eta(z) p(t - z) g(t, z) dt.$$

To conclude that F is an entire function, it suffices to show that for any bounded disc $D \subset \mathbb{C}$ centered at the origin, there is an integrable function u_D such that the integrand is bounded by u_D uniformly for all $z \in D$ (see [23, IV theorem 5.8]). Let r be the radius of such a disc D; then for any $z = x + \mathbf{i}y \in D$ the estimate

$$e^{-\pi(t-x)^2} \le g_D(t) := \begin{cases} e^{-\frac{\pi}{4}t^2}, & |t| \ge 2r, \\ 1, & \text{otherwise}, \end{cases}$$

holds. Furthermore, there is a polynomial $\widetilde{p}$ such that $p(t-z) \le \widetilde{p}(t)$ for all $z \in D$. Therefore

$$|h(t)\eta(z)p(t-z)g(t,z)| \le \sup_{z\in D} |\eta(z)| h(t)\widetilde{p}(t) g_D(t) =: u_D(t).$$

Since g_D decays exponentially and h and $\widetilde{p}$ each have polynomial growth, we get the desired result. □

The following theorem states that the Fourier transform of the spectrogram turns out to be the product of the ambiguity functions of the window g and the signal f (see [18, 19]). This result allows us to write down a reconstruction formula for our problem.

We will present a proof of the statement for the case where f is a tempered distribution. To that end we first of all have to give a meaningful definition of $\mathcal{A}f$ for f a tempered distribution.

For any $F : \mathbb{R}^2 \to \mathbb{C}$ we define two linear transforms by

$$TF(x,y) := F(x, x-y) \quad \text{and} \quad SF(x,y) := F(y,x). \tag{A.1}$$

Clearly $T^{-1} = T$ and $S^{-1} = S$ hold. For $F \in \mathcal{S}'(\mathbb{R}^2)$ let $TF \in \mathcal{S}'(\mathbb{R}^2)$ be defined by

$$(TF, \Theta)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)} := (F, T\Theta)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)},$$

and $SF \in \mathcal{S}(\mathbb{R}^2)$ analogously.

Note that this notation makes sense: If F is a regular tempered distribution we have

$$(TF, \Theta) = \int_{\mathbb{R}}\int_{\mathbb{R}} F(x,y)T\Theta(x,y)\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} TF(x,y)\Theta(x,y)\,dx\,dy$$

since $T^{-1} = T$ and T describes a linear coordinate transform with Jacobean determinant -1.

For $f \in \mathcal{S}'(\mathbb{R})$ we can define a tempered distribution by $\mathcal{A}f := S \circ \mathcal{F}_1 \circ T\big(f \otimes \overline{f}\big)$, where $\mathcal{F}_1$ denotes the Fourier transform w.r.t. the first variable of a bivariate tempered distribution, i.e.,

$$(\mathcal{F}_1 F, \Theta)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)} = \left(F, (x,y) \mapsto \int_{\mathbb{R}} \Theta(t,y) e^{2\pi \mathbf{i} xt}\,dt\right)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)}.$$

We call $\mathcal{A}f$ the ambiguity function of f. For $f \in \mathcal{S}(\mathbb{R})$ the calculation

$$\mathcal{A}f(x,y) = \big[\mathcal{F}_1 \circ T(f \otimes \overline{f})\big](y,x) = \int_{\mathbb{R}} f(t)\overline{f(t-x)}e^{-2\pi \mathbf{i} yt}\,dt \tag{A.2}$$

shows that $\mathcal{A}f$ is indeed an extension of the definition of the ambiguity function (see Theorem 2.3).

In the following $\mathcal{F}$ will denote the Fourier transform of bivariate functions. By duality $\mathcal{F}$ can be defined on tempered distributions:

$$(\mathcal{F}F, \Theta)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)} = \left(F, (x,y) \mapsto \int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(s,t)e^{2\pi\mathbf{i}(xs+yt)}ds\,dt\right)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)}.$$

THEOREM A.2. *Let $f \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$; then $\mathcal{F}|V_g f|^2 = S\mathcal{A}f \cdot S\mathcal{A}g$, i.e.,*

$$(\mathcal{F}|V_g f|^2, \Theta)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)} = (S\mathcal{A}f, S\mathcal{A}g \cdot \Theta)_{\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)} \tag{A.3}$$

holds for all $\Theta \in \mathcal{S}(\mathbb{R}^2)$.

PROOF. First note that $V_g f$ has at most polynomial growth (see [32, theorem 11.2.3.]). So $|V_g f|^2$ also has polynomial growth and therefore is in $\mathcal{S}'(\mathbb{R}^2)$, and its Fourier transform is well-defined.

To simplify notation we will use duality brackets without explicitly stating in which spaces we take duality. From the context it will be clear if we mean duality either in $\mathcal{S}'(\mathbb{R})\times\mathcal{S}(\mathbb{R})$ or in $\mathcal{S}'(\mathbb{R}^2)\times\mathcal{S}(\mathbb{R}^2)$. Since compactly supported functions are dense in $\mathcal{S}(\mathbb{R}^2)$ and $\mathcal{F}: \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R}^2)$ is unitary, it suffices to show

$$\big(\mathcal{F}|V_g f|^2, \mathcal{F}\Theta\big) = (S\mathcal{A}f, S\mathcal{A}g \cdot \mathcal{F}\Theta)$$

for all $\Theta \in C_c^\infty(\mathbb{R}^2)$, where we denote by $C_c^\infty(\mathbb{R}^2)$ the space of infinitely often differentiable functions on $\mathbb{R}^2$ with compact support.

For the moment let us assume that g is also compactly supported. The spectrogram can be written as

$$\begin{aligned}|V_g f(x,y)|^2 &= (f, e^{-2\pi\mathbf{i}y\cdot}\overline{g}(\cdot - x)) \cdot \overline{(f, e^{-2\pi\mathbf{i}y\cdot}\overline{g}(\cdot - x))}\\ &= (f \otimes \overline{f}, (s,t) \mapsto e^{-2\pi\mathbf{i}ys}\overline{g}(s-x)e^{2\pi\mathbf{i}yt}g(t-x)).\end{aligned}$$

We obtain

$$\begin{aligned}(\mathcal{F}(|V_g f|^2), \mathcal{F}\Theta) &= \int_{\mathbb{R}}\int_{\mathbb{R}} |V_g f|^2\Theta(x,y)dx\,dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \Theta(x,y)(f \otimes \overline{f}, (s,t)\\ &\quad\mapsto e^{-2\pi\mathbf{i}ys}\overline{g}(s-x)e^{2\pi\mathbf{i}yt}g(t-x))dx\,dy.\end{aligned} \tag{A.4}$$

What we want to do next is to interchange integration and evaluation by the distribution $f \otimes \overline{f}$ in the equation above. To this end we approximate the integral by a sequence of Riemann sums and use the linearity of $f \otimes \overline{f}$.

Let functions J_n for $n \in \mathbb{N}$ and J be defined by

$$J_n(s,t) := n^{-2} \sum_{k,l \in \mathbb{Z}} \Theta\left(\frac{k}{n}, \frac{l}{n}\right) e^{-2\pi \mathbf{i} \frac{l}{n} s} \overline{g}\left(s - \frac{k}{n}\right) e^{2\pi \mathbf{i} \frac{l}{n} t} g\left(t - \frac{k}{n}\right),$$

$$J(s,t) := \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta(x,y) e^{-2\pi \mathbf{i} y s} \overline{g}(s-x) e^{2\pi \mathbf{i} y t} g(t-x) \, dx \, dy.$$

For $M > 0$ such that $\operatorname{supp} \Theta \subset [-M/2, M/2]^2$ and $\operatorname{supp} g \subset [-M/2, M/2]$, clearly both $\operatorname{supp} J_n$ and $\operatorname{supp} J$ are subsets of $[-M, M]^2$. Furthermore, the indices k and l in the definition of J_n will in fact only run over the finite set $\mathbb{Z} \cap [-Mn, Mn]$.

Note that (A.4) is an integral of a continuous and compactly supported function and therefore

$$(\mathcal{F}|V_g f|^2, \mathcal{F}\Theta) = \lim_{n \to \infty} (f \otimes \overline{f}, J_n) \tag{A.5}$$

holds. Clearly J_n converges to J pointwise. To interchange taking the limit and evaluation by $f \otimes \overline{f}$ we will show that J_n converges to J w.r.t. Schwartz space topology, i.e.,

$$\sup_{s,t} \left| s^{\beta_1} t^{\beta_2} \frac{\partial^{\alpha_1 + \alpha_2}}{\partial s^{\alpha_1} \partial t^{\alpha_2}} (J_n(s,t) - J(s,t)) \right|$$

goes to zero for any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N} \cup \{0\}$.

The polynomial factor can be omitted as there is a mutual compact support of $(J_n)_{n \in \mathbb{N}}$ and J. Using $\mathfrak{D} := \frac{\partial^{\alpha_1 + \alpha_2}}{\partial s^{\alpha_1} \partial t^{\alpha_2}}$, let Ψ be defined by

$$\Psi(x,y,s,t) := \mathfrak{D}(e^{-2\pi \mathbf{i} y s} \overline{g}(s-x) e^{2\pi \mathbf{i} y t} g(t-x)).$$

Then obviously

$$\mathfrak{D} J(s,t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \Theta(x,y) \Psi(x,y,s,t) \, dx \, dy,$$

$$\mathfrak{D} J_n(s,t) = n^{-2} \sum_{k,l} \Theta\left(\frac{k}{n}, \frac{l}{n}\right) \Psi\left(\frac{k}{n}, \frac{l}{n}, s, t\right).$$

Again for any fixed (s,t) the values $\mathfrak{D} J_n(s,t)$ can be interpreted as Riemann approximations for the integral $\mathfrak{D} J(s,t)$, and we can infer pointwise convergence. By showing that $(\mathfrak{D} J_n)_{n \in \mathbb{N}}$ is equicontinuous on the compact set $[-M, M]^2$, we can conclude that $\mathfrak{D} J_n \to \mathfrak{D} J$ uniformly. Since Ψ is a smooth function there exists a $c > 0$ such that

$$|\Psi(x,y,s,t) - \Psi(x,y,s',t')| \le c |(s,t) - (s',t')|$$

for all $(x, y, s, t), (x, y, s', t') \in [-M/2, M/2]^2 \times [-M, M]^2$. Equicontinuity holds by the estimate

$$\begin{aligned}
&|\mathfrak{D} J_n(s,t) - \mathfrak{D} J_n(s',t')| \\
&\quad \leq n^{-2} \sum_{k,l} \left| \Theta\left(\frac{k}{n}, \frac{l}{n}\right) \right| \left| \Psi\left(\frac{k}{n}, \frac{l}{n}, s, t\right) - \Psi\left(\frac{k}{n}, \frac{l}{n}, s', t'\right) \right| \\
&\quad \leq n^{-2} \sum_{k,l} \|\Theta\|_{L^\infty(\mathbb{R}^2)} \cdot c \cdot |(s,t) - (s',t')| \\
&\quad \leq \|\Theta\|_{L^\infty(\mathbb{R}^2)} M^2 c |(s,t) - (s',t')|
\end{aligned}$$

where we used the fact that for every n the indices run over the finite set $|k|, |l| \leq \frac{Mn}{2}$.

Defining $h_\tau(\cdot) := g(\cdot)\overline{g}(\cdot - \tau)$ we obtain

$$J(s,t) = [\mathcal{F}_2 \Theta(\cdot, s-t) * h_{s-t}(\cdot)](s),$$

and therefore

$$TJ(s,t) := J(s, s-t) = [\mathcal{F}_2 \Theta(\cdot, t) * h_t(\cdot)](s).$$

The Fourier transform of the first variable gives

$$\mathcal{F}_1 \circ TJ(s,t) = \mathcal{F}\Theta(s,t) \cdot \widehat{h_t}(s).$$

Now $\widehat{h_t}$ turns out to be the ambiguity function of g:

$$\widehat{h_t}(s) = \mathcal{F}(g(\cdot)\overline{g}(\cdot - t))(s) = \mathcal{A}g(t,s).$$

Putting it all together, we get

$$\begin{aligned}
(\mathcal{F}|V_g f|^2, \mathcal{F}\Theta) &= (f \otimes \overline{f}, J) \\
&= (T(f \otimes \overline{f}), TJ) = (\mathcal{F}_1 \circ T(f \otimes \overline{f}), \mathcal{F}_1 \circ TJ) \\
&= (\mathcal{F}_1 \circ T(f \otimes \overline{f}), S\mathcal{A}g \cdot \mathcal{F}\Theta) = (S\mathcal{A}f, S\mathcal{A}g \cdot \mathcal{F}\Theta).
\end{aligned}$$

It remains to prove that the result holds true for any Schwartz function g. We will do this by a density argument: For $g \in \mathcal{S}(\mathbb{R})$ one can find a sequence $(g_n)_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ of compactly supported functions converging to g.

Since for any $f \in \mathcal{S}'(\mathbb{R})$ there exist $C > 0$ and $L > 0$ such that

$$|(f,h)| \leq C \sum_{\alpha,\beta \leq L} \left\| \frac{d^\alpha}{d\cdot^\alpha}\left(\cdot^\beta \cdot h(\cdot)\right) \right\|_{L^\infty(\mathbb{R})} \quad \text{for all } h \in \mathcal{S}(\mathbb{R})$$

(see [29, chapter 8.3]) we can estimate

$$\begin{aligned}
|V_{g-g_n} f(x,y)| &= \left|\left(f, e^{2\pi \mathbf{i} y \cdot}\overline{(g_n-g)}(\cdot - x)\right)\right| \\
&\le C \sum_{\alpha,\beta \le L} \left\| \frac{d^\alpha}{d\cdot^\alpha}\left(\cdot^\beta e^{2\pi \mathbf{i} y \cdot}\overline{(g_n-g)}(\cdot - x)\right)\right\|_{L^\infty(\mathbb{R})} \\
&= C \sum_{\alpha,\beta \le L} \left\| \frac{d^\alpha}{d\cdot^\alpha}\left((\cdot + x)^\beta e^{2\pi \mathbf{i} y \cdot}\overline{(g_n-g)}(\cdot)\right)\right\|_{L^\infty(\mathbb{R})} \\
&\le p(x,y) \cdot \max_{\alpha,\beta \le L} \left\| \frac{d^\alpha}{d\cdot^\alpha} \cdot^\beta \left(g_n(\cdot) - g(\cdot)\right)\right\|_{L^\infty(\mathbb{R})}
\end{aligned}$$

for some polynomial p.

In particular, for any compact $K \subset \mathbb{R}^2$ there is a constant C_K independent of n such that the function on the right-hand side of the inequality above can be bounded by C_K for all $(x,y) \in K$.

The STFT is continuous; therefore the function

$$|V_{g_n} f(x,y)| \le |V_{g_n - g} f(x,y)| + |V_g f(x,y)|$$

can also be bounded by a constant independent of n on any compact K. Obviously $|V_{g_n} f|^2$ converges to $|V_g f|^2$ pointwise. By dominated convergence we get

$$\begin{aligned}
\lim_{n\to\infty} (\mathcal{F}|V_{g_n} f|^2, \mathcal{F}\Theta) &= \lim_{n\to\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} |V_{g_n} f(x,y)|^2 \Theta(x,y)\,dx\,dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |V_g f(x,y)|^2 \Theta(x,y)\,dx\,dy = (\mathcal{F}|V_g f|^2, \mathcal{F}\Theta).
\end{aligned}$$

Since $g \mapsto g \otimes \overline{g}$ is continuous as a mapping from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R}^2)$ and so are S, $\mathcal{F}_1$, and T on $\mathcal{S}(\mathbb{R}^2)$, so is their composition $\mathcal{A}g = S \circ \mathcal{F}_1 \circ T(f \otimes \overline{f})$, which implies convergence of $(\mathcal{A}_{g_n})_{n\in\mathbb{N}}$ to $\mathcal{A}_g$. Multiplication by a fixed Schwartz function is again a continuous operator on $\mathcal{S}(\mathbb{R}^2)$; therefore

$$\lim_{n\to\infty} (S\mathcal{A}f, S\mathcal{A}g_n \cdot \mathcal{F}\Theta) = (S\mathcal{A}f, S\mathcal{A}g \cdot \mathcal{F}\Theta). \tag{A.6}$$ □

As a consequence of Theorem A.2, we obtain that a window function g whose ambiguity function has no zeros allows phase retrieval.

THEOREM A.3. *Let $g \in \mathcal{S}(\mathbb{R})$ be such that its ambiguity function $\mathcal{A}g$ has no zeros. Then for any $f, h \in \mathcal{S}'(\mathbb{R})$ with $|V_g f| = |V_g h|$, there exists $\alpha \in \mathbb{R}$ such that $h = e^{\mathbf{i}\alpha} f$. If $f \in \mathcal{S}(\mathbb{R})$, then*

$$f(t) \cdot \overline{f(0)} = \mathcal{F}_2^{-1}(S\mathcal{F}|V_g f|^2/\mathcal{A}g)(t,t), \quad t \in \mathbb{R}, \tag{A.7}$$

holds true, where S is defined by (A.1) *and $\mathcal{F}_2^{-1}$ denotes the inverse Fourier transform w.r.t. the second variable.*

PROOF. For $\Theta \in C_c^\infty(\mathbb{R}^2)$ so is the function $(S\mathcal{A}g)^{-1} \cdot \Theta$. By Theorem A.2, the tempered distributions $\mathcal{A}f$ and $\mathcal{A}h$ coincide on the dense subspace $C_c^\infty(\mathbb{R}^2)$ and are therefore equal. For arbitrary $\phi, \psi \in \mathcal{S}(\mathbb{R})$ we get

$$\big(\mathcal{A}f, S \circ \mathcal{F}_1 \circ T(\phi \otimes \overline{\psi})\big) = (f \otimes \overline{f}, \phi \otimes \overline{\psi}) = (f, \phi) \cdot \overline{(f, \psi)}$$

and further

$$(f, \phi) \cdot \overline{(f, \psi)} = (h, \phi) \cdot \overline{(h, \psi)}.$$

The choice $\psi = \phi$ implies $|(f, \phi)| = |(h, \phi)|$.

Let ψ be such that $(h, \psi) \neq 0$; then we obtain the equation

$$(h, \phi) = \frac{\overline{(f, \psi)}}{\overline{(h, \psi)}}(f, \phi).$$

Since the fraction has modulus one the statement holds.

For $f \in \mathcal{S}(\mathbb{R})$ equation (A.3) implies

$$S\mathcal{F}|V_g f|^2 = \mathcal{A}f \cdot \mathcal{A}g$$

pointwise. Looking at (A.2) shows $f(t) \cdot \overline{f(t-x)} = \mathcal{F}_2^{-1}\mathcal{A}f(x, t)$. Combining these observations yields

$$f(t) \cdot \overline{f(0)} = \mathcal{F}_2^{-1}\mathcal{A}f(t, t) = \mathcal{F}_2^{-1}(S\mathcal{F}|V_g f|^2/\mathcal{A}g)(t, t). \qquad \square$$

Remark A.4. If we restrict the signals f and h to be in $L^2(\mathbb{R})$, the ambiguity function $\mathcal{A}g$ can in fact vanish on a set of measure zero and the statement of Theorem A.3 still holds.

By calculating the ambiguity function for the Gaussian, we can conclude that the Gabor transform does phase retrieval:

LEMMA A.5. *Let $\varphi(\cdot) = e^{-\pi \cdot^2}$ be the Gaussian. Then we have*

$$\mathcal{A}\varphi(x, y) = c \cdot e^{-\pi \mathbf{i} xy} \cdot e^{-\pi/2 \cdot (x^2+y^2)}$$

for some positive constant c.

PROOF. Using the substitution $\tau = t - x/2$ gives

$$\begin{aligned}
\mathcal{A}\varphi(x, y) &= \int_{\mathbb{R}} e^{-\pi t^2} e^{-\pi(t-x)^2} e^{-2\pi \mathbf{i} yt}\, dt \\
&= \int_{\mathbb{R}} e^{-\pi(\tau+x/2)^2} e^{-\pi(\tau-x/2)^2} e^{-2\pi \mathbf{i} y(\tau+x/2)^2}\, d\tau \\
&= e^{-\pi \mathbf{i} yx} e^{-\pi/2 \cdot x^2} \int_{\mathbb{R}} e^{-2\pi\tau^2} e^{-2\pi \mathbf{i} y\tau}\, d\tau.
\end{aligned}$$

It is well-known that the Fourier transform of a Gaussian is again a Gaussian. We will still do the calculation to get the constants: Let $g(\cdot) := e^{-2\pi \cdot^2}$. Then

$$\widehat{g}'(y) = -\int_{\mathbb{R}} 2\pi \mathbf{i} t \cdot e^{-2\pi t^2} e^{-2\pi \mathbf{i} y t}\, dt = \frac{\mathbf{i}}{2} \cdot \widehat{g'}(y) = -\pi y \cdot \widehat{g}(y).$$

Therefore with $c = \int_{\mathbb{R}} g > 0$, we have $\widehat{g}(y) = c \cdot e^{-\pi/2 \cdot y^2}$. □

Therefore by Theorem A.3 the Gabor transform does phase retrieval:

COROLLARY A.6. *Let $\varphi(\cdot) := e^{-\pi \cdot^2}$ be the Gaussian window. Let $f, h \in \mathcal{S}'(\mathbb{R})$ be such that $|V_\varphi f| = |V_\varphi h|$; then there exists $\alpha \in \mathbb{R}$ such that $h = e^{\mathbf{i}\alpha} f$.*

Appendix B Poincaré and Cheeger Constants

In this section we relate the Poincaré constant to a geometric quantity, the so-called Cheeger constant. This concept goes back to Jeff Cheeger [16]. We will further show that on a bounded domain that is equipped with a weight arising from a Gabor measurement there always holds a Poincaré inequality. Moreover, we find that when the weight w is chosen to be a Gaussian, there exists a finite constant $C > 0$ independent of $R > 0$ such that

$$C_{\text{poinc}}(p, B_R(0), w) \leq C.$$

B.1 Cheeger Constant

The goal is to estimate the Poincaré constant from above in terms of the Cheeger constant. First, recall the definition of Lipschitz and locally Lipschitz functions:

DEFINITION B.1. Let $A \subset \mathbb{R}^d$ and $f : A \to \mathbb{R}$. Then f is called

(i) *Lipschitz* (on A) if there exists a $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y| \quad \text{for all } x, y \in A,$$

(ii) *locally Lipschitz* (on A) if f is Lipschitz on any compact subset of A.

Let $\mathcal{H}^{d-1}$ denote the $(d-1)$-dimensional Hausdorff measure. A definition and some basic properties about Hausdorff measures as well as a proof of the following formula can be found in [25, chap. 3.4.3].

THEOREM B.2 (Coarea formula, or change-of-variables formula). *Let $f : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz and $g : \mathbb{R}^d \to \mathbb{R}$ be an integrable function. Then the restriction $g|_{f^{-1}\{t\}}$ is integrable w.r.t. $\mathcal{H}^{d-1}$ for almost all $t \in \mathbb{R}$ and*

$$\int_{\mathbb{R}^d} g(x)|\nabla f(x)| dx = \int_{\mathbb{R}} \int_{f^{-1}\{t\}} g(s) d\mathcal{H}^{d-1}(s) dt.$$

Later, in Proposition B.5, we use the coarea formula to establish a link between the Poincaré's inequality and the Cheeger constant. However, we will need the coarea formula to hold under slightly different assumptions:

LEMMA B.3. *Let $D \subset \mathbb{R}^d$ be an open set and w a nonnegative, integrable function on D. Let u be a measurable and real-valued function and assume that there exists a set $E \subset D$ such that*

(i) *u restricted to $D \setminus E$ is locally Lipschitz,*
(ii) *there exists a sequence $D_1 \subset D_2 \subset \cdots \subset D \setminus E$ of bounded open sets such that $\bigcup_n D_n = D \setminus E$ and $\overline{D_n} \subset D \setminus E$ for all $n \in \mathbb{N}$, and*
(iii) *w vanishes for $\mathcal{H}^{d-1}$ almost all $x \in E$.*

Then

$$\int_D w(x)|\nabla u(x)|dx = \int_{\mathbb{R}} \int_{u^{-1}\{t\}} w(s) d\mathcal{H}^{d-1}(s) dt.$$

PROOF. By extending w by 0 outside of D we consider w as a function defined on the whole space $\mathbb{R}^d$. By Assumptions (i) and (ii), the restriction of u on the compact set $\overline{D_n}$ is Lipschitz and thus also u restricted to D_n is Lipschitz and therefore has an extension u_n that is Lipschitz on $\mathbb{R}^d$ (compare [25, chap. 3.1.]). With $w_n := w \cdot \chi_{D_n}$ we can apply Theorem B.2 on $f = u_n$ and $g = w_n$ for any n and obtain

$$\int_{\mathbb{R}^d} w_n(x)|\nabla u_n(x)|dx = \int_{\mathbb{R}} \int_{u_n^{-1}\{t\}} w_n(s) d\mathcal{H}^{d-1}(s) dt. \tag{B.1}$$

Since $u_n(x) = u(x)$ for $x \in D_n$ and D_n is open, Rademacher's theorem implies $\nabla u_n(x) = \nabla u(x)$ makes sense for almost all $x \in D_n$. By monotone convergence we can take the limit $n \to \infty$ in the left-hand side of equation (B.1):

$$\begin{aligned} \lim_n \int_{\mathbb{R}^d} w_n(x)|\nabla u_n(x)|dx &= \lim_n \int_{D_n} w(x)|\nabla u(x)|dx \\ &= \int_{D \setminus E} w(x)|\nabla u(x)|dx. \end{aligned}$$

Assumption (iii) in particular implies that w vanishes for almost every $x \in E$ (w.r.t. d-dimensional Lebesgue measure). Therefore we can conclude

$$\lim_n \int_{\mathbb{R}^d} w_n(x)|\nabla u_n(x)|dx = \int_D w(x)|\nabla u(x)|dx.$$

Let us have a closer look at the right-hand side of equation (B.1) Since u_n and u coincide on D_n, we have

$$\begin{aligned}\lim_n \int_{\mathbb{R}} \int_{u_n^{-1}\{t\}} w_n(s) d\mathcal{H}^{d-1}(s) dt &= \lim_n \int_{\mathbb{R}} \int_{u_n^{-1}\{t\} \cap D_n} w(s) d\mathcal{H}^{d-1}(s) dt \\ &= \lim_n \int_{\mathbb{R}} \int_{u^{-1}\{t\} \cap D_n} w(s) d\mathcal{H}^{d-1}(s) dt \\ &= \int_{\mathbb{R}} \int_{u^{-1}\{t\} \cap (D \setminus E)} w(s) d\mathcal{H}^{d-1}(s) dt,\end{aligned}$$

where we again used monotone convergence. Assumption (iii) tells us that E is a zero set w.r.t. $w(\cdot) d\mathcal{H}^{d-1}(\cdot)$, and so we finally obtain the claimed equality. □

We consider now a domain $D \subset \mathbb{R}^2$ that can be bounded or unbounded, together with a nonnegative and integrable weight w. We define a measure μ on D by

(B.2) $$\mu(C) := \int_C w(x) dx, \quad C \subset D \text{ measurable w.r.t. Lebesgue measure.}$$

For $A \subset D$ a one-dimensional manifold, we use the notation

(B.3) $$\nu(A) := \int_A w(s) d\sigma(s),$$

where σ denotes the surface measure on A. Furthermore, let us define a system of subsets of D by

(B.4) $$\mathcal{C} = \mathcal{C}(D, w) := \left\{ \begin{matrix} \emptyset \neq C \subset D \text{ open: } \partial C \cap D \text{ is a one-dimensional} \\ \text{manifold and } \mu(C) \leq \frac{1}{2}\mu(D) \end{matrix} \right\}.$$

The *Cheeger constant* h w.r.t. D and w is defined by

(B.5) $$h = h(D, w) := \inf_{C \in \mathcal{C}} \frac{\nu(\partial C \cap D)}{\mu(C)}.$$

Remark B.4. If D is not connected, there is a component C of D such that $C \in \mathcal{C}$. Since $\partial C \cap D = \emptyset$ we clearly have $h = 0$ in that case.

For a measurable, real-valued function u on D we denote the sublevel, superlevel, and level sets of u by

$$\mathcal{S}_t := \{x \in D \colon u(x) < t\}, \quad \mathcal{U}_t := \{x \in D \colon u(x) > t\},$$
$$\text{and} \quad \mathcal{A}_t := \{x \in D \colon u(x) = t\}.$$

In Proposition B.5 we will now establish a first connection between the Cheeger constant and a Poincare-type inequality:

PROPOSITION B.5. *Let $D \subset \mathbb{R}^2$ be a domain and w a weight on D. Let h denote the Cheeger constant of D and w and let μ be defined as in equation* (B.2). *Let u be a nonnegative function on D and assume that there exists $E \subset D$ such that* *conditions* (i)–(iii) *of Lemma* B.3 *hold. Let* $\mathcal{U}_t$ *and* $\mathcal{A}_t$ *denote the superlevel and the level sets w.r.t.* u. *Furthermore, suppose that*

(1) $\mu(\mathcal{U}_0) \leq \frac{1}{2}\mu(D)$,
(2) $\mathcal{A}_t$ *is a one-dimensional manifold for almost all* $t > 0$,
(3) $\mathcal{A}_t = \partial\mathcal{U}_t \cap D$ *for almost all* $t > 0$, *and*
(4) $\mathcal{U}_t$ *is open for almost all* $t > 0$.

Then the inequality

$$h\int_D u(x)d\mu(x) \leq \int_D |\nabla u(x)|d\mu(x) \tag{B.6}$$

holds true.

PROOF. Let $\Gamma \subset (0,\infty)$ be such that

$$\mathcal{A}_t \text{ is a one-dimensional manifold} \quad \text{and} \quad \mathcal{A}_t = \partial\mathcal{U}_t \cap D \quad \text{for all } t \in \Gamma$$

and that $(0,\infty) \setminus \Gamma$ is of Lebesgue measure zero.

Applying Lemma B.3 and using the fact that $\mathcal{H}^1$ coincides with the surface measure on one-dimensional manifolds gives us

$$\begin{aligned}\int_D |\nabla u(x)|w(x)dx &= \int_0^\infty\int_{\mathcal{A}_t} w(s)d\mathcal{H}^1(s)dt = \int_\Gamma\int_{\mathcal{A}_t} w(s)d\mathcal{H}^1(s)dt\\ &= \int_\Gamma \nu(\mathcal{A}_t)dt = \int_\Gamma \nu(\partial\mathcal{U}_t \cap D)dt,\end{aligned}$$

where ν is defined as in (B.3). Now we can estimate

$$\begin{aligned}\int_D |\nabla u(x)|d\mu(x) &\geq \int_{\Gamma\cap\{t:\mu(\mathcal{U}_t)>0\}} \frac{\nu(\partial\mathcal{U}_t \cap D)}{\mu(\mathcal{U}_t)} \cdot \mu(\mathcal{U}_t)dt\\ &\geq h\int_{\Gamma\cap\{t:\mu(\mathcal{U}_t)>0\}} \mu(\mathcal{U}_t)dt = h\int_{(0,\infty)} \mu(\mathcal{U}_t)dt\\ &= h\int_D u(x)d\mu(x).\end{aligned}$$

□

Before we prove the main result of this section, Theorem B.7, we need one more lemma:

LEMMA B.6. *Let* $D \subset \mathbb{R}^2$ *be a domain,* w *a weight on* D, *and* $p \in [1,2]$. *Then for any* $F = u + \mathbf{i}v \in L^p(D,w) \cap L^1(D,w)$ *and any* $a, b \in \mathbb{R}$, *we have*

$$\|F - F_D^w\|_{L^p(D,w)}^p \leq 2^p \cdot \big(\|u-a\|_{L^p(D,w)}^p + \|v-b\|_{L^p(D,w)}^p\big).$$

PROOF.

Step 1. Assume $f \in L^p(D,w) \cap L^1(D,w)$ is a real-valued function with $f_D^w = 0$. We first show that

$$\|f\|_{L^p(D,w)} \leq 2\|f + c\|_{L^p(D,w)} \tag{B.7}$$

for arbitrary $c \in \mathbb{R}$. This statement can be found in [22] but we still give a proof. W.l.o.g. we may assume that c is positive. Let μ be defined as in (B.2). Then we have

$$\int_{\{x\in D: f(x)>0\}} |f(x)|^p \, d\mu(x) \le \int_{\{x\in D: f(x)>0\}} |f(x)+c|^p \, d\mu(x)$$

and

$$\int_{\{x\in D: f(x)<-2a\}} |f(x)+c|^p \, d\mu(x) \ge 2^{-p} \int_{\{x\in D: f(x)<-2c\}} |f(x)|^p \, d\mu(x).$$

Furthermore, using $\int_{\{x\in D: f(x)\le 0\}} |f(x)| d\mu(x) = \int_{\{x\in D: f(x)>0\}} |f(x)| d\mu(x)$ we obtain

$$\begin{aligned}
\int_{\{x\in D: -2c\le f(x)\le 0\}} |f(x)|^p \, d\mu(x) &\le (2c)^{p-1} \int_{\{x\in D: -2c\le f(x)\le 0\}} |f(x)| d\mu(x) \\
&\le (2a)^{p-1} \int_{\{x\in D: f(x)>0\}} |f(x)| d\mu(x) \\
&\le 2^{p-1} \int_{\{x\in D: f(x)>0\}} |f(x)+c|^p d\mu(x).
\end{aligned}$$

Combining these estimates and noting $2^{p-1}+1 \le 2^p$, we see that the inequality (B.7) holds.

Step 2. For any dimension d and $p \le 2$ it holds that

$$\|x\|_2 \le \|x\|_p \quad \text{for all } x \in \mathbb{R}^d,$$

where $\|x\|_p^p := \sum_{j=1}^d |x_j|^p$. Using this inequality and applying (B.7) on u and v, respectively, we obtain

$$\begin{aligned}
\|F - F_D^w\|_{L^p(D,w)}^p &= \int_D \big((u(x)-u_D^w)^2 + (v(x)-v_D^w)^2\big)^{p/2} d\mu(x) \\
&\le \int_D \big|u(x)-u_D^w\big|^p + \big|v(x)-v_D^w\big|^p \, d\mu(x) \\
&\le 2^p \int_D |u(x)-a|^p + |v(x)-b|^p \, d\mu(x).
\end{aligned}$$ □

Finally, we establish a weighted Poincaré inequality for certain meromorphic functions. Looking at (B.6) it is not surprising that the corresponding Poincaré constant can be controlled by the reciprocal of the Cheeger constant.

THEOREM B.7. *Let $D \subset \mathbb{R}^2$ be a domain, w a weight on D, and $p \in [1,2]$. Let h denote the Cheeger constant of D and w. Assume that h is positive. Then a weighted Poincaré inequality holds and $C_{\text{poinc}}(D,w,p) \le \frac{4p}{h}$, i.e.,*

$$\|F - F_D^w\|_{L^p(D,w)} \le \frac{4p}{h} \|\nabla F\|_{L^p(D,w)} \tag{B.8}$$

for all $F \in W^{1,p}(D,w) \cap L^1(D,w) \cap \mathcal{M}(D)$.

PROOF. Let u and v denote the real and imaginary parts of F, and let μ be defined as in (B.2). Let m_u be a median of u, i.e.,

$$\mu(\mathcal{U}^u_{m_u}) \leq \frac{1}{2}\mu(D) \quad \text{and} \quad \mu(\mathcal{S}^u_{m_u}) \leq \frac{1}{2}\mu(D),$$

where $\mathcal{U}^u_t$ and $\mathcal{S}^u_t$ denote super- and sublevel sets of u.

To see why such a number exists, observe that the mapping $t \mapsto \mu(\mathcal{U}^u_t)$ is continuous and takes its values in the interval $[0, \mu(D)]$. Therefore, there has to be a number m_u such that $\mu(\mathcal{U}^u_{m_u}) = \frac{1}{2}\mu(D)$ and since

$$\mu(\mathcal{S}^u_{m_u}) \leq \mu(D) - \mu(\mathcal{U}^u_{m_u}) = \frac{1}{2}\mu(D),$$

m_u is a median of u.

Let m_v be a median of v, defined analogously. By Lemma B.6 we get

$$\|F - F^w_D\|^p_{L^p(D,w)} \leq 2^p \cdot \left(\|u - m_u\|^p_{L^p(D,w)} + \|v - m_v\|^p_{L^p(D,w)}\right). \tag{B.9}$$

We will only estimate the first term in the right-hand side of the inequality above. The second one can be dealt with in the exact same way. Let u_+ and u_- denote the positive and the negative parts of the function $u - m_u$. Then we have

$$\psi := (u - m_u) \cdot |u - m_u|^{p-1} = u^p_+ - u^p_-$$

and

$$\|\psi\|^p_{L^p(D,w)} = \|u - m_u\|^p_{L^p(D,w)} = \|u^p_+\|_{L^1(D,w)} + \|u^p_-\|_{L^1(D,w)}.$$

We want to apply Proposition B.5 on both u^p_+ and u^p_-. First, we check that the assumptions (i)–(iii) of Lemma B.3 are satisfied:

(i). Let E denote the set of points $x \in D$ such that x is a pole of F. The restriction of ψ on $D \setminus E$ is a smooth function and therefore locally Lipschitz. Since for $x, y \in D \setminus E$ we have

$$\left|u^p_\pm(x) - u^p_\pm(y)\right| \leq |\psi(x) - \psi(y)|,$$

u^p_+ and u^p_- are also locally Lipschitz on $D \setminus E$.

(ii). Obviously, setting

$$D_n := (D \cap B_n(0)) \setminus \bigcup_{x \in E} \overline{B_{1/n}(x)}$$

for $n \in \mathbb{N}$ is a valid choice.

(iii). Since E is a discrete set, this property holds for any weight w.

Next we verify that Assumptions (1)–(4) of Proposition B.5 hold for u^p_+ and u^p_-: We write $\mathcal{U}^+_t$, $\mathcal{U}^-_t$, and $\mathcal{U}^\psi_t$ for the superlevel sets of u^p_+, u^p_-, and ψ and accordingly for the sublevel and level sets of the same functions.

(1). This property follows directly from the definition of ψ.

(4). Since ψ is continuous on $D \setminus E$, its super- and sublevel sets are open in $D \setminus E$. The set of poles E is discrete; therefore any open set in $D \setminus E$ also is open in D. The property then follows by the observation that for $t > 0$ the following two equalities hold:

$$\mathcal{U}_t^+ = \mathcal{U}_t^\psi \quad \text{and} \quad \mathcal{U}_t^- = \mathcal{S}_{-t}^\psi.$$

(3). Let $x \in \partial\mathcal{U}_t^\psi \cap D$; then for any $\varepsilon > 0$ the ball $B_\varepsilon(x)$ contains points y, y' such that $\psi(y) > t$ and $\psi(y') \leq t$. By continuity of ψ, we infer $\psi(x) = t$, i.e., $x \in \mathcal{A}_t^\psi$. Thus the inclusion $\partial\mathcal{U}_t^\psi \cap D \subset \mathcal{A}_t$ holds for all t.

Assume next that the set

$$J := \{t \in \mathbb{R}\colon \mathcal{A}_t^\psi \supsetneq \partial\mathcal{U}_t^\psi \cap D\}$$

is of positive measure. For $t \in J$ there exists an $x \in \mathcal{A}_t^\psi$ such that $x \notin \partial\mathcal{U}_t^\psi \cap D$. Therefore for sufficiently small $\varepsilon > 0$, we have $B_\varepsilon(x) \cap \mathcal{U}_t^\psi = \varnothing$, so ψ has a local maximum in x, which implies $\nabla\psi(x) = 0$. Sard's theorem tells us that $\psi(\{x : \nabla\psi(x) = 0\}) \supseteq J$ is a zero set, which contradicts our assumption. This proves that $\mathcal{A}_t^\psi = \partial\mathcal{U}_t^\psi \cap D$ for almost all $t \in \mathbb{R}$. A similar argument shows that also $\mathcal{A}_t^\psi = \partial\mathcal{S}_t^\psi \cap D$ holds true for almost all t.

The observation

$$\begin{aligned} \partial\mathcal{U}_t^+ \cap D &= \partial\mathcal{U}_t^\psi \cap D = \mathcal{A}_t^\psi = \mathcal{A}_t^+, \\ \partial\mathcal{U}_t^- \cap D &= \partial\mathcal{S}_{-t}^\psi \cap D = \mathcal{A}_{-t}^\psi = \mathcal{A}_t^-, \end{aligned}$$

concludes the argument.

(2). Clearly it suffices to show that $\mathcal{A}_t^\psi$ is a one-dimensional manifold for almost all $t \in \mathbb{R}$.

Let $t \notin J'$, where

$$J' := \{t \in \mathbb{R} : \exists x \in \mathcal{A}_t^\psi \text{ s.t. } \nabla\psi(x) = 0\}.$$

Again by Sard's theorem J' is a set of measure zero. For arbitrary $x \in \mathcal{A}_t^\psi$, by the implicit function theorem $\mathcal{A}_t^\psi$ is locally the graph of a smooth function that implies that both $\mathcal{A}_t^+$ and $\mathcal{A}_t^-$ are one-dimensional manifolds for almost all $t > 0$.

Proposition B.5 can now be applied on u_+^p and u_-^p:

$$\begin{aligned} &h\|u - m_u\|_{L^p(D,w)}^p \\ &= h\int_D u_+^p(x) + u_-^p(x)d\mu(x) \\ &\leq \int_D |\nabla u_+^p(x)| + |\nabla u_-^p(x)|d\mu(x) = \int_D |\nabla\psi(x)|d\mu(x). \end{aligned} \tag{B.10}$$

We want to show that $\|u - m_u\|_{L^p(D,w)} \leq \frac{p}{h} \cdot \|\nabla u\|_{L^p(D,w)}$. For $p = 1$ we are done. For any $p > 1$ we have

$$|\nabla \psi(x)| = p|u(x) - m_u|^{p-1}|\nabla u(x)|.$$

Using Hölder's inequality we obtain

$$\begin{aligned}
&\int_D |\nabla \psi(x)| d\mu(x) \\
&\quad = p \int_D |u(x) - m_u|^{p-1} \cdot |\nabla u(x)| d\mu(x) \\
&\quad \leq p\left(\int_D |u(x) - m_u|^{(p-1)p'} d\mu(x)\right)^{1/p'} \cdot \left(\int_D |\nabla u(x)|^p \, d\mu(x)\right)^{1/p} \\
&\quad = p\|u - m_u\|_{L^p(D,w)}^{p-1} \|\nabla u\|_{L^p(D,w)}.
\end{aligned} \tag{B.11}$$

Note that by Cauchy-Riemann equations we have for any $x \in D \setminus E$ that

$$|\nabla F(x)| = \sqrt{2}|\nabla u(x)| = \sqrt{2}|\nabla v(x)|.$$

Combining estimates (B.9), (B.10), and (B.11) we finally obtain

$$\begin{aligned}
\|F - F_D^w\|_{L^p(D,w)}^p &\leq 2^p \cdot \left(\frac{p}{h}\right)^p \int_D |\nabla u(x)|^p + |\nabla v(x)|^p \, d\mu(x) \\
&= 2^{1+p/2}\left(\frac{p}{h}\right)^p \|\nabla F\|_{L^p(D,w)}^p.
\end{aligned}$$

Since $(2^{p/2+1})^{1/p} \leq 2^{3/2} < 4$, inequality (B.8) holds true. □

B.2 Positivity of the Cheeger Constant for Finite Domains

The goal of this section is to prove the following statement:

THEOREM B.8. *Let $D \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary. Furthermore, let $f \in \mathcal{S}'(\mathbb{R})$ such that $V_\varphi f$ has no zeros on ∂D and $1 \leq p < \infty$. Then*

$$h_{p,D}(f) > 0.$$

To this end we will need the fact that in the definition of the Cheeger constant it suffices to consider connected sets C.

LEMMA B.9. *Let w be a nonnegative weight on a domain $D \subset \mathbb{R}^2$, and let h denote the Cheeger constant* (*see equation* (B.5)). *Let $\mathcal{C}$ be defined as in equation* (B.4). *Then*

$$h = \inf_{\substack{C \in \mathcal{C} \\ C \text{ connected}}} \frac{\int_{\partial C} w \, d\sigma}{\int_C w}. \tag{B.12}$$

PROOF. The inequality

$$h = \inf_{\substack{C \in \mathcal{C} \\ C \text{ connected}}} \frac{\int_{\partial C} w \, d\sigma}{\int_C w}$$

is trivial.

It is an easy exercise to see that for positive numbers $(a_l)_{l\in\mathbb{N}}$ and $(b_l)_{l\in\mathbb{N}}$ the following inequality holds:

$$\frac{\sum_l a_l}{\sum_l b_l} \geq \inf_l \frac{a_l}{b_l}. \tag{B.13}$$

For arbitrary $C \in \mathcal{C}$, since C is open, we can write C as a disjoint union of at most countably many connected, open sets $C_l,\ l \in \mathbb{N}$. Applying inequality (B.13) on $a_l = \int_{\partial C_l \cap D} w\, d\sigma$ and $b_l = \int_{C_l} w$ gives

$$\begin{aligned} \frac{\int_{\partial C \cap D} w\, d\sigma}{\int_C w} = \frac{\sum_l \int_{\partial C_l \cap D} w\, d\sigma}{\sum_l \int_{C_l} w} &\geq \inf_l \frac{\int_{\partial C_l \cap D} w\, d\sigma}{\int_{C_l} w} \\ &\geq \inf_{\substack{C \in \mathcal{C} \\ C \text{ connected}}} \frac{\int_{\partial C \cap D} w\, d\sigma}{\int_C w}. \end{aligned}$$

Taking the infimum over all $C \in \mathcal{C}$ yields the desired result. $\square$

First we will show that Theorem B.8 holds in the case $w \equiv 1$. Recall that for $D \subset \mathbb{R}^d$ open, a function $u \in L^1(D)$ is of bounded variation ($u \in \mathrm{BV}(D)$) if

$$\sup_{\phi \in C_c^1(D,\mathbb{R}^d):\, |\phi| \leq 1} \int_D u \operatorname{div} \phi \leq \infty,$$

where $C_c^1(D,\mathbb{R}^d)$ denotes the set of continuously differentiable functions from D to $\mathbb{R}^d$ whose support is a compact subset of D. We will make use of the following properties of functions of bounded variation (see [25, chap. 5] for details): For any $u \in \mathrm{BV}(D)$ there exists a Radon measure μ on D and a μ-measurable function $\sigma : D \to \mathbb{R}^d$ such that $|\sigma| = 1$ μ-a.e. and

$$\int_D u \operatorname{div} \phi = -\int_D \phi \cdot \sigma \, d\mu \quad \text{for all } \phi \in C_c^1(D,\mathbb{R}^d).$$

We will use the notation $|\nabla u| = \mu$ in the following. Equipped with the norm

$$\|\cdot\|_{\mathrm{BV}(D)} := \|\cdot\|_{L^1(D)} + |\nabla \cdot|(D),$$

$\mathrm{BV}(D)$ becomes a Banach space.

To show that Theorem B.8 holds in the case $w \equiv 1$, let us consider the functional

$$\mathcal{F}\colon u \mapsto |\nabla u|(D), \quad u \in \mathrm{BV}(D), \tag{B.14}$$

and the minimization problem

$$\text{minimize } \mathcal{F} \text{ in } V := \left\{ u \in \mathrm{BV}(D)\colon \|u\|_{L^1(D)} = 1,\ |\operatorname{supp} u| \leq \frac{1}{2}|D| \right\}. \tag{B.15}$$

Proving that $h(D,1) > 0$ by showing that (B.15) has a solution is inspired by [14, 34], where they considered a slightly different problem, namely proving positivity of the quantity

$$\inf_{C \subset D \text{ open, } \partial C \text{ smooth}} \frac{\ell(\partial C)}{|C|}.$$

Note that this situation corresponds to estimating Poincaré constants for functions that satisfy Dirichlet conditions:

$$\|u\|_{L^p(D)} \leq C \ \|\nabla u\|_{L^p(D)} \quad \text{for all } u \text{ vanishing on } \partial D.$$

PROPOSITION B.10. *Let $D \subset \mathbb{R}^2$ be a bounded and connected Lipschitz domain, and let $\mathcal{C} := \{C \subset D \colon \partial C \cap D \text{ is smooth}, \ |C| \leq \frac{1}{2}|D|\}$. Let $\mathcal{F}$ and V be defined as in* (B.14)–(B.15). *Then*

(i) *There exists $u^* \in V$ such that*

$$\mathcal{F}(u^*) = \inf_{u \in V} \mathcal{F}(u) > 0.$$

(ii) *For any $C \in \mathcal{C}$ we have $\mathcal{F}(\chi_C) = \ell(\partial C \cap D)$.*

PROOF.

(i) Choose a minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset V$, i.e.,

$$\lim_n \mathcal{F}(u_n) = \inf_{u \in V} \mathcal{F}(u).$$

Then $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{BV}(D)$ and therefore by [25, chap. 5.2.3, theorem 4] there is a subsequence that we still call $(u_n)_{n\in\mathbb{N}}$ and a $u^* \in \mathrm{BV}(D)$ such that $u_n \to u^*$ in $L^1(D)$. Obviously $\|u^*\|_{L^1(D)} = 1$.

To show that $u^* \in V$, it remains to verify that $|\operatorname{supp} u^*| \leq \frac{1}{2}|D|$. Assume that $|\operatorname{supp} u^*| > \frac{1}{2}|D|$. Then for $\varepsilon > 0$ sufficiently small we have $|D_\varepsilon| > \frac{1}{2}|D|$, where we set $D_\varepsilon := \{x \colon |u^*(x)| > \varepsilon\}$. Since L^1-convergence implies almost uniform convergence, there must be a set $E_\varepsilon \subset D$ such that

$$|E_\varepsilon| < \frac{1}{2}|D| - |D_\varepsilon| \quad \text{and} \quad u_n \to u^* \text{ uniformly on } D \setminus E_\varepsilon.$$

In particular, convergence is uniform on $D_\varepsilon \setminus E_\varepsilon$. Therefore there exists $N \in \mathbb{N}$ such that $|u_n(x) - u^*(x)| < \varepsilon/2$ for all $x \in D_\varepsilon \setminus E_\varepsilon$. Applying the inverse triangle inequality yields $|u_n(x)| > \varepsilon/2$ for these n and x. By construction we have $|D_\varepsilon \setminus E_\varepsilon| > \frac{1}{2}|D|$, which contradicts the assumption that $(u_n)_{n\in\mathbb{N}} \subset V$.

By [25, chap. 5.2.1, theorem 1] the functional $\mathcal{F}$ is lower-semicontinuous w.r.t. L^1-norm. Thus we obtain

$$\inf_{u \in V} \mathcal{F}(u) \leq \mathcal{F}(u^*) \leq \liminf_n \mathcal{F}(u_n) = \lim_n \mathcal{F}(u_n) = \inf_{u \in V} \mathcal{F}(u).$$

It remains to show that $\mathcal{F}(u^*)$ is strictly positive. Assume $\mathcal{F}(u^*) = 0$. Then $|\nabla u^*|$ is the zero measure, which by [4, prop. 3.2(a)] amounts to u^*

being constant on the connected set D. Since there is no constant function u^* such that

$$|\operatorname{supp} u^*| \leq \frac{1}{2}|D| \quad \text{and} \quad \|u^*\|_{L^1(D)} = 1,$$

we have a contradiction.

(ii) Any $C \in \mathcal{C}$ can be extended to a set $C' \supseteq C$ such that

$$C' \cap D = C \quad \text{and} \quad \partial C' \text{ is smooth.}$$

Note that if $\partial C \subset D$ we can choose $C' = C$. Since C' has a smooth boundary, the length of $\partial C' \cap D$ can be measured by the total variation of the gradient of $\chi_{C'}$, i.e.,

$$|\nabla \chi_{C'}|(D) = \mathcal{H}^{n-1}(\partial C' \cap D),$$

see [25, chap. 5.1, exam. 2]. Therefore we obtain

$$\begin{aligned} |\nabla \chi_C|(D) = |\nabla \chi_{C'}|(D) &= \mathcal{H}^{n-1}(\partial C' \cap D) \\ &= \mathcal{H}^{n-1}(\partial C \cap D) = \ell(\partial C \cap D). \end{aligned} \qquad \square$$

As a direct consequence we obtain the following theorem:

THEOREM B.11. *Let $D \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then*

$$h(D, 1) > 0,$$

where $h(D, 1)$ is defined as in (B.5).

PROOF. We use the notation of Proposition B.10. Since $\mathcal{F}(\chi_C) = \ell(\partial C \cap D)$ for any $C \in \mathcal{C}$, we obtain

$$\frac{\ell(\partial C \cap D)}{|C|} = \frac{\mathcal{F}(\chi_C)}{\|\chi_C\|_{L^1(D)}} = \mathcal{F}\big(\|\chi_C\|_{L^1(D)}^{-1} \cdot \chi_C\big) \geq \mathcal{F}(u^*) > 0. \qquad \square$$

Let us get to the general case where w emerges from a Gabor measurement, i.e., $w = |V_\varphi f|^p$. On a bounded domain D one can construct a rather simple, equivalent weight $w_r \sim w$, which we will analyze.

PROOF OF THEOREM B.8. The idea of the proof is to construct a weight equivalent to $|V_\varphi f|^p$ that locally is either constant or looks like $z \mapsto |z|^q$ for some positive number q. We then seek to exploit results on Cheeger constants for the case $w \equiv 1$ as well as isoperimetric inequalities w.r.t. weights of the form $|z|^p$.

Up to multiplication with a nonzero function η and a reflection in the plane, the function $V_\varphi f$ is an entire function (see Theorem 2.4) and therefore can only have a finite number of zeros $(\zeta_i)_{i=1}^N$ in D. We have for every $z = x + \mathbf{i}y \in D$ that

$$V_\varphi f(z) = \prod_{i=1}^{N} (\overline{z} - \overline{\zeta_i})^{m_i} \cdot g(\overline{z}), \tag{B.16}$$

where $m_i \in \mathbb{N}$ denotes the multiplicity of the zero ζ_i, and g is a continuous function without zeros on $\overline{D}$. Due to the compactness of $\overline{D}$ the function $z \mapsto |g(\overline{z})|$ assumes a nonzero minimum and a maximum on $\overline{D}$.

For $r > 0$ let $D_0^r := \bigcup_{i=1}^N B_r(\zeta_i)$ and define

$$w_r(z) := \begin{cases} 1, & z \in D \setminus D_0^r, \\ |z - \zeta_i|^{m_i \cdot p}, & z \in B_r(\zeta_i) \text{ for some } i \in \{1, \dots, N\}. \end{cases}$$

This definition is ambiguous if $z \in B_r(\zeta_i) \cap B_r(\zeta_j)$ for $i \neq j$. But clearly there is a $r_0 > 0$ such that all these intersections will be empty for $r < r_0$.

For $\delta > 0$ let us define the set

$$D_\delta := \{x \in D : \operatorname{dist}(x, \partial D) < \delta\}.$$

Obviously for $\delta \to 0$ we have that $|D_\delta| \to 0$. Since $V_\varphi f$ has no zeros on the boundary ∂D we can choose δ such that

$$|D_\delta| \leq \frac{1}{2}|D| \quad \text{and} \quad \bigcup_{i=1}^N B_\delta(\zeta_i) \cap D_\delta = \varnothing.$$

From now on we consider the weight w_r for fixed $0 < r < \min\{r_0, \delta, 1\}$. Since $w_r \sim |V_\varphi f|^p$ in D and by Lemma B.9 it suffices to show that the quantity

$$\frac{\int_{\partial C} w_r \, d\sigma}{\int_C w_r}$$

can be uniformly bounded from below by a positive constant for all connected (due to Lemma B.9), open sets $C \subset D$ with smooth boundary such that $\int_C |V_\varphi f|^p \leq \frac{1}{2} \int_C |V_\varphi f|^p$.

Let us fix a C with these properties. We can now look at the following two cases separately:

Case A. $\ell(\partial C \cap D \setminus \bigcup_{i=1}^N B_{r/2}(\zeta_i)) \geq r/2$.

Case B. $\ell(\partial C \cap D \setminus \bigcup_{i=1}^N B_{r/2}(\zeta_i)) < r/2$.

Within Case B we further distinguish the following subcases:

Case B.1. $\partial C \cap \partial D \neq \varnothing$.

Case B.2. $\partial C \cap \partial D = \varnothing$ and $C \cap B_{r/2}(\zeta_i) \neq \varnothing$ for some i.

Case B.3. $\partial C \cap \partial D = \varnothing$ and $C \cap \bigcup_{i=1}^N B_{r/2}(\zeta_i) = \varnothing$.

ad A. Let $m := \max_i\{m_i\}$. Since $w_r(z) \geq (r/2)^{mp}$ for $z \in D \setminus \bigcup_{i=1}^N B_{r/2}(\zeta_i)$, we get the estimate

$$\frac{\int_{\partial C \cap D} w_r \, d\sigma}{\int_C w_r} \geq \frac{(r/2)^{mp+1}}{\int_D w_r}.$$

ad B.1. By construction we have $C \subset D_\delta$. Since $w_r \equiv 1$ in D_δ and $|C| \leq \frac{1}{2}|D|$ we obtain

$$\frac{\int_{\partial C \cap D} w_r \, d\sigma}{\int_C w_r} \geq h(D,1).$$

By Theorem B.11 $h(D,1)$ is positive.

ad B.2. We can infer that $C \subset B_r(\zeta_i)$ for suitable i and let $q = m_i p$. Let us assume for simplicity that $\zeta_i = 0$. Let μ and ν be defined as in (B.2) and (B.3) w.r.t. the weight $|\cdot|^{m_i p}$.
Since

$$\mu(B_{2r}(2r)) \geq \mu(B_r(0)) \geq \mu(C)$$

by continuity of $s \mapsto \mu(B_s(s))$, there exists $s \in (0, 2r]$ such that $\mu(B_s(s)) = \mu(C)$. We can now appeal to a result about weighted isoperimetric problems [21, see theorem 3.16] that guarantees $\nu(\partial B_s(s)) \leq \nu(\partial C)$. Since

$$\nu(B_s(s)) \geq s\pi(\sqrt{2}s)^q \quad \text{and} \quad \mu(B_s(s)) \leq 2s^2\pi(2s)^q,$$

we obtain

$$\frac{\nu(\partial C)}{\mu(C)} \geq \frac{\nu(\partial B_s(s))}{\mu(B_s(s))} \gtrsim s^{-1}.$$

The function s^{-1} is bounded from below by a positive constant on the interval $(0, 2r]$, and we are done in this case.

ad B.3. We estimate

$$\frac{\int_{\partial C \cap D} w_r \, d\sigma}{\int_C w_r} \geq \frac{(r/2)^{mp} \ell(\partial C)}{|C|} \geq \frac{(r/2)^{mp}}{|D|^{1/2}} \frac{\ell(\partial C)}{|C|^{1/2}}.$$

By the isoperimetric inequality the fraction $\ell(\partial C)/|C|^{1/2}$ has a positive lower bound independent of C. □

B.3 Cheeger Constant of a Gaussian

In this subsection we will study the Cheeger constant of the Gaussian $\varphi = e^{-\pi \cdot^2}$ on disks centered at 0.

THEOREM B.12. *For $p \in [1, \infty)$ there exists a constant $\delta > 0$, depending on p but independent of $R > 0$, such that*

$$h_{p, B_R(0)}(\varphi) \geq \delta.$$

PROOF. By Lemma A.5 there exists a positive constant r such that

$$|V_\varphi \varphi(x,y)|^p = r e^{-p\pi/2(x^2+y^2)}.$$

For $q \geq \pi/2$ let $w_q(x,y) := e^{-q(x^2+y^2)}$. Proving the statement amounts to showing that $h(B_R(0), w_q)$ is uniformly bounded away from zero for any fixed $q \geq \pi/2$. The restriction is, however, not necessary and it suffices to assume $q > 0$.

Let $\beta > 0$ be such that $\int_{\mathbb{R}^2} \beta w_q = 1$. For $C \subset \mathbb{R}^2$ and A a one-dimensional manifold, we will use the notations

$$\mu(C) := \beta \int_C w_q \quad \text{and} \quad \nu(A) := \beta \int_A w_q \, d\sigma,$$

where σ denotes the surface measure on A. For $R > 0$ let us define

$$\begin{aligned} \mathcal{C}_R &:= \left\{ \begin{matrix} C \subset B_R(0) \text{ open, connected: } \partial C \cap B_R(0) \text{ is smooth} \\ \text{and } \mu(C) \le \frac{1}{2}\mu(B_R(0)) \end{matrix} \right\}, \\ \mathcal{C}_R^i &:= \{C \in \mathcal{C}_R \colon \partial C \cap \partial B_R(0) = \varnothing\}, \\ \mathcal{C}_R^b &:= \{C \in \mathcal{C}_R \colon \partial C \cap \partial B_R(0) \neq \varnothing\}. \end{aligned}$$

Clearly $\mathcal{C}_R = \mathcal{C}_R^i \cup \mathcal{C}_R^b$.

Our proof will heavily rely on the fact that on probability spaces with log-concave measures an isoperimetric inequality holds true [9]; i.e., there exists $c > 0$ such that

$$\nu(\partial C) \ge c I(\mu(C)) \quad \text{for all } C \subset \mathbb{R}^2 \text{ with smooth boundary,} \tag{B.17}$$

where $I := \gamma \circ \Gamma^{-1}$ with

$$\gamma(t) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad \Gamma(t) := \int_{-\infty}^t \gamma(s) ds.$$

The function $I : [0,1] \to [0, 1/\sqrt{2\pi}]$ is strictly positive on $(0,1)$ and satisfies q$I(0) = I(1) = 0$. An elementary calculation yields $I''(t) = -1/\gamma(\Gamma^{-1}(t)) \le 0$; therefore I is concave. Since $I\left(\frac{1}{2}\right) = 1/\sqrt{2\pi}$ we have for any C such that $\mu(C) \le \frac{1}{2}$ that

$$\nu(\partial C) \ge c \cdot I(\mu(C)) \ge c \cdot \sqrt{\frac{2}{\pi}} \mu(C). \tag{B.18}$$

Note that since $\mathbb{R}^2$ has no boundary, equation (B.18) tells us that $h(\mathbb{R}^2, w_q) \ge c\sqrt{2/\pi}$.

First let $C \in \mathcal{C}_R^i$. Since $\mu(C) \le \frac{1}{2}\mu(B_R(0)) \le \frac{1}{2}$, we have

$$\frac{\nu(\partial C \cap B_R(0))}{\mu(C)} = \frac{\nu(\partial C)}{\mu(C)} \ge c \cdot \sqrt{\frac{2}{\pi}}.$$

Thus it remains to look at sets $C \in \mathcal{C}_R^b$.

For $R > 0$ let $\rho = \rho(R) > 0$ be such that $\mu(B_\rho(0)) = \frac{3}{4}\mu(B_R(0))$. Since C is connected, there is exactly one connected component A_0 of ∂C such that $A_0 \cap \partial B_R(0) \neq \varnothing$.

We will now have a look at the ratio $(R-\rho(R))/R$. Let $R < 1/\sqrt{q}$ and let $\alpha := \sqrt{3/4e}$; then

$$\mu(B_{\alpha R}(0)) = \beta \int_{B_{\alpha R}(0)} e^{-q|z|^2}\,dz$$
$$\leq \beta\alpha^2 R^2\pi = e\alpha^2\cdot\beta e^{-1}R^2\pi \leq e\alpha^2\mu(B_R(0)) = \frac{3}{4}\mu(B_R(0)),$$

and thus we obtain for $R \in (0, 1/\sqrt{q})$

$$\frac{R-\rho(R)}{R} \geq 1-\alpha > 0.$$

Note that $(R-\rho(R))/R$ is a nonnegative and continuous function of $R > 0$. Since $\rho(R)$ converges to a finite limit for $R \to \infty$, there exists a $\kappa > 0$ that only depends on q such that

$$R-\rho(R) \geq \kappa R \quad \text{for all } R > 0. \tag{B.19}$$

We will now distinguish three cases:

Case A. $A_0 \cap B_\rho \neq \varnothing$.

Case B. $A_0 \cap B_\rho = \varnothing$ and $C_0 \cap B_\rho \neq \varnothing$.

Case C. $A_0 \cap B_\rho = \varnothing$ and $C_0 \cap B_\rho = \varnothing$.

In the first two cases we will show that there exists a positive λ that does not depend on R and C such that

$$\ell(A_0 \cap \partial B_R(0)) \leq \lambda\cdot\ell(A_0 \cap B_R(0)).$$

This implies that $\nu(A_0 \cap \partial B_R(0)) \leq \lambda\nu(A_0 \cap B_R(0))$ and therefore

$$\nu(A_0) = \nu(A_0 \cap B_R(0)) + \nu(A_0 \cap \partial B_R(0)) \leq (1+\lambda)\nu(A_0 \cap B_R(0)).$$

Now we can estimate

$$\frac{\nu(\partial C \cap B_R(0))}{\mu(C)} = \frac{\nu(A_0 \cap B_R(0)) + \nu(\partial C \setminus A_0)}{\mu(C)}$$
$$\geq (1+\lambda)^{-1}\frac{\nu(\partial C)}{\mu(C)} \geq (1+\lambda)^{-1}c\sqrt{\frac{2}{\pi}},$$

where we used (B.18).

ad A. By (B.19) we have

$$\ell(A_0 \cap B_R(0)) \geq 2|R-\rho(R)| \geq 2\kappa R.$$

Since $\ell(A_0 \cap \partial B_R(0)) \leq 2R\pi$, we can choose $\lambda = \pi/\kappa$.

ad B. If C is such that $A_0 \cap \partial B_R(0)$ is not contained in any open half-plane H such that $0 \in \partial H$, then there has to be a connected component of $A_0 \cap B_R(0)$ with euclidean length at least R.

If $A_0 \cap \partial B_R(0)$, however, is contained in some half-plane H, there is an arch Λ of $B_R(0)$ in H whose endpoints are contained in $A_0 \cap \partial B_R(0)$ and $\Lambda \supseteq A_0 \cap \partial B_R(0)$. Then $\ell(A_0 \cap B_R(0)) \geq l$, where l denotes the distance between the endpoints of Λ. From basic geometry we know that $\ell(\Lambda)/2R = \arcsin(l/2R)$. Since $\arcsin(x) \leq \frac{\pi}{2}x$ for $x \in [0,1]$, we obtain

$$\frac{\ell(A_0 \cap \partial B_R(0))}{\ell(A_0 \cap B_R(0))} \leq \frac{\ell(\Lambda)}{l} = \frac{\arcsin(l/2R)}{l/2R} \leq \frac{\pi}{2}.$$

ad C. Let C' denote the open and bounded set with boundary A_0 and set $E := C' \setminus \overline{C}$. Since $B_\rho(0)$ is contained in C' we have $\mu(C') \geq \frac{3}{4}\mu(B_R(0))$ and

$$\mu(E) = \mu(C' \setminus C) = \mu(C') - \mu(C) \geq \frac{3}{4}\mu(B_R(0)) - \frac{1}{2}\mu(B_R(0)) = \frac{1}{4}\mu(B_R(0)).$$

In case $\mu(E) \leq \frac{1}{2}$ we estimate using (B.18)

$$\begin{aligned}\nu(\partial C \cap B_R(0)) \geq \nu(\partial E) \geq c\sqrt{\frac{2}{\pi}}\mu(E) &\geq c\sqrt{\frac{2}{\pi}}\frac{1}{4}\mu(B_R(0)) \\ &\geq c\sqrt{\frac{2}{\pi}}\frac{1}{2}\mu(C).\end{aligned}$$

If $\mu(E) > \frac{1}{2}$ we can apply (B.18) on the unbounded set $E' := \mathbb{R}^2 \setminus \overline{E}$. Since $B_R(0) \supseteq E$ it follows that $\mu(B_R(0)) > \frac{1}{2}$ and we obtain

$$\begin{aligned}&\nu(\partial C \cap B_R(0)) \geq \nu(\partial E) = \nu(\partial E') \geq c\sqrt{\frac{2}{\pi}}\mu(E') \\ &\quad = c\sqrt{\frac{2}{\pi}}(1-\mu(E)) \geq c\sqrt{\frac{2}{\pi}}(2\mu(B_R(0)) - \mu(B_R(0))) \geq c\sqrt{\frac{2}{\pi}}\mu(C). \quad \square\end{aligned}$$

Appendix C Spectral Clustering Algorithm

In this section we provide some details on the partitioning algorithm used for our experiment in Section 2.2. Spectral clustering methods are based on relating optimal partitioning of a graph to the eigenvector corresponding to the second eigenvalue of the so-called graph Laplacian.

Suppose we are given a set of finitely many points $V := \{v_1, \dots, v_l\} \subset \mathbb{R}^d$ and a similarity measure w on V, i.e.,

$$\begin{aligned}&w : V \times V \to [0,\infty), \quad w \text{ is symmetric,} \\ &\text{and} \quad w(v_i, v_i) = 0 \text{ for all } i \in \{1, \dots, l\}.\end{aligned}$$

A weighted, undirected graph G is associated to the pair (V, w) in a very natural way: The vertices of G are exactly the points $v_1, \dots, v_l$. Two vertices v_i, v_j are connected if and only if $w(v_i, v_j) > 0$, and in this case their connecting edge has weight $w(v_i, v_j)$. The matrix $W := (w(v_i, v_j))_{i,j=1}^l$ is called the *weight matrix* of G. For any $i \in \{1, \dots, l\}$ and $C \subset V$, we will use the notations

$$d_i := \sum_{j=1}^{l} W_{i,j}, \quad \mathrm{vol}(C) := \sum_{j: v_j \in C} d_j, \quad \mathrm{cut}_G(C) := \sum_{i,j: v_i \in C, v_j \in V \setminus C} W_{i,j},$$

for the *degree* of the i^{th} vertex, the *volume* of C, and the *cut* of C in V. The *Cheeger ratio* of a set $C \subset V$ is defined by

$$h_G(C) := \frac{\mathrm{cut}_G(C)}{\min\{\mathrm{vol}(C), \mathrm{vol}(V \setminus C)\}}$$

and the *Cheeger constant* of G by $h_G := \min_{C \subset V} h_G(C)$.

Let us from now on assume the graph is connected, i.e., $d_i > 0$ for all i. To compute a partition such that the corresponding Cheeger ratio is quasi-optimal, we will draw onto the results in [12]. Let I denote the $l \times l$ identity matrix and D the diagonal matrix with entries $d_1, \ldots, d_l$; then the *normalized graph Laplacian* is given by the matrix

$$L := I - D^{-\frac{1}{2}} W D^{-\frac{1}{2}}.$$

The vector $(1, \ldots, 1)^{\mathsf{T}}$ is an eigenvector of L with corresponding eigenvalue 0. By the assumption that G is connected, all other eigenvalues of L will be positive. The partition is computed by thresholding an eigenvector corresponding to the smallest positive eigenvalue of L.

Let u be an element of the eigenspace of the smallest positive eigenvalue of L, and let

$$C^* \text{ be a minimizer of } h_G(\cdot) : \{C_t : t \in \mathbb{R}\} \to [0, \infty),$$

where $C_t := \{v_i : u_i > t\}$. Then for $h_G^* := h_G(C^*)$ it holds that

$$h_G \le h_G^* \le 2 \cdot \sqrt{h_G}.$$

Here we have samples of $|V_\varphi f(x, y)|$ available for $(x, y) \in Z \subset \Delta \cdot \mathbb{Z}^2 + d$, where $d \in \mathbb{R}^2$ and $\Delta > 0$. On the set Z we define a similarity measure w by

$$w(z, z') := \begin{cases} \frac{1}{2}(|V_\varphi f|^p(z) + |V_\varphi f|^p(z')), & \text{if } |z - z'| = \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Let $C \subset Z$. Since $|V_\varphi f|$ is smooth, we obtain (for small Δ) that

$$\begin{aligned} \mathrm{cut}_G(C) = \sum_{\substack{z \in C \\ z' \in Z \setminus C}} w(z, z') &= \sum_{\substack{z \in C, z' \in Z \setminus C \\ |z - z'| = \Delta}} \frac{1}{2}(|V_\varphi f|^p(z) + |V_\varphi f|^p(z')) \\ &\approx \sum_{\substack{z \in C, z' \in Z \setminus C \\ |z - z'| = \Delta}} |V_\varphi f|^p \left(\frac{z + z'}{2} \right). \end{aligned}$$

Therefore, up to the factor Δ, $\mathrm{cut}_G(D)$ can be interpreted as a discrete version of the boundary integral in the nominator in equation (2.1). Similarly, $\mathrm{vol}(C)$ can be interpreted as an approximation of $\int_C |V_\varphi f|$ that occurs in the denominator.

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